School of Computer Science and Communication, KTH

2D1260 Finite Element Methods: Written Examination

Tuesday 2007-01-16, kl 8-13

Coordinator: Johan Hoffman

Aids: none. **Time:** 5 hours.

Answers may be given in English or in Swedish. All answers should be explained and calculations shown unless stated otherwise. A correct answer without explanation can be left without points. Do not leave integrals or systems of equations unsolved unless explicitly allowed. *Each of the 5 problems gives 10 p, resulting in a total of 50 p: 20 p for grade 3, 30 p for grade 4, and 40 p for grade 5.*

Problem 1: Consider the problem:

$$\begin{aligned} -\Delta u(x) &= 1 \quad x \in \Omega \subset \mathbb{R}^2 \\ u(x) &= 0 \quad\;\; x \in \partial\Omega \end{aligned}$$

with $x = (x_1, x_2)$ and Ω defined in Fig. 1.

(a) Formulate a finite element method (FEM) using a continuous piecewise linear approximation (cG(1)) defined on the mesh in Fig. 1.
(b) Compute the corresponding matrix and vector. You do not have to solve the resulting system of equations. *Hint: Many element integrals are the same.*

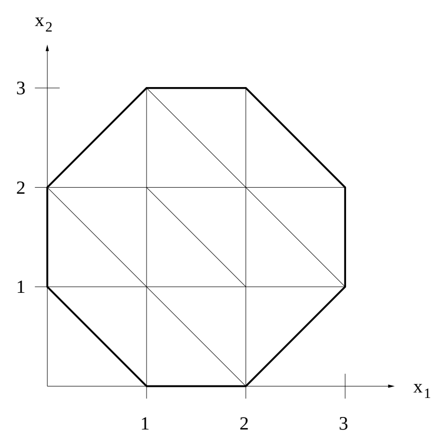

Figure 1: Triangulation (mesh) of domain Ω.

Note: *The exam continues on the next page!*

Problem 2: Consider the problem:

$$
\begin{aligned}
-\Delta u(x) + \alpha u(x) &= f(x), \quad x \in \Omega \subset \mathbb{R}^3, \\
\beta \partial_n u(x) + \gamma u(x) &= g(x), \qquad x \in \Gamma,
\end{aligned}
$$

with $\partial_n u = \nabla u \cdot n$, n the outward normal of the boundary Γ, and α,β,γ are non-negative constants.

State the Lax-Milgram theorem. Determine if the assumptions of the Lax-Milgram theorem are satisfied in the following cases:

(a) $\alpha = 0,\ \beta = 0,\ \gamma = 1,\ g = 0,\ f \in L_2(\Omega)$
(b) $\alpha = 0,\ \beta = 1,\ \gamma = 0,\ g = 0,\ f \in L_2(\Omega)$
(c) $\alpha = 1,\ \beta = 1,\ \gamma = 0,\ g \in L_2(\Gamma),\ f = 0$

For each case (a)-(c); derive a bilinear form $a : V \times V \to \mathbb{R}$ and a linear form $L : V \to \mathbb{R}$, and specify the Hilbert space V and the norm $\|\cdot\|_V$.

Hint: The following Trace Inequality may be useful: There exists a constant C, such that for all $v \in H^1(\Omega)$, we have that

$$\|v\|_{L_2(\Gamma)} \leq C\|v\|_{H^1(\Omega)}.$$

Problem 3: Consider the problem:

$$
\begin{aligned}
-(a(x)u'(x))' &= f(x) \quad x \in (0,1) \\
u(0) &= u(1) = 0
\end{aligned}
$$

The energy norm $\|\cdot\|_E$ for this problem is defined as $\|v\|_E = \|v'\|_a$, with the weighted L_2 norm

$$\|w\|_a = \left(\int_0^1 a(x) w^2(x)\, dx \right)^{1/2}$$

(a) Formulate the cG(1) method for the problem (FEM with a continuous piecewise linear approximation on a subdivision $\mathcal{T}_h$ of (0,1)).
(b) Prove that the cG(1) solution U is the best possible solution in the space $V_h = \{$continuous piecewise linear functions v on $\mathcal{T}_h$ with $v(0) = v(1) = 0\}$, with respect to the norm $\|\cdot\|_E$.
(c) Prove the a priori error estimate: $\|u - U\|_E \leq C_i \|hu''\|_a$
(d) Prove the a posteriori error estimate: $\|u - U\|_E \leq C_i \|hR(U)\|_{a^{-1}}$

The residual $R(U) = f + (aU')'$ is defined on each subinterval $I_i = (x_{i-1}, x_i)$, where x_i are the nodes, and C_i is an interpolation constant.

Note: *The exam continues on the next page!*

Problem 4: Consider the problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = 0, \quad \forall (x,t) \in \Omega \times (0,T], \\
&u(x,t) = 0, \quad \forall x \in \partial\Omega, \\
&u(x,0) = u_0(x).
\end{aligned}$$

Prove the following stability estimate:

$$\frac{1}{2}\|u(T)\|^2 + \int_0^T \|\nabla u\|^2 \, dt = \frac{1}{2}\|u_0\|^2,$$

where $\|\cdot\|$ is the standard L_2-norm of functions defined on Ω.

Problem 5: Answer the following questions related to standard FEM algorithms (it may be helpful to illustrate your answers with pictures):

(a) What is a least squares stabilized finite element method?
(b) Describe how a mapping to a reference element is used to compute element integrals, in the case of triangular elements (you do not have to carry out any computations, just illustrate the idea and function of the algorithm).
(c) What is a hanging node?
(d) Describe the steps in a red-green mesh refinement algorithm for triangles.
(e) What is the empty circle property of a Delaunay triangulation?

Good Luck!

Johan

Solutions to exam

Problem 1: See pages 360-363 in the CDE book.

(a) Find $U \in V_h$ such that

$$\int_\Omega \nabla U(x) \cdot \nabla v(x) \; dx = \int_\Omega v(x) \; dx \quad \forall v \in V_h \tag{1}$$

(b) $V_h = \{$continuous piecewise linear functions v on $\mathcal{T}_h$ such that v=0 on $\partial\Omega\}$, with $\mathcal{T}_h$ the triangulation of Ω in Fig. 2, with $h = 1$. There are 4 degrees of freedom; the nodes N_1, N_2, N_3, N_4.

A basis for V_h is $\{\phi_i\}_{i=1}^4$; with $\phi_i \in V_h$, and $\phi_i(N_j) = 1$ if $j = i$ and $\phi_i(Nj) = 0$ if $j \neq i$. Set $U(x) = \sum_{i=1}^4 \xi_i \phi_i(x)$, then (1) is equivalent to $A\xi = b$ where A is a 4×4 matrix, and b is a 4-vector, defined by

$$A_{ij} = \int_\Omega \nabla\phi_i(x) \cdot \nabla\phi_j(x) \; dx, \quad b_i = \int_\Omega \phi_i(x) \; dx$$

A_{11} involves integration over elements $e_1, e_2, e_3, e_5, e_6, e_7$, where e_1, e_7 are of the type in Fig.15.8 at page 362 in the CDE book, with integral $\int_{e_1} \nabla\phi_1 \cdot \nabla\phi_1 \; dx = 1$, and e_2, e_3, e_5, e_6 are of the type in Fig.15.9, with integral $\int_{e_2} \nabla\phi_1 \cdot \nabla\phi_1 \; dx = 1/2$. Thus

$$A_{11} = \int_{e_1} + \int_{e_2} + \int_{e_3} + \int_{e_5} + \int_{e_6} + \int_{e_7} = 1 + 1/2 + 1/2 + 1/2 + 1/2 + 1 = 4$$

A_{22} involves integration over elements e_3, e_4, e_7, e_8, e_9, where e_3, e_4, e_9 are of the type in Fig.15.8 at page 362 in the CDE book, with integral $\int_{e_3} \nabla\phi_2 \cdot \nabla\phi_2 \; dx = 1$, and e_7, e_8 are of the type in Fig.15.9, with integral $\int_{e_7} \nabla\phi_2 \cdot \nabla\phi_2 \; dx = 1/2$. Thus

$$A_{22} = \int_{e_3} + \int_{e_4} + \int_{e_7} + \int_{e_8} + \int_{e_9} = 1 + 1 + 1/2 + 1/2 + 1 = 4$$

We further have that $A_{33} = A_{22}$ and $A_{44} = A_{11}$.

A_{12} involves integration over elements e_3, e_7, which are of the type in Fig.15.10 at page 363 in the CDE book, with integral $\int_{e_3} \nabla\phi_1 \cdot \nabla\phi_2 \; dx = -1/2$. Thus

$$A_{12} = \int_{e_3} + \int_{e_7} = -1/2 - 1/2 = -1$$

A_{13} involves integration over elements e_6, e_7, which are of the type in Fig.15.10 at page 363 in the CDE book, with integral $\int_{e_6} \nabla\phi_1 \cdot \nabla\phi_3 \; dx = -1/2$. Thus

$$A_{13} = \int_{e_6} + \int_{e_7} = -1/2 - 1/2 = -1$$

$A_{14} = 0$ since there is no overlap of ϕ_1 and ϕ_4.

$A_{21} = A_{12}$

$A_{23} = 0$ by Problem 15.21 in the book.

A_{24} involves integration over elements e_8, e_9, which are of the type in Fig.15.10 at page 363 in the CDE book, with integral $\int_{e_8} \nabla\phi_2 \cdot \nabla\phi_4 \, dx = -1/2$. Thus

$$A_{24} = \int_{e_8} + \int_{e_9} = -1/2 - 1/2 = -1$$

$A_{31} = A_{13} = -1$

$A_{32} = A_{23} = 0$

A_{34} involves integration over elements e_8, e_{12}, which are of the type in Fig.15.10 at page 363 in the CDE book, with integral $\int_{e_8} \nabla\phi_3 \cdot \nabla\phi_4 \, dx = -1/2$. Thus

$$A_{34} = \int_{e_8} + \int_{e_{12}} = -1/2 - 1/2 = -1$$

$A_{41} = A_{14} = 0$

$A_{42} = A_{24} = -1$

$A_{43} = A_{34} = -1$

$$b_1 = \int_\Omega \phi_1(x) \, dx = \text{volume under } \phi_1 = 6 \times \frac{\frac{h^2}{2} \times 1}{3} = h^2 = 1, \text{ with } b_4 = b_1,$$

and $b_2 = \displaystyle\int_\Omega \phi_1(x) \, dx = \text{volume under } \phi_2 = 5 \times \frac{\frac{h^2}{2} \times 1}{3} = \frac{5}{6}h^2 = \frac{5}{6}$, with $b_3 = b_2$.

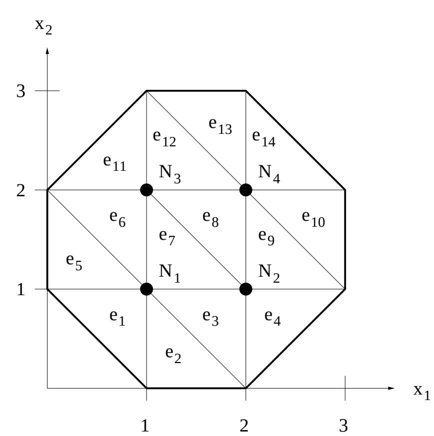

Figure 2: Triangulation (mesh) of domain Ω.

Problem 2: Theorem 21.1 in the CDE book.

(a) Section 21.4.3 in CDE book. Use the Poincare-Friedrich inequality (Theorem 21.4) to prove that $a(\cdot,\cdot)$ is elliptic. Continuity of $a(\cdot,\cdot)$ follows by

$$\begin{aligned} |a(v,w)| &= \int_\Omega \nabla v \cdot \nabla w \; dx \le \|\nabla v\| \; \|\nabla w\| \\ &\le (\|\nabla v\|^2 + \|v\|^2)^{1/2}(\|\nabla w\|^2 + \|w\|^2)^{1/2} = \|v\|_V \|w\|_V \end{aligned}$$

(b) The assumptions of the Lax-Milgram theorem are not satisfied: With $V = H^1(\Omega)$ we can prove that $a(\cdot,\cdot)$ and $L(\cdot)$ are continuous, but we cannot prove that $a(\cdot,\cdot)$ is elliptic; we cannot bound the L_2-norm of the solution using Poincare-Friedrich inequality (Theorem 21.4) since we do not know anything about the solution on the boundary (we have only Neumann boundary conditions).

(c) Section 21.4.4 in the CDE book. Continuity of $a(\cdot,\cdot)$ follows by

$$\begin{aligned} |a(v,w)| &= \int_\Omega (\nabla v \cdot \nabla w + vw) \; dx \le \|\nabla v\| \; \|\nabla w\| + \|v\| \; \|w\| \\ &= (\|\nabla v\|, \|v\|) \cdot (\|\nabla w\|, \|w\|) \le \|v\|_V \|w\|_V \end{aligned}$$

and continuity of $L(\cdot)$ follows by

$$|L(v)| = \int_\Gamma gv \; ds \le \|g\|_{L_2(\Gamma)} \|v\|_{L_2(\Gamma)} \le \|g\|_{L_2(\Gamma)} C \|v\|_{H^1(\Omega)},$$

so that $\kappa_3 = C\|g\|_{L_2(\Gamma)}$.

Problem 3:

(a) Find $U \in V_h$ such that $\int_0^1 aU'v' \; dx = \int_0^1 fv \; dx$ for all $v \in V_h$

(b) Section 8.2.1

(c) Section 8.2.1

(d) Section 8.2.2

Problem 4:

Section 16.3

Problem 5: See section 18.3 in the CDE book, and the lecture notes from lecture 2, slides avaliable at:
`http://www.csc.kth.se/utbildning/kth/kurser/2D1260/fem06/`