School of Computer Science and Communication, KTH

2D1260 Finite Element Methods: Written Examination

Saturday 2006-10-21, kl 8-13

Coordinator: Johan Hoffman

Aids: none. **Time:** 5 hours.

Answers may be given in English or in Swedish. All answers should be explained and calculations shown unless stated otherwise. A correct answer without explanation can be left without points. You do not have to solve the resulting systems of equations in Problem 1(b)-(c). *Each of the 5 problems gives 10 p, resulting in a total of 50 p: 20 p for grade 3, 30 p for grade 4, and 40 p for grade 5.*

Problem 1: Consider the problem:

$$\begin{aligned} -\Delta u(x) &= 1, \quad x \in \Omega \subset \mathbb{R}^2, \\ u(x) &= 0, \qquad x \in \partial\Omega, \end{aligned}$$

with $x = (x_1, x_2)$ and Ω the square defined in Fig. 1.

(a) Formulate a finite element method (FEM) using a continuous piecewise linear approximation (cG(1)) defined on the mesh in Fig. 1.
(b) Compute the corresponding matrix and vector.
(c) Compute the matrix and vector, with a non homogeneous Dirichlet condition

$$u(x) = 1, \quad \text{for } x_1 = 2, 0 < x_2 < 2 \quad \text{(that is, not for the corner nodes)},$$

with still homogeneous Dirichlet boundary conditions for the rest of the boundary.

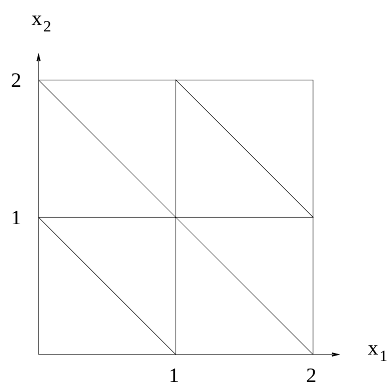

Figure 1: Triangulation (mesh) of domain Ω.

Note: *The exam continues on the next page!*

Problem 2: Consider the problem:

$$\begin{aligned} -\Delta u(x) + u(x) &= f(x) \quad x \in \Omega \subset \mathbb{R}^3 \\ \partial_n u(x) &= g(x) \qquad x \in \partial\Omega \end{aligned} \tag{1}$$

with $\partial_n u = \nabla u \cdot n$, and n the outward normal of the boundary $\partial\Omega$.

(a) State the Lax-Milgram theorem.
(b) For the problem (1), derive a bilinear form $a : V \times V \to \mathbb{R}$ and a linear form $L : V \to \mathbb{R}$, and specify the Hilbert space V and the norm $\|\cdot\|_V$.
(c) Show that the assumptions of the Lax-Milgram theorem are satisfied for this problem, and specify sufficient conditions on f and g.

Problem 3: Consider an abstract variational problem: Find $u \in V$ such that

$$a(u, v) = L(v)$$

for all $v \in V$, with V a Hilbert space, and $a(\cdot, \cdot)$ and $L(\cdot)$ are bilinear and linear forms on V satisfying the conditions in the Lax-Milgram theorem. The abstract Galerkin method for this problem is formulated as: Find $U \in V_h$ such that

$$a(U, v) = L(v)$$

for all $v \in V_h$, with V_h a finite dimensional subspace of V.

(a) Prove the Galerkin orthogonality: $a(u - U, v) = 0$, for all $v \in V_h$.
(b) Prove that: $\|u - U\|_V \leq C\|u - v\|_V$, and specify the constant $C > 0$.
(c) Define the energy norm $\|\cdot\|_E$. What is the constant C in (b) if $\|\cdot\|_V = \|\cdot\|_E$?
(d) Now consider the case of $V = H_0^1(0, 1)$, and $V_h = \{$continuous piecewise linear functions v on $\mathcal{T}_h$ with $v(0) = v(1) = 0\}$, with $\mathcal{T}_h$ a subdivision of the interval (0,1). Define

$$a(u, v) = \int_0^1 u'(x)v'(x)\, dx, \quad L(v) = \int_0^1 f(x)v(x)\, dx.$$

The energy norm $\|\cdot\|_E$ for this problem is defined as $\|v\|_E = \|v'\|$, with $\|\cdot\|$ the L_2 norm. Prove the a priori error estimate: $\|u - U\|_E \leq C_i\|hu''\|$
(e) Prove the a posteriori error estimate: $\|u - U\|_E \leq C_i\|hR(U)\|$

The residual $R(U) = f + U''$ is defined on each subinterval $I_i = (x_{i-1}, x_i)$, where x_i are the nodes, and C_i is an interpolation constant.

Note: *The exam continues on the next page!*

Problem 4: For $a(x) > 0$ and $c(x) \geq 0$, consider the problem:

$$-(a(x)u'(x))' + c(x)u(x) = f(x) \quad x \in (0,1)$$
$$u(0) = u(1) = 0$$

(a) Formulate the cG(1) method for the problem (FEM with a continuous piecewise linear approximation on a subdivision $\mathcal{T}_h$ of (0,1)).
(b) Prove the a posteriori error estimate:

$$\|u - U\|_{L_2(0,1)} \leq SC_i \|h^2 R(U)\|_{L_2(0,1)}$$

where U is the cG(1) solution, and

$$S = \max_{\xi \in L_2(0,1)} \frac{\|\varphi''\|_{L_2(0,1)}}{\|\xi\|_{L_2(0,1)}}$$

with φ the solution to the dual problem:

$$-(a(x)\varphi'(x))' + c(x)\varphi(x) = \xi(x) \quad x \in (0,1)$$
$$\varphi(0) = \varphi(1) = 0$$

(c) Prove that if $a > 0$ and $c \geq 0$ are constants, then $S \leq a^{-1}$.
(d) How should you choose $\xi(x)$, the data to the dual problem, to prove an a posteriori error estimate of the mean value of the solution u:

$$\frac{1}{|\omega|} \int_\omega u(x)\, dx, \quad \text{with } |\omega| = \int_\omega dx = \text{ the area of } \omega$$

Problem 5: Answer the following questions related to standard FEM algorithms (it may be helpful to illustrate some of your answers with pictures):

(a) What is a least squares stabilized finite element method?
(b) Describe how a mapping to a reference element is used to compute element integrals, in the case of triangular elements (you do not have to carry out any computations, just illustrate the idea and function of the algorithm).
(c) What is a hanging node?
(d) Describe the steps in a red-green mesh refinement algorithm for triangles.
(e) Describe the steps in an adaptive algorithm for local mesh refinement based on a posteriori error estimation.

Good Luck!

Johan

Solutions to exam

Problem 1: See pages 360-363 in the CDE book.

(a) Find $U \in V_h$ such that

$$\int_\Omega \nabla U(x) \cdot \nabla v(x)\ dx = \int_\Omega v(x)\ dx \quad \forall v \in V_h \tag{2}$$

(b) $V_h = \{$continuous piecewise linear functions v on $\mathcal{T}_h$ such that v=0 on $\partial\Omega\}$, with $\mathcal{T}_h$ the triangulation of Ω in Fig. 2, with $h = 1$. There is 1 degree of freedom; the node N_1.

A basis for V_h is $\{\phi_1\}$; with $\phi_1 \in V_h$, and $\phi_1(N_1) = 1$ and $\phi_1 = 0$ in all other vertices (nodes). Set $U(x) = \xi_1\phi_1(x)$, then (2) is equivalent to $A\xi = b$ where A and b are scalars, given by

$$A_{11} = \int_\Omega \nabla\phi_1(x) \cdot \nabla\phi_1(x)\ dx, \quad b_1 = \int_\Omega \phi_1(x)\ dx$$

A_{11} involves integration over elements $e_2, e_3, e_4, e_5, e_6, e_7$, where e_2, e_7 are of the type in Fig.15.8 at page 362 in the CDE book, with integral $\int_{e_2} \nabla\phi_1 \cdot \nabla\phi_1\ dx = 1$, and e_3, e_4, e_5, e_6 are of the type in Fig.15.9, with integral $\int_{e_3} \nabla\phi_1 \cdot \nabla\phi_1\ dx = 1/2$. Thus

$$A_{11} = \int_{e_2} + \int_{e_3} + \int_{e_4} + \int_{e_5} + \int_{e_6} + \int_{e_7} = 1 + 1/2 + 1/2 + 1/2 + 1/2 + 1 = 4$$

$$b_1 = \int_\Omega \phi_1(x)\ dx = \text{volume under } \phi_1 = 6 \times \frac{\frac{h^2}{2} \times 1}{3} = h^2 = 1.$$

(c) We still have only one degree of freedom, but we will get a contribution to the right hande side from the non homogeneous boundary condition at node N_2. Set $U(x) = \xi_1\phi_1(x) + \xi_2\phi_2(x)$, with $\xi_2 = U(N_2) = 1$. Then (2) is equivalent to a 2×2 system:

$$\begin{aligned} A_{11}\xi_1 + A_{12}\xi_2 &= b_1 \\ A_{21}\xi_1 + A_{22}\xi_2 &= b_2 \end{aligned}$$

where we can move the known data ξ_2 to the right hand side:

$$\begin{aligned} A_{11}\xi_1 &= b_1 - A_{12}\xi_2 \\ A_{21}\xi_1 &= b_2 - A_{22}\xi_2 \end{aligned}$$

We only have one unknown, so we only need one of the equations:

$$A_{11}\xi_1 = b_1 - A_{12}\xi_2$$

$A_{11} = 4$ as before.

A_{12} involves integration over elements e_4, e_7, which are of the type in Fig.15.10 at page 363 in the CDE book, with integral $\int_{e_4} \nabla\phi_1 \cdot \nabla\phi_2 \, dx = -1/2$. Thus

$$A_{12} = \int_{e_4} + \int_{e_7} = -1/2 - 1/2 = -1,$$

$b_1 = 1$ as before, so the total right hand side vector b is

$$b = b_1 - A_{12}\xi_2 = 1 - (-1) \times 1 = 2$$

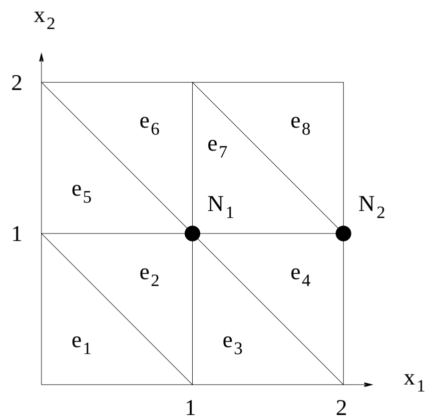

Figure 2: Triangulation (mesh) of domain Ω.

Problem 2:

(a) Theorem 21.1 in the CDE book.
(b)-(c) Section 21.4.1 and 21.4.4. in the CDE book.

Problem 3:

(a) Section 21.3

(b) Theorem 21.3

(c) Section 21.1 and Theorem 21.3.

(d) Section 8.2.1 with $a = 1$.

(e) Section 8.2.2 with $a = 1$.

Problem 4:

(a) Find $U \in V_h$ such that $\int_0^1 (aU'v' + cUv)\ dx = \int_0^1 fv\ dx \quad \forall v \in V_h$
(with V_h as in Problem 3)

(b) Section 15.5.2 in the CDE book.

(c) Multiply the dual problem by $-\varphi''$ and integrate from 0 to 1:

$$\int_0^1 (a\varphi''\varphi'' - c\varphi\varphi'')\ dx = -\int_0^1 \varphi''\xi\ dx$$

$$a\|\varphi''\|^2 + c\int_0^1 (\varphi')^2\ dx = -\int_0^1 \varphi''\xi\ dx \leq \|\varphi''\|\ \|\xi\| \leq \frac{a}{2}\|\varphi''\|^2 + \frac{1}{2a}\|\xi\|^2$$

$$\frac{a}{2}\|\varphi''\|^2 + c\|\varphi'\|^2 \leq \frac{1}{2a}\|\xi\|^2 \ \Rightarrow\ \|\varphi''\|^2 + \frac{2c}{a}\|\varphi'\|^2 \leq \frac{1}{a^2}\|\xi\|^2$$

$$\Rightarrow\ \|\varphi''\| \leq \frac{1}{a}\|\xi\| \ \Rightarrow\ \frac{\|\varphi''\|}{\|\xi\|} \leq \frac{1}{a}$$

(d) $\xi = \frac{\chi_\omega}{|\omega|}, \quad \chi_\omega(x) = 1$ if $x \in \omega,$ and 0 else

Problem 5: See section 18.3 in the CDE book, and the lecture notes from lecture 2, slides avaliable at:
`http://www.csc.kth.se/utbildning/kth/kurser/2D1260/fem06/`