Lecture 8 :

Convection - Diffusion - Reaction

$$\begin{cases} \dot{u} + \nabla\cdot(\beta u) + \alpha u - \nabla\cdot(\varepsilon \nabla u) = f & \text{in } \Omega \times I \\ u = g_- & \text{on } (\Gamma \times I)_- \\ u = g_+ \quad \text{or} \quad \varepsilon \partial_n u = g_+ & \text{on } (\Gamma\times I)_+ \\ u(\cdot,0) = u_0 & \text{on } \Omega \end{cases}$$

$\beta = (\beta_1, \beta_2, \beta_3)$ convection field

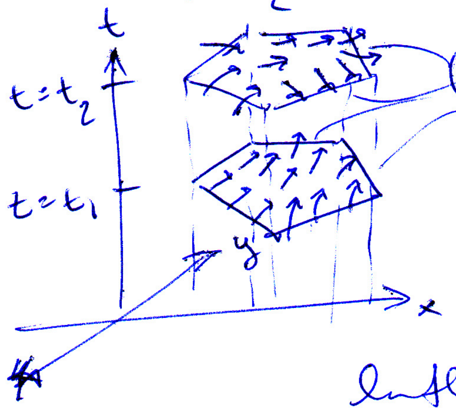
α = absorption coefficient

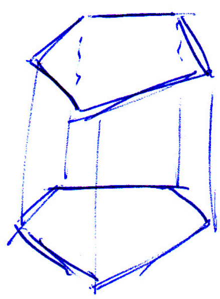
ε = diffusion coefficient

($\Omega \subset \mathbb{R}^3$, $\Gamma = \partial\Omega$, $I = (0,T]$)

$(\Gamma\times I)_- = \{(x,t) \in \Gamma\times I : \beta\cdot n < 0\}$ inflow boundary

$(\Gamma\times I)_+ = \{(x,t) \in \Gamma\times I : \beta\cdot n > 0\}$ outflow boundary





$\Omega \times I$

Inflow/outflow boundary may change with time !

$\nabla\cdot(\beta u) = \beta\cdot\nabla u + (\nabla\cdot\beta) u$

We may use $\beta\cdot\nabla u \Rightarrow \alpha \rightsquigarrow (\nabla\cdot\beta) + \alpha$

$\nabla\cdot(\beta u) \,/\, \beta\cdot\nabla u$: divergence/non-divergence form.

Divergence form $\nabla\cdot(\beta u)$

Assume: $\varepsilon\partial_n u = 0$ on $\Gamma\times I$ (insulation)

$\beta\cdot n = 0$ on $\Gamma\times I$ (no convection through the boundary)

$f = 0$ (no heat source)

$\alpha = 0$ (no absorption)

$$\Rightarrow \frac{d}{dt}\int_\Omega u\,dx = \int_\Omega \dot{u}\,dx = \int_\Omega \big(\nabla\cdot(\varepsilon\nabla u) - \nabla\cdot(\beta u)\big)\,dx$$

[Gauss Thm: (13.13) $\int_\Omega \nabla\cdot v\,dx = \int_\Gamma v\cdot n\,ds$]

$$= \int_\Gamma \Big(\varepsilon\partial_n u - \underbrace{(\beta u)\cdot n}_{u(\beta\cdot n)}\Big)\,ds = 0 \Rightarrow \int_\Omega u\,dx \text{ conserved!}$$

Non-divergence form: $\beta\cdot\nabla u$

$$\Rightarrow \frac{d}{dt}\int_\Omega u\,dx = \int_\Omega \dot{u}\,dx = \int_\Omega \nabla\cdot(\varepsilon\nabla u) - \beta\cdot\nabla u\,dx$$

$$= \int_\Omega \underbrace{\nabla\cdot(\varepsilon\nabla u) - \nabla\cdot(\beta u)}_{=0 \text{ as above}} + (\nabla\cdot\beta)\,u\,dx$$

$$= \int_\Omega (\nabla\cdot\beta)\,u\,dx$$

Conservative only if $\nabla\cdot\beta = 0$

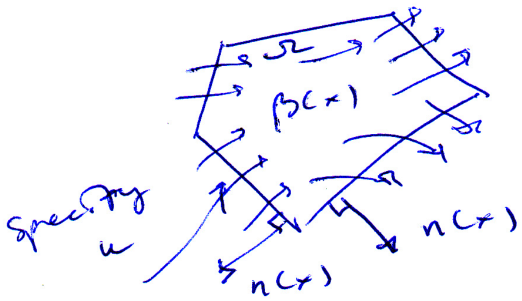
(β divergence free)

Stationary problem

$$\begin{cases} \beta \cdot \nabla u + \alpha u - \nabla \cdot (\varepsilon \nabla u) = f & \text{in } \Omega \\ u = g_- & \text{on } \Gamma_- \\ u = g_+ \text{ or } \varepsilon \partial_n u = g_+ & \text{on } \Gamma_+ \end{cases}$$

$\Gamma_- = \{ x \in \Gamma : \beta \cdot n < 0 \}$ inflow boundary

$\Gamma_+ = \{ x \in \Gamma : \beta \cdot n > 0 \}$ outflow boundary



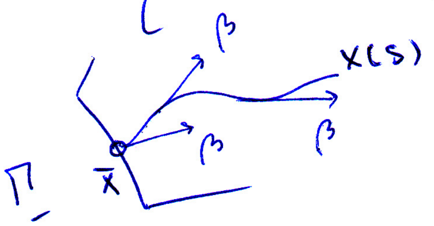
specify u or $\varepsilon \partial_n u$ (if $\varepsilon > 0$)

$\frac{\varepsilon}{|\beta|}$ small – convection dominated (hyperbolic)

$\frac{\varepsilon}{|\beta|}$ large – diffusion dominated (elliptic – see Ch. 15)

$\varepsilon \to 0 \Rightarrow$ reduced problem:

$$\begin{cases} \beta \cdot \nabla u + \alpha u = f & \text{in } \Omega \\ u = g_- & \text{on } \Gamma_- \end{cases}$$



$$\begin{cases} \frac{dx}{ds} = \beta(x(s)) & s > 0 \\ x(0) = \bar{x} \end{cases}$$

Assume non-closed streamlines $x(s)$: $x(s) \neq \bar{x}$ for $s > 0$.

Corresponds to ODE along streamlines $x(s)$:

$$\frac{d}{ds}\big(u(x(s))\big) + \alpha(x(s))\, u(x(s)) = (\beta \cdot \nabla u + \alpha u)(x(s)) = f(x(s)) \qquad s > 0$$

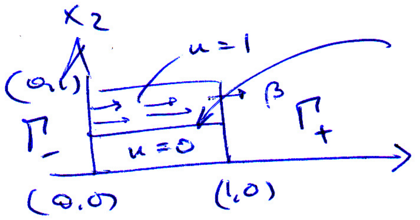
$$\left(\text{chain rule: } \frac{d}{ds} u(x(s)) = \frac{dx}{ds} \cdot \nabla u \right)$$

Initial data for ODE: $u(x(0)) = g_-(\bar{x})$

Information propagates sharply along streamlines!

Ex:
$$\begin{cases} \dfrac{\partial u}{\partial x_1} = 0 & x \in \Omega = [0,1] \times [0,1] \\ u(0,x_2) = \begin{cases} 0 & 0 < x_2 < 1/2 \\ 1 & 1/2 \le x_2 < 1 \end{cases} \end{cases} \qquad \left(\begin{array}{l} \beta = (1,0) \\ \alpha = 0 \\ f = 0 \end{array} \right)$$

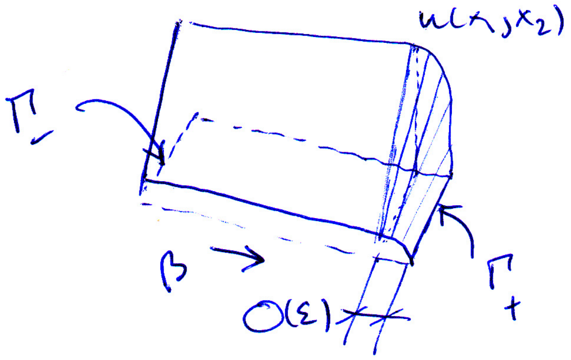
Solution: $u(x_1,x_2) = \begin{cases} 0 & 0 < x_2 < 1/2, \ 0 < x_1 < 1 \\ 1 & 1/2 \le x_2 < 1, \ 0 < x_1 < 1 \end{cases}$

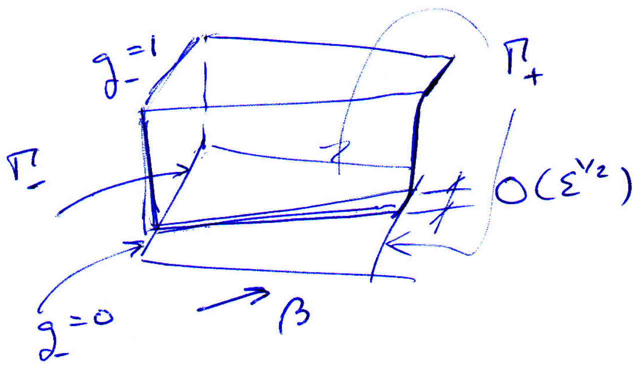


Discontinuity along streamline $x(s) = (s, \frac{1}{2})$

* The case $\dfrac{\varepsilon}{|\beta|}$ small similar to $\varepsilon \approx 0$.
* $\varepsilon > 0$ spreads out discontinuities $\Rightarrow$ continuous solution

(1) characteristic layer: $O(\sqrt{\varepsilon})$ (across streamline)
(2) boundary layer: $O(\varepsilon)$ (with Dirichlet b.c. Γ_+)



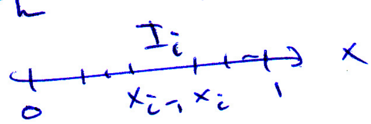


Ex (Problem 18.6)
$$\begin{cases} -\varepsilon u'' + u' = 0 & \text{on } \Omega = (0,1) \\ u(0) = 1, \ u(1) = 0 & (\Gamma_- = 0, \ \Gamma_+ = 1) \end{cases}$$

cG(1): Find $U \in V_h$: $\int_0^1 \varepsilon U' v' \, dx + \int_0^1 U' v \, dx = 0 \quad \forall v \in \hat{V}_h$

$$\begin{cases} V_h = \{ v \in H^1(0,1) : v(0) = 1, \ v(1) = 0 \} \\ \hat{V}_h = \{ v \in H^1(0,1) : v(0) = v(1) = 0 \} = H^1_0(0,1) \end{cases}$$

$\mathcal{T}_h$: uniform mesh with M interior nodes:



$h = \dfrac{1}{M+1}$

$$U(x)=\sum_{j=1}^{M} \xi_j \phi_j(x) \longrightarrow A\xi = b \qquad A=(a_{ij}),\ b=(b_i)$$

$$\begin{cases} a_{ij} = \int_0^1 \varepsilon \phi_j'(x)\phi_i'(x)\,dx + \int_0^1 \phi_j'(x)\phi_i(x)\,dx \\ b_i = -\int_0^1 \varepsilon u(0)\phi_0'(x)\phi_i'(x)\,dx + \int_0^1 u(0)\phi_0'(x)\phi_i(x)\,dx = -\int_0^1 (\varepsilon\phi_0'\phi_i' + \phi_0'\phi_i)\,dx \end{cases}$$

$$a_{ii} = \int_{I_i}\left(\varepsilon \frac{1}{h}\frac{1}{h} + \frac{1}{h}\frac{x-x_{i-1}}{h}\right)dx + \int_{I_{i+1}}\left(\varepsilon\left(\frac{-1}{h}\right)\left(\frac{-1}{h}\right) + \left(\frac{-1}{h}\right)\frac{x_i-x}{h}\right)dx = \frac{\varepsilon}{h}+\frac{1}{2}+\frac{\varepsilon}{h}-\frac{1}{2} = \frac{2\varepsilon}{h}$$

$$\left.\begin{aligned} a_{i-1\,i} &= \int_{I_i}\left(\varepsilon\left(\frac{-1}{h}\right)\frac{1}{h} + \left(\frac{-1}{h}\right)\frac{x-x_{i-1}}{h}\right)dx = -\frac{\varepsilon}{h}-\frac{1}{2} \\ a_{i\,i+1} &= \int_{I_{i+1}}\left(\varepsilon\frac{1}{h}\left(\frac{-1}{h}\right) + \frac{1}{h}\frac{x_i-x}{h}\right)dx = -\frac{\varepsilon}{h}+\frac{1}{2} \end{aligned}\right\}$$ A non-symmetric!

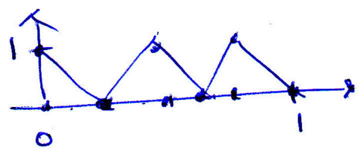
Equation i: $\sum_{j=1}^{n} \xi_j a_{ij} = \xi_{i-1}\left(-\frac{\varepsilon}{h}-\frac{1}{2}\right) + \xi_i \frac{2\varepsilon}{h} + \xi_{i+1}\left(-\frac{\varepsilon}{h}+\frac{1}{2}\right) = 0$ (for $i>1$).

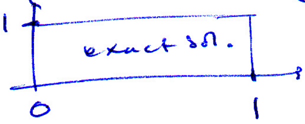
$\left(\frac{\varepsilon}{h}\right)$ large $\rightarrow -\xi_{i-1} + 2\xi_i - \xi_{i+1} = 0$ (Poisson-like...)

$\left(\frac{\varepsilon}{h}\right)$ small $\rightarrow -\frac{1}{2}\xi_{i-1} + \frac{1}{2}\xi_{i+1} = 0 \iff \xi_{i+1} = \xi_{i-1}$

$\Rightarrow$ solution ξ oscillating (M even) or A singular (M odd)

M even : $\varepsilon = 0$:

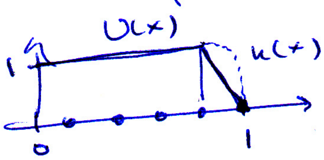




Stabilization: add artificial viscosity : $\varepsilon = \frac{h}{2}$

$\Rightarrow$ New equations : $\boxed{-\xi_{i-1} + \xi_i = 0}$ (upwind method)

$\Rightarrow$



- Galerkin FEM optimal for diffusion dominated problems
- —— " —— not optimal for convection —— " ——
- For non-smooth exact solutions U contains spurious oscillations when using standard FEM when $\frac{\varepsilon}{h}$ small.
- Artificial viscosity $\varepsilon \sim h \rightarrow$ stability (no oscillations) but bad accuracy (no resolution of layers).

Streamline diffusion method

(a) introduction of weighted least squares terms.
(b) —— " —— artificial viscosity based on residual.

(D) $Au = f$ (differential equation)

(G) Find $U \in V_h$: $(AU, v) = (f, v) \quad \forall v \in V_h$

(LS) Find $U \in V_h$: $\|AU - f\|^2 = \min_{v \in V_h} \|Av - f\|^2$

corresponds to $(AU, Av) = (f, Av) \quad \forall v \in V_h$

(GLS): Find $U \in V_h$: $(AU, v) + (\delta AU, Av) = (f, v) + (\delta f, Av) \quad \forall v \in V_h$

(weighted combination of (G) & (LS) : $\delta > 0$)

Petrov-Galerkin method : $(V_h \neq \hat{V}_h)$

Find $U \in V_h$: $(AU, v + \delta Av) = (f, v + \delta Av) \quad \forall v \in V_h$

(Sd): Find $U \in V_h$: $(AU, v + \delta Av) + (\hat{\varepsilon} \nabla U, \nabla v) = (f, v + \delta Av)$

$\hat{\varepsilon} = \gamma_1 h^2 |R(U)|$ (artificial viscosity) $\quad \forall v \in V_h$

Assume $(Av, v) \geq c\|v\|^2 \quad c > 0$ ~~[illegible]~~

Set $v = U$ in (Sd) $\Rightarrow (AU, U) + (\delta AU, AU) + (\hat{\varepsilon} \nabla U, \nabla U) = (f, U) + (\delta f, AU)$

$$\Rightarrow c\|U\|^2 + \|\sqrt{\delta} AU\|^2 + \|\sqrt{\hat{\varepsilon}} \nabla U\|^2 \leq \|f\| \|U\| + \|\sqrt{\delta} f\| \|\sqrt{\delta} AU\|$$

$$\leq \frac{c}{2}\|U\|^2 + \frac{1}{2c}\|f\|^2 + \frac{1}{2}\|\sqrt{\delta} AU\|^2 + \frac{1}{2}\|\sqrt{\delta} f\|^2$$

$$\Rightarrow \frac{c}{2}\|U\|^2 + \frac{1}{2}\|\sqrt{\delta} AU\|^2 + \|\sqrt{\hat{\varepsilon}} \nabla U\|^2 \leq \frac{1}{2c}\|f\|^2 + \frac{1}{2}\|\sqrt{\delta} f\|^2$$

$$\Rightarrow \left(\|U\| + \|\sqrt{\delta} AU\| + \|\sqrt{\hat{\varepsilon}} \nabla U\|\right)^2 \leq C\|f\|^2$$

$$\Rightarrow \boxed{\|U\| + \|\sqrt{\delta} AU\| + \|\sqrt{\hat{\varepsilon}} \nabla U\| \leq C\|f\|}$$

Set $v = U$ in (G): $(AU, U) = (f, U)$

$\Rightarrow c\|U\|^2 \le (f, U) \le \frac{c}{2}\|U\|^2 + \frac{1}{2c}\|f\|^2$

$\Rightarrow c\|U\|^2 \le \frac{1}{c}\|f\|^2 \Rightarrow \boxed{\|U\| \le C\|f\|}$

For (G) we only have control of $\|U\|$, whereas for (Sd) we have control of: $\|U\|, \|\sqrt{\delta}AU\|, \|\sqrt{\hat{\varepsilon}}\nabla U\|$

Ex: $Au = \beta \cdot \nabla u + \alpha u$

Add small diffusion: $Au - \nabla \cdot (\varepsilon \nabla u) = f$

$\Rightarrow \beta \cdot \nabla u + \alpha u - \nabla \cdot (\varepsilon \nabla u) = f$

(G) Find $U \in V_h$: $(AU, v) + (\varepsilon \nabla U, \nabla v) = (f, v) \ \forall v \in V_h$

$\ldots \Rightarrow (v = U)$ $\boxed{\|\sqrt{\varepsilon}\nabla U\| + \|U\| \le C\|f\|}$

(Sd) Find $U \in V_h$: $(AU, v) + (\delta AU, Av) + (\hat{\varepsilon}\nabla U, \nabla v) = (f, v) \ \forall v \in V_h$

$\ldots \Rightarrow$ $\boxed{\|\sqrt{\hat{\varepsilon}}\nabla U\| + \|\sqrt{\delta}AU\| + \|U\| \le C\|f\|}$

with $\hat{\varepsilon} = \max\{\varepsilon, \gamma, h^2|f - AU|\}$

- For small ε we have no control of derivatives in (G) $\Rightarrow$ oscillations may occur.
- In (Sd) we always have controll of $\|\sqrt{\delta}AU\|$ & $\|\sqrt{\hat{\varepsilon}}\nabla U\|$.

Navier–Stokes equations

$$\begin{cases} \dot{u} + u\cdot\nabla u + \nabla p - \nu\Delta u = f & \text{in } \Omega\times(0,T] \\ \nabla\cdot u = 0 & \text{in } \Omega\times(0,T] \\ u = 0 \quad \text{on } \Gamma\times(0,T] & \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

V.F. Find $(u,p)\in V\times Q$:

$$(\dot{u},v) + (u\cdot\nabla u, v) - (p,\nabla\cdot v) + (\nu\nabla u,\nabla v) + (\nabla\cdot u, q) = (f,v) \quad \text{for all } (v,q)\in V\times Q$$

FEM: Find $(U,P)\in V_h\times Q_h$:

$$(\dot{U},v) + (U\cdot\nabla U, v) - (P,\nabla\cdot v) + (\nu\nabla U,\nabla v) + (\nabla\cdot U, q) = (f,v) \quad \forall (v,q)\in V_h\times Q_h$$

GLS: Find $(U,P)\in V_h\times Q_h$:

$$(\dot{U},v) + (U\cdot\nabla U, v) - (P,\nabla\cdot v) + (\nu\nabla U,\nabla v) + (\nabla\cdot U, q) + (\delta(U\cdot\nabla U+\nabla P), U\cdot\nabla v+\nabla q) = (f, v+\delta(U\cdot\nabla v+\nabla q))$$

for all $(v,q)\in V_h\times Q_h$

Set $v=u$, $q=p$ in (V.F.) using that $(u\cdot\nabla u, u)=0$ (with $f=0$)

$$(\dot{u},u) + (\nu\nabla u,\nabla u) = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\sqrt{\nu}\nabla u\|^2 = 0 \Rightarrow \|u(T)\|^2 + \int_0^T \|\nu\nabla u\|^2 dt = \|u_0\|^2$$

Set $(v,q) = (U,P)$ in (GLS) $\Rightarrow$

$$\|U(T)\|^2 + \int_0^T \|\nu\nabla U\|^2 dt + \int_0^T \|\sqrt{\delta}(U\cdot\nabla U+\nabla P)\|^2 dt = \|u_0\|^2$$
