Solutions to Exam in EL2745 Principles of Wireless Sensor Networks, March 14, 2013

1. **BPSK modulation over fading channels**

   (a) See Lectures 4 and 5

   (b) The general form for BPSK follows the equation:

   $$s_n(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f t + \pi(1-n)), \quad n = 0, 1.$$

   This yields two phases 0 and π. Specifically, binary data is often conveyed via the following signals:

   $$\begin{aligned} s_0(t) &= \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f t + \pi) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f t) \\ s_1(t) &= \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f t) \end{aligned}$$

   Hence, the signal-space can be represented by

   $$\phi(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f t)$$

   where 1 is represented by $\sqrt{E_b}\phi(t)$ and 0 is represented by $-\sqrt{E_b}\phi(t)$. Now we comment on the channel model. The transmitted signal that gets corrupted by noise n typically refereed as added white Gausssian noise. It is called white since the spectrum of the noise is flat for all frequencies. Moreover, the values of the noise n follows a zero mean gaussian probability distribution function with variance $\sigma^2 = N_0/2$. So for above model, the received signal take the form

   $$\begin{aligned} y(t) &= s_0(t) + n \\ y(t) &= s_1(t) + n \end{aligned}$$

   The conditional probability distribution function (PDF) of y for the two cases are:

   $$\begin{aligned} f(y|s_0) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y+\sqrt{E_b})^2}{N_0}} \\ f(y|s_1) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y-\sqrt{E_b})^2}{N_0}} \end{aligned}$$

   Assuming that s_0 and s_1 are equally probable, the threshold 0 forms the optimal decision boundary. Therefore, if the received signal y is greater than 0, then the receiver assumes s_1 was transmitted and vise versa. With this threshold the probability of error given s_1 is transmitted is

$$p(e|s_1) = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{0} e^{-\frac{(y-\sqrt{E_b})^2}{N_0}} dy = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{E_b}{N_0}}}^{\infty} e^{-z^2} dz$$
$$= \mathbf{Q}\left(\sqrt{\frac{2E_b}{N_0}}\right) = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right),$$

where $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-x^2} dx$ is the complementary error function. Similarly, the probability of error given s_0 is transmitted is

$$p(e|s_0) = \frac{1}{\sqrt{\pi N_0}} \int_{0}^{\infty} e^{-\frac{(y+\sqrt{E_b})^2}{N_0}} dy = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{E_b}{N_0}}}^{\infty} e^{-z^2} dz$$
$$= \mathbf{Q}\left(\sqrt{\frac{2E_b}{N_0}}\right) = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right).$$

Hence, the total probability of error is

$$P_b = p(s_1)p(e|s_1) + p(s_0)p(e|s_0) = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right).$$

Note that the probabilities $p(s_0)$ and $p(s_1)$ are equally likely.

(c) Let $P(\gamma)$ be the probability of error for a digital modulation as a function of E_b/N_0, γ, in the Gaussian channel. Let the channel amplitude be denoted by the random variable α, and let the average SNR normalized per bit be denoted by $\gamma^\star = \mathbf{E[\alpha^2]E_b/N_0}$. Then to obtain $P(e)$ for a Rayleigh fading channel $P(\gamma)$ must be integrated over the probability that a given γ is encountered:

$$P(e) = \int_0^{\infty} P(\gamma)p(\gamma)d\gamma,$$

For Rayleigh fading,

$$p(\gamma) = \frac{1}{\gamma^\star} e^{-\gamma/\gamma^\star}.$$

In the case of coherent BPSK, the integration can actually be computed yielding

$$P(e) = \frac{1}{2}\left[1 - \sqrt{\frac{\gamma^\star}{1+\gamma^\star}}\right].$$

At high SNR like OQPSK systems, the approximation $(1+x)^{1/2} \sim 1+x/2$ can be used, giving

$$P(e) \sim \frac{1}{4\gamma^\star}$$

compared with $P(e) = \mathbf{Q}(\sqrt{2\gamma^\star})$ for the Gaussian channels.

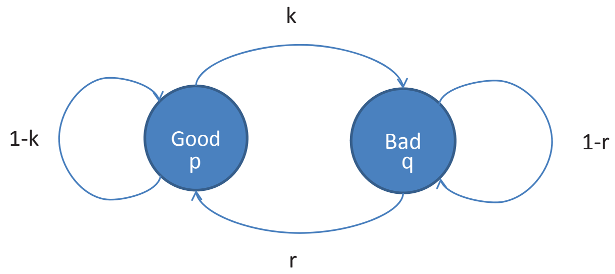

Figure 1: 2-state Markov chain describing to Gilbert Elliott model.

(d) By the Stirling approximation we have

$$f(\theta) = f(\mu) + (\theta - \mu)\frac{f(\mu+h) - f(\mu-h)}{2h} + \frac{1}{2}(\theta-\mu)^2\frac{f(\mu+h) - 2f(\mu) + f(\mu-h)}{h^2} + \cdots,$$

then, taking the expectation we have

$$\mathbf{E}f(\theta) \sim f(\mu) + \frac{1}{2}\frac{f(\mu+h) - 2f(\mu) + f(\mu-h)}{h^2}\sigma^2.$$

It has been shown that $h = \sqrt{3}$ yields a good result. So we obtain $\mathbf{Q}(\gamma)$. Given a log-normal random variable z with mean μ_z and variance σ_z^2, we calculate the average probability of error as the average of $\mathbf{Q}(\gamma)$. Namely,

$$\mathbf{E}\{\mathbf{Q}(z)\} \sim \frac{2}{3}\mathbf{Q}(\mu_z) + \frac{1}{6}\mathbf{Q}(\mu_z + \sqrt{3}\sigma_z) + \frac{1}{6}\mathbf{Q}(\mu_z - \sqrt{3}\sigma_z).$$

2. **Energy harvesting in WSNs**

(b) We form a GE model as illustrated in Fig. 1.

(b) Steady state probabilities are derived via

$$\pi_G + \pi_B = 1,$$
$$\pi_G = (1-k)\pi_G + r\pi_B$$

which yields

$$\pi_G = \frac{r}{k+r},$$

and

$$\pi_B = 1 - \frac{r}{k+r} = \frac{k}{k+r}.$$

(c) steady state probability of energy generation is given by

$$p_g = p\pi_G + q\pi_B = 0.9\frac{0.2}{0.1+0.2} + 0.3\frac{0.1}{0.1+0.2} = 0.7$$

(d) According to its definition, the harvesting burst is a random variable with the geometric distribution. We have

$$p(\text{burst of length } t) = k(1-k)^{t-1} = 0.1(0.9)^{t-1}$$

The mean value of harvesting burst then is given by

$$\text{AHB} = \sum_{t\geq 1} tk(1-k)^{t-1} = \frac{1}{k} = 10\,.$$

1. **Neighbor discovery in WSNs**

(a) The probability that node i successfully discovers a given neighbor in a given slot is given by

$$p_s = p(1-p)^{n-1}$$

(b) The optimal transmission probability $p^\star$ is achieved by setting the derivative of the concave function p_s to 0. That is

$$p_s' = 0 \Leftrightarrow (1-p)^{n-1} - p(n-1)(1-p)^{n-2} = 0$$

and hence, $p^\star = 1/n$. Substituting $p^\star$ in p_s we obtain

$$p_s = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1} \approx \frac{1}{ne}.$$

(c) Let $\mathbf{E}[W_0]$ denotes the mean time until discovering the first neighbor. It is easy to verify that W_0 is a geometric random variable with the parameter np_s, namely

$$\mathbf{E}[W_0] = \sum_{i=0}^{\infty} inp_s(1-np_s)^{i-1} = \frac{1}{np_s}.$$

Now let $\mathbf{E}[W_1]$ be the mean time of the first frame *i.e*, the time between when first neighbor discovered until the slot that second neighbor is discovered. Here W_1 is a geometric random variable with parameter $(n-1)p_s$. So we calculate $\mathbf{E}[W_1]$ as the following

$$\mathbf{E}[W_1] = \sum_{i=0}^{\infty} i(n-1)p_s(1-(n-1)p_s)^{i-1} = \frac{1}{(n-1)p_s}.$$

Following the same line of reasoning, we conclude that the average duration of frame i is equal to

$$\mathbf{E}[W_i] = \frac{1}{(n-i)p_s}\,.$$

(d) The mean time for discovering n neighbors is a summation of $\mathbf{E}[W_i]$'s. Namely,

$$\mathbf{E}[W] = \sum_{i=0}^{N-1} \mathbf{E}[W_i] = \sum_{i=0}^{N-1} \frac{1}{(N-i)p_s} = \frac{1}{p_s} \sum_{i=1}^{N} \frac{1}{i}.$$

Considering that $N = 4$ and the slot time is 81 ms, we have

$$p_s = \frac{1}{4}\left(1-\frac{1}{4}\right)^3 = \frac{3^3}{4^4}$$

$$\mathbf{E}[t] = 81 \cdot \mathbf{E}[W] = 81 \frac{1}{p_s} \sum_{i=1}^{4} \frac{1}{i}$$

$$= 81 \cdot 4 \left(\frac{4}{3}\right)^3 \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) = 1.6 \text{ s}.$$

2. **Distributed detection/estimation**

(a) From the problem, we can make the simple transformation

$$\tilde{y}_k = \tilde{x} + \tilde{n}_k,$$

where $\tilde{y}_k = y_k - 2$, $\tilde{x} = x - 1$ and $\tilde{n}_k = n_k - 1$. It follows that $\tilde{x}$ is in the interval [-1,1], while $\tilde{n}_k$ is uniformly distributed in interval [-1,1]. Thus, we find that

$$\Pr(m_k = 1) = \int_{-\tilde{x}}^{1} p(\tilde{n}) d\tilde{n} = \frac{1}{2}(1+\tilde{x})$$
$$\Pr(m_k = 0) = \int_{-1}^{-\tilde{x}} p(\tilde{n}) d\tilde{n} = \frac{1}{2}(1-\tilde{x}).$$

Then

$$\mathbf{E}(m_k) = \frac{1}{2}(1+\tilde{x}) \tag{1}$$
$$\mathbf{E}(m_k - \mathbf{E}(m_k))^2 = \frac{1}{4}(1-\tilde{x}^2) \tag{2}$$

(b) From Eq.(2), we obtain

$$\mathbf{E}(m_k - \mathbf{E}(m_k))^2 = \frac{1}{4}(1-\tilde{x}^2) \leq \frac{1}{4}$$

(c) From Eqs.(1) and (2) and the given fusion function, we have

$$\mathbf{E}(\hat{\tilde{x}}) = \frac{2}{N} \sum_{k=1}^{N} \mathbf{E}(m_k) - 1$$
$$= \frac{2}{N} \sum_{k=1}^{N} \frac{1}{2}(1+\tilde{x}) - 1 = \tilde{x},$$

whereas $\hat{x} = \hat{\hat{x}} + 1$ and

$$
\begin{aligned}
\mathbf{E}(\hat{\hat{x}} - \tilde{x})^2 &= \mathbf{E}\left(\left(\frac{2}{N}\sum_{k=1}^{N} m_k - 1\right) - \tilde{x}\right)^2 \\
&= \frac{4}{N^2}\mathbf{E}\left(\sum_{k=1}^{N} m_k - \sum_{k=1}^{N}\frac{1}{2}(1+\tilde{x})\right)^2 \\
&= \frac{4}{N^2}\mathbf{E}\left(\sum_{k=1}^{N}(m_k - \mathbf{E}(m_k))\right)^2 . \qquad (3)
\end{aligned}
$$

Since m_k's are independent, we calculate Eq.(3) as

$$
\begin{aligned}
\frac{4}{N^2}\mathbf{E}\left(\sum_{k=1}^{N}(m_k - \mathbf{E}(m_k))\right)^2 &= \frac{4}{N^2}\sum_{k=1}^{N}\mathbf{E}(m_k - \mathbf{E}(m_k))^2 \\
&= \frac{1}{N}(1-\tilde{x})^2 \le \frac{1}{N}
\end{aligned}
$$

(d) Based on the results above, we need more than $1/\varepsilon$ nodes to satisfy the variance bound.

3. **Networked Control System**

(a) Since $\tau < h$, at most two controllers samples need be applied during the k-th sampling period: $u((k-1)h)$ and $u(kh)$. The dynamical system can be rewritten as

$$
\begin{aligned}
&\dot{x}(t) = Ax(t) + Bu(t), \qquad t \in [kh+\tau, (k+1)h+\tau) \\
&y(t) = Cx(t)\,, \\
&u(t^+) = -x(t-\tau), \qquad t \in \{kh+\tau, \quad k = 0,1,2,\ldots\}
\end{aligned}
$$

where $u(t^+)$ is a piecewise continuous and changes values only at $kh+\tau$. By sampling the system with period h, we obtain

$$
\begin{aligned}
&x((k+1)h) = \Phi x(kh) + \Gamma_0(\tau)u(kh) + \Gamma_1(\tau)u((k-1)h) \\
&y(hk) = Cx(kh)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi &= e^{Ah} = e^{ah} = 1\,, \\
\Gamma_0(\tau) &= \int_0^{h-\tau} e^{As}Bds = h - \tau\,, \\
\Gamma_1(\tau) &= \int_{h-\tau}^{h} e^{As}Bds = \tau\,.
\end{aligned}
$$

given that $A = 0, B = 1, C = 1$.

(b) Time Division Multiple Access (TDMA) MACs typically introduce a constant delay. The IEEE 802.15.4 MAC in the slotted (or beacon enabled) modality supports TDMA in the contention free period (CFP).

(c) Let $z(kh) = [x^T(kh), u^T((k-1)h)]^T$ be the augmented state vector, then the augmented closed loop system is

$$z((k+1)h) = \tilde{\Phi} z(kh)\,,$$

where

$$\tilde{\Phi} = \begin{bmatrix} \Phi - \Gamma_0(\tau) & \Gamma_1(\tau) \\ -1 & 0 \end{bmatrix}.$$

Using the results obtained in (a), we can obtain

$$\tilde{\Phi} = \begin{bmatrix} 1-(h-\tau) & \tau \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \tau & \tau \\ -1 & 0 \end{bmatrix},$$

where we used that $h = 1$.

(d) The characteristic polynomial of this matrix is

$$\lambda^2 - \tau\lambda + \tau = \left(\lambda - \frac{\tau}{2}\right)^2 + \tau - \frac{\tau^2}{4}$$

Thus

$$\lambda = \frac{\tau}{2} \pm \sqrt{\tau - \frac{\tau^2}{4}}\,,$$

whose absolute values are $\sqrt{\tau}$ that means the system is stable when $\tau < h = 1$.

(e) We use the following result to study the stability of the system:

Theorem 1 *Consider the system given in Fig. 2. Suppose that the closed-loop system without packet losses is stable. Then*

- *if the open-loop system is marginally stable, then the system is exponentially stable for all* $0 < r \leq 1$.
- *if the open-loop system is unstable, then the system is exponentially stable for all*

$$\frac{1}{1-\gamma_1/\gamma_2} < r \leq 1\,,$$

where $\gamma_1 = \log[\lambda^2_{\max}(\Phi - \Gamma)]$, $\gamma_2 = \log[\lambda^2_{\max}(\Phi)]$

Here we have

$$\Phi = e^{Ah} = 1\,,$$

$$\Gamma = \int_0^h e^{As} B ds = h = 1\,.$$

Thus, $\gamma_1 = -\infty$ that implies the system is stable for all packet loss rate.

These conditions are only sufficient. If they are not satisfied, then one has to look for other theoretical results that ensure stability in the presence of packet losses. Another option is to tune the protocol parameters and the controller parameters (if they can be tuned). One example of protocol parameter is the duration of the time slots in a TDMA protocol or the number of retransmissions in a CSMA protocol. One example of control parameter is the constant K in a state feedback control low $u(kh) = -Kx(kh)$.

A typical MAC category introducing packet losses is CSMA or ALHOA. The reason of packet losses can be the collision at the receiver of packets transmitted by different senders at the same time.