DN2660 The Finite Element Method: Written Examination
Thursday 2012-10-18, 8-13
Coordinator: Johan Jansson

Aids: none Time: 5 hours

Answers must be given in English. All answers should be explained and calculations shown unless stated otherwise. A correct answer without explanation can be given zero points, while a good explanation with an incorrect answer can give some points. Maximum is 30 points.

Good luck,
Johan

Problem 1 - Galerkin's method

Consider the equation:

$$\begin{aligned} -\nabla \cdot (a(x)\nabla u(x)) + b(x) \cdot \nabla u(x) = f(x), &\quad x \in \Omega \\ u(x) = 0, &\quad x \in \Gamma \end{aligned}$$

where $a(x), b(x)$ and $f(x)$ are known coefficients and $b(x)$ is vector-valued.

Recall the formula:

$$\int_\Omega D_{x_i} vw dx = \int_\Gamma vwn_i ds - \int_\Omega vD_{x_i} w dx, \quad i = 1, 2, ..., d$$

1. (2p) Formulate a finite element method (Galerkin's method) for the equation using piecewise linear approximation (cG(1)).
2. (1p) Explain what the Galerkin orthogonality means, both in general and for this equation.
3. (1p) Formulate a Robin boundary condition and use it to enforce the homogenous Dirichlet boundary condition.
4. (2p) What is an L_2 projection? What is the relation to Galerkin's method?

Solution 1

1. (2p)

   Multiply by test function $v \in V$ with $v(x) = 0, x \in \Gamma$ and use integration by parts: $(-\nabla \cdot (a(x)\nabla u(x)), v(x)) = (a(x)\nabla u(x), \nabla v(x)) - \int_\Gamma (\nabla u(x) \cdot n)v(x)ds$ where the boundary term is zero due to the condition on v (1p)

   $$\begin{aligned} (a(x)\nabla u(x), \nabla v(x)) + (b(x)u(x), v(x)) - (f(x), v(x)) = 0, \quad & x \in \Omega, \forall v \in V \\ (R(u), v) = (a(x)\nabla u(x), \nabla v(x)) + (b(x)u(x), v(x)) - (f(x), v(x)) = 0, \quad & x \in \Omega, \forall v \in V \\ u(x) = 0, \quad & x \in \Gamma \end{aligned}$$

   Seek approximation $U = \sum_{i=1}^N \xi_i \phi_i \in V_h$ with $(R(U), v) = 0, \quad \forall v \in V_h$. (1p)

   Thus:

   $$\begin{aligned} (R(U), v) = (a(x)\nabla U(x), \nabla v(x)) + (b(x)U(x), v(x)) - (f(x), v(x)) = 0, \quad & x \in \Omega, \quad \forall v \in V_h \\ U(x) = 0, \quad & x \in \Gamma \end{aligned}$$

2. (1p)

   The Galerkin Orthogonality is the condition $(R(U), v) = 0, \forall v \in V_h$ we enforce on U. See above for the formulation for this equation.

3. (1p) A Robin boundary condition is formulated as such:

   $$-\nabla u(x) \cdot n = \gamma(u - g_D) + g_N$$

   By setting $\frac{1}{\gamma} = 0$ (i.e. γ large), we can enforce the homogenous Dirichlet condition $u = 0$, by choosing $g_D = 0$.

4. (2p) An L_2 projection is solving the equation $R(u) = u - f = 0$ by Galerkin's method, seeking an approximation $U = \sum_{i=1}^N \xi_i \phi_i \in V_h$ of u with $(R(U), v) = 0, \quad \forall v \in V_h$. It can be viewed as projecting f into the finite element function space V_h, which is also the best possible approximation in the L_2 norm.

Problem 2 - Stability

Consider the heat equation with zero source:

$$\begin{aligned} \dot{u} - \Delta u = 0, \quad & x \in \Omega, \quad t \in [0, T] \\ u(0, x) = u_0(x) & \\ u(t, x) = 0, \quad & x \in \Gamma \end{aligned}$$

The $cG(1)dG(0)$ method for this equation is:

$$(U_n, v) = (U_{n-1}, v) - k_n(\nabla U(t_n, x), \nabla v)), \quad \forall v \in V_h \times W_k$$

1. (2p) Derive the stability estimate:

   $$\|U_n\| \leq \|U_{n-1}\|$$

   Explain what a stability estimate is in general, and give an interpretation what this particular stability estimate says about the discrete temperature U.

2. (2p) Explain the basic concept behind a streamline diffusion stabilized finite element method.

Solution 2

1. (2p)

   See module 7 for derivation of the stability estimate.

   Generally a stability estimate bounds the solution or derivatives of the solution (u, ∇u) in terms of data (f, u_0). If we have a stability estimate we can be sure that the solution does not grow uncontrollably and we can use this property in further error estimation.

   In this specific case we can see that the norm of the discrete temperature $\|U(t)\|$ can never increase in time.

2. (2p)

   See module 7 for an explanation of the concept behind streamline diffusion.

Problem 3 - Assembly of a linear system

1. (3p) Formulate a general assembly algorithm of a linear system given a bilinear form $a(u, v)$ and linear form $L(v)$ representing a linear boundary value partial differential equation (PDE) in 2D/3D, with a piecewise linear Galerkin approximation (cG(1)). Include explanations of the following concepts:

   - Mesh
   - Map from reference cell
   - Formula for computation of a matrix and vector element
   - Quadrature

2. (2p) Define a basic linear boundary value PDE in 1D or 2D. Apply Galerkin's method, construct a simple mesh and compute a matrix element by hand (you don't have to use a general assembly algorithm here).

Solution 3

1. (3p)

   See module 4 for a description of a general assembly algorithm.

2. (2p)

   See CDE chapter 8 for an example of assembly of a boundary value PDE in 1D.

Problem 4 - Error estimation

Consider the equation:

$$\begin{aligned} -u'' + u = f, \quad x \in [0,1] \\ u(0) = u(1) = 0 \end{aligned}$$

1. (2p) Show that Galerkin's method is optimal for the equation and derive an a priori error estimate in the energy norm $\|w\|_E$.

2. (1p) Explain what an a posteriori error estimate is, give a definition of the energy norm for the equation and explain why the energy norm is often used.

3. (3p) Derive an a posteriori error estimate using duality for a general quantity of the error (e, ψ) in the form: $|(e, \psi)| \leq C_i h^2 \|R(U)\| \|\phi''\|$, where ϕ is the dual solution and $R(U)$ the residual. Use continuous piecewise linear approximation and the interpolation estimate $\|\phi - \pi\phi\| \leq C_i h^2 \|\phi''\|$.

Solution 4

1. (2p) We define the energy norm for this equation: $\|w\|_E = \sqrt{a(w,w)} = \sqrt{\int_0^1 (w')^2 + w^2 dx}$.
   We then proceed with the a priori error estimate:

$$\begin{aligned} \|e\|_E^2 = (e, e)_E = (u - U, u - U)_E = \\ (u - U, u - U)_E + (u - U, v - v)_E = \\ (u - U, u - v)_E + (u - U, v - U)_E = \\ (u - U, u - v)_E \leq \|e\|_E \|u - v\|_E \Rightarrow \\ \|e\|_E \leq \|u - v\|_E, \quad \forall v \in V_h \end{aligned}$$

   This proves that there is no better approximation than U in V_h in the energy norm (if we can define the energy norm).

Continuing, remembering that interpolant $\pi u \in V_h$ and using interpolation estimate $\|u - \pi u\|_E \leq Ch\|u'\|_E$:

$$\|e\|_E \leq \|u - v\|_E, \quad \forall v \in V_h \Rightarrow$$
$$\|e\|_E \leq \|u - \pi u\|_E \leq Ch\|u'\|_E$$

Which means that the energy norm of the error converges to zero with first order rate.

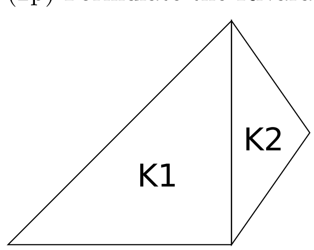
2. (1p) See module 5 for a discussion what an a posteriori error estimate is. The energy norm is defined above. The energy norm is defined using the bilinear form (which is the differential operator) like so: $\|e\|^2 = a(e, e)$, which allows the use of the Galerkin orthogonality in the form: $a(e, v) = 0, \forall v \in V_h$.

3. (3p) See module 5. (1p) for the error representation $(e, \psi) = \int_\Omega -U'\phi' - U\phi + f\phi dx$. (1p) for integrating by parts and enabling the use of the interpolation estimate. (1p) for the rest of the structure of the derivation.

Problem 5 - Adaptivity

1. (3p) Formulate an adaptive finite element method based on an a posteriori error estimate with local mesh refinement given a tolerance TOL on a quantity or norm of the error $e = u - U$. Discuss why adaptivity is important.

2. (2p) Formulate the Rivara recursive bisection algorithm. Consider the mesh:



Mark the triangle K2 for refinement and perform the Rivara algorithm by hand, show all steps.

Solution 5

1. (3p)

   See module 6 for a formulation of an adaptive algirithm. (2p)

   Adaptivity is important because it can greatly improve efficiency. If we don't have adaptivity we must refine the mesh uniformly (everywhere) to be sure that the error converges. If the error contribution is localized, this efficiency difference could be enormous. (1p)

2. (2p)

   See module 6 for a formulation of the Rivara algorithm. (1p)

   First we call bisect(K2), where we will bisect the longest edge of K2, the edge e_{12} between K1 and K2, creating two new cells K3 and K4. We check if all cells incident to the edge are conforming, and see that K1 is not conforming because there is a hanging node on the edge e_{12}.

   We thus call bisect(K1), where we will bisect the longest edge (diagonal edge) of K1, thus creating two new cells, K5 and K6 where K6 is incident to e_{12}. We see that K6 is not conforming because there is still a hanging node on the edge e_{12}.

   We thus call bisect(K6), where we bisect the edge e_{12}, thus creating two new cells K7 and K8, thus eliminating the hanging node on e_{12}. We now have no further hanging nodes and the original bisect(K2) call will return. (1p)

Problem 6 - Abstract formulation

1. (2p) Explain what the Lax-Milgram theorem says, what it requires to be satisfied, and what it can be used for.

2. (2p) Define a linear, time-independent boundary value PDE of your choice and show why or why not the Lax-Milgram theorem is satisfied (an argument is sufficient to show that it's not satisfied).

Solution 6

1. (2p)

   See module 8 for a formulation and explanation of the Lax-Milgram theorem. The theorem can be used to prove existence and uniqueness of solutions to linear, elliptic boundary value PDE.

2. (2p)

   See CDE chapter 21.