Linear programming
EG2200 Lecture 7, autumn 2014
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Course objectives

Formulate short-term planning problems of hydro-thermal power systems.
Applied mathematical programming

- Short-term planning of hydro-thermal power systems is an application of mathematical programming (optimisation)
- In this course we will teach how to formulate optimisation problems, but now how to solve them. Solutions methods are taught by the Dept. of Mathematics, for example
  - SF1861 Optimisation, 6 hp
  - SF2812 Applied Linear Optimisation, 7.5 hp
  - SF2822 Applied Nonlinear Optimisation, 7.5 hp
- When formulating optimisation problems, it is necessary to be familiar with the basic concepts in mathematical programming
Optimisation

- Optimisation theory (also referred to as mathematical programming) is a discipline in applied mathematics.
- General example:

\[
\begin{align*}
\text{minimise} & \quad f(x) \\
\text{subject to} & \quad x \in \mathcal{X},
\end{align*}
\]

where

- \(x\) = vector of optimisation variables,
- \(\mathcal{X}\) = set of feasible solutions.
Feasible solutions

The set of feasible solutions is defined using various mathematical expressions.

- Constraints (defines relations between variables)
  
  Example: \( g(x) \leq b \).

- Variable limits
  
  Examples: \( x \leq x \leq \bar{x} \),
  
  \( x \) integer.
Minimisation or maximisation

Notice that minimisation and maximisation problems are exchangable, because

\[
\text{minimise } f(x) \iff \text{maximise } -f(x)
\]

Example:

\[
\begin{align*}
\text{maximise } x & \iff \text{minimise } -x \\
\text{s.t. } 0 \leq x \leq 10. & \iff \text{s.t. } 0 \leq x \leq 10.
\end{align*}
\]
Linear programming (LP)

- Class of optimisation problems with linear objective function and constraints.
- Standard form:
  
  \[
  \begin{align*}
  \text{minimise} & \quad c^T x \\
  \text{s.t.} & \quad Ax = b, \\
  & \quad 0 \leq x.
  \end{align*}
  \]

- Large LP problems (more than 100 000 variables!) can be solved reasonably fast.
- Commercial software available
  - GAMS, Matlab, Excel…
Example A.1

Formulation of LP problem in standard form

- Alice is buying something for her mother’s guests
- 2 litres of fruit are needed to fill the fruit bowl
- Alice’s mother wants each of the five guests to have at least two items, i.e., Alice needs to buy at least 10 items
- Alice has 100 SEK and can keep the change
- A pear costs 3 SEK, each pear has the volume 1/6 litres
- An apple costs 5 SEK, each apple has the volume 0.3 litres
Example A.1

LP formulation

- Introduce
  \[ x_1 = \text{number of pears}, \]
  \[ x_2 = \text{number of apples}. \]

- Formulate optimisation problem:
  \[
  \begin{align*}
  \text{maximise} & \quad 100 - 3x_1 - 5x_2 \quad \{\text{profit}\} \\
  \text{subject to} & \quad \frac{1}{6}x_1 + 0.3x_2 \geq 2, \quad \{\text{volume constraint}\} \\
  & \quad x_1 + x_2 \geq 10, \quad \{\text{quantity constraint}\} \\
  & \quad x_1 \geq 0, \ x_2 \geq 0. \quad \{\text{variable limits}\}
  \end{align*}
  \]
Example A.1

Minimisation

- An LP problem in standard form should be a minimisation problem.

- Maximising profit $\Leftrightarrow$ Minimising cost
  
  $$\text{maximise} \quad 100 - 3x_1 - 5x_2 \quad \{\text{profit}\}$$

  $$\text{minimise} \quad 3x_1 + 5x_2 \quad \{\text{cost}\}$$

- Notice that constant values in the objective function will not have any impact on the solution, as it is not affected by the choice of values for the optimisation variables!

- Introduce
  
  $$z = \text{objective function.}$$
Example A.1

Slack variables

• An LP problem in standard form should have equality constraints instead of inequalities constraints!

• Introduce slack variables:
  
  \[ x_3 = \text{extra volume,} \]
  
  \[ x_4 = \text{extra quantity.} \]

• Reformulate the constraints:
  
  \[ \frac{1}{6}x_1 + 0.3x_2 - x_3 = 2, \quad \{\text{volume constraint}\} \]
  
  \[ x_1 + x_2 - x_4 = 10, \quad \{\text{quantity constraint}\} \]

• Add variable limits:
  
  \[ x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ x_4 \geq 0. \quad \{\text{variable limits}\} \]
Example A.1

LP formulation in standard form

\[
\begin{align*}
\text{min} \quad & z = 3x_1 + 5x_2 \quad \quad \{\text{cost}\} \\
\text{s.t.} \quad & \frac{1}{6}x_1 + 0.3x_2 - x_3 = 2, \quad \{\text{volume constraint}\} \\
& x_1 + x_2 - x_4 = 10, \quad \{\text{quantity constraint}\} \\
& x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0. \quad \{\text{variable limits}\}
\end{align*}
\]
Example A.1

Optimal solution

Optimum:
\[ x_1 = 7.5 \]
\[ x_2 = 2.5 \]

Optimal value of the objective function:
\[ z = 35 \]
Extreme points

• The “corners” in the feasible set are called extreme points!
• The optimal solution of an LP problem can always be found in one of the extreme points.
Standard form

- The standard form is useful for mathematical analysis and is also required by some solvers.
- In engineering, a clear problem formulation is more important, i.e., optimisation problems should be formulated so that it is easy to recognise the underlying engineering problem.
  - Use easily recognised symbols instead of $x_1, x_2, \ldots$
  - Choose maximisation or minimisation depending on what is natural for the engineering problem.
  - Choose between equality or inequality constraints depending on what is natural for the engineering problem.
Example A.2

Infeasible problem

- Alice’s father says “Do not buy more than 1 kg.”
- A pear weights \(\frac{1}{6}\) kg.
- An apple weights 0.3 kg.

Solution

Add another constraint:

\[
\frac{1}{6}x_1 + 0.3x_2 \leq 1. 
\]

\{weight constraint\}
Example A.2

Optimal solution

The set of feasible solutions is empty, i.e., the problem is infeasible.
Infeasible problems

- An infeasible problem means that there are conflicting constraints that cannot be fulfilled at the same time.
- Short-term planning problems should during normal circumstances have a feasible solution!
- How do you find the problem?
  - Some solvers can indicate which constraints are infeasible.
  - Add slack variables. For example, in problem A.2 we can reformulate the new constraint as
    \[
    \frac{1}{6}x_1 + 0.3x_2 + x_5 - x_6 \leq 1. \quad \text{(weight constraint)}
    \]
    \[
    x_5 \geq 0, \ x_6 \geq 0.
    \]
Example A.3

Non-binding constraints

• There are only 13 pears left in the shop.

Solution
Add another constraint:

\[ x_1 \leq 13. \]  \{pear limitaiton\}
Example A.3

Optimal solution

Optimum:
\[ x_1 = 7.5 \]
\[ x_2 = 2.5 \]

Optimal value:
\[ z = 35 \]

The new constraint does not change the optimal solution!
Example A.4

Unbounded problems

• Alice’s mother says: “You will get 1 SEK for each item you bring home.”

Solution

New objective function:

$$\max \quad z = x_1 + x_2$$

{income}
Example A.4

Optimal solution

Optimum:
\[ x_1 = \infty \]
\[ x_2 = \infty \]

Optimal value:
\[ z = \infty \]
Unbounded problems

- An unbounded problem means that there are not sufficient constraints.
- Short-term planning problems should not be unbounded!
- How do you find the problem?
  - Most likely some constraints are missing or not formulated correctly!
  - Check variable values! Is some variable outside the range one could expect?
  - Add temporary constraints.
Example A.5

Change in objective function

- A pear costs 4 SEK instead of 3 SEK.

Solution

New objective function:

$$\min \quad z = 4x_1 + 5x_2 \quad \{\text{cost}\}$$
Example A.5

Optimal solution

Optimum:
\[ x_1 = 7.5 \]
\[ x_2 = 2.5 \]

Optimal value:
\[ z = 42.5 \]

Same optimal solution, new optimal value!
Example A.6

Degenerated solution

- A pear costs 5 SEK instead of 3 SEK.

Solution

New objective function:

\[
\min z = 5x_1 + 5x_2 \quad \{\text{cost}\}
\]
Example A.7

Optimal solution

Optimum:
\[ x_1 \in [0, 7.5] \]

\[ x_2 = 10 - x_1 \]

Optimal value:
\[ z = 50 \]

Many optimal solutions, same optimal value!
Degenerated problems

- A degenerated problem does not have a unique optimal solution.
- Many short-term planning problems are degenerated!
- The solution to a degenerated problem might differ from solver to solver and can even depend on the order of the variables and constraints!
- To verify that two solutions of the same degenerated problem are equivalent, one must study the optimal value of the objective function and not the optimal solution of the optimisation variables!
Example A.7

Flat optimum

- Compare the solution of the following cases
  - A pear costs 4.90 SEK and an apple costs 5 SEK.
  - A pear costs 5 SEK and an apple costs 4.90 SEK.
Example A.7

Optimal solution

Optimum:
\[ x_1 = 7.5 \]
\[ x_2 = 2.5 \]

or

\[ x_1 = 0 \]
\[ x_2 = 10 \]

Optimal value:
\[ z = 49.25 \]

or

\[ z = 49 \]
Flat optimum

- A flat optimum means that there are extreme points that are not optimal, but which generate a value of the objective function that is close to the optimal value.
- Some solvers might not search for the exact optimal solution, but for a solution that is “good enough”.
**LP duality**

All LP problems (primal problem) have a corresponding dual problem.

<table>
<thead>
<tr>
<th>Primal problem</th>
<th>Dual problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \min c^T x )</td>
<td>( \max b^T \lambda )</td>
</tr>
<tr>
<td>( \text{s.t. } Ax = b, )</td>
<td>( \text{s.t. } A^T \lambda \leq c. )</td>
</tr>
<tr>
<td>( x \geq 0, )</td>
<td>( (\lambda \text{ unrestricted}) )</td>
</tr>
</tbody>
</table>

where

- \( x = \) primal variables,
- \( \lambda = \) dual variables.
Strong duality

Theorem:

If the primal problem has an optimal solution then the dual problem also has an optimal solution and the optimal values are equal.

- The proof is left for mathematics courses…
- The practical value of LP duality is that the dual variables (which are obtained as part of the solution algorithm for LP problems) can be used for sensitivity studies of the optimal solution.
Marginal values

- The right hand side of the primal problem appears as the objective function of the dual problem.
- A small change in the objective function of the dual problem will not change the solution to the dual problem (cf example A.5) $\Rightarrow$ easy to calculate new optimal value of the objective function.
- According to the strong duality theorem, the new optimal value of the primal problem is equal to the new optimal value of the dual problem.
- Hence, we can use the dual variables to compute how a small change in the right hand side of an LP problem will affect the optimal value of the objective function.

The dual variables can be interpreted as the marginal value of the right hand side in a constraint, as they state how the objective function will change for a small change in the right hand side: $\Delta z = \lambda^T \Delta b$. 
Example A.9

Application of dual variables

• Assume that Alice would deceive her mother and buy only 1.9 litres of fruit. How much would she earn on this?*

* The lecturer would like to emphasise that the objective of this example is absolutely not to encourage such behaviour!
Example A.9

Solution

• The dual problem is

\[
\begin{align*}
\text{max} & \quad 2\lambda_1 + 10\lambda_2 \\
\text{s.t.} & \quad \frac{1}{6}\lambda_1 + \lambda_2 \leq 3, \\
& \quad 0.3\lambda_1 + \lambda_2 \leq 5, \\
& \quad \lambda_1 \geq 0, \lambda_2 \geq 0.
\end{align*}
\]

• The restriction that the dual variables must be non-negative is due to the inequality constraints when Alice’s primal problem is formulated without slack variables.
Example A.9

Optimal solution

Optimum:
\[ \lambda_1 = 15 \]
\[ \lambda_2 = 0.5 \]
Optimal value:
\[ z = 35 \]
Example A.9

Sensitivity analysis

• If Alice only buys 1.9 litres fruit then the right hand side of the volume constraint changes by $-0.1$.
• The optimal value will then change by $-0.1 \lambda_1 = -1.5$.
• This means that Alice will save a cost of 1.5 SEK.
Mixed integer linear programming (MILP)

- Class of optimisation problems with linear objective function and constraints. However, some variables may only assume integer values.

\[
\begin{align*}
\text{minimise} & \quad c^T x \\
\text{s.t.} & \quad Ax = b, \\
& \quad x \in \{0, 1, \ldots\}
\end{align*}
\]

- Well-behaved MILP problems can be solved reasonably fast.
- Difficult MILP problems will take considerably longer time to solve than an LP problem of the same size!

⇒ Avoid integer variables if not necessary!
Example A.10

Integer solution

• Customers are only allowed to buy whole fruits.

Solution

Add variable limits:

\[ x_1, x_2 \text{ integers.} \]
Example A.10

Optimal solution

Optimum:
\[ x_1 = 7 \]
\[ x_2 = 3 \]

Optimal value:
\[ z = 36 \]
Piecewise linear functions

- Sometimes we want to include an approximation of a nonlinear function in an LP problem.
- In a piecewise linear function the variable is divided in segments. The lower limit of each segment is equal to 0; hence, we get that

\[ x = \sum x_j, \]

where \( x_j \) is the value of the \( j \):th segment.

- Notice that we do not allow any combinations of \( x_j \); we cannot start using a segment until the previous segment is fully utilised, i.e., if \( x_j > 0 \) then we should have \( x_j = \bar{x}_{j-1} \), where \( \bar{x}_{j-1} \) is the maximum value for segment \( j - 1 \).
Example A.10

Quantity discount

- For the first five pears the price is 5 SEK/pear. For additional pears, Alice gets a discount and only pays 3 SEK/pear.
Example A.10

Solution

Reformulate the problem with a piecewise linear function and a introduce a binary variable:

\[
\begin{align*}
\text{min} & \quad 5x_{1,1} + 3x_{1,2} + 5x_2 & \quad \{\text{cost}\} \\
\text{s.t.} & \quad \frac{1}{6}x_{1,1} + \frac{1}{6}x_{1,2} + 0.3x_2 \geq 2, & \quad \{\text{volume constraint}\} \\
& \quad x_{1,1} + x_{1,2} + x_2 \geq 10, & \quad \{\text{quantity constraint}\} \\
& \quad x_{1,1} \geq 5s, \\
& \quad x_{1,2} \leq M \cdot s, \\
& \quad x_{1,1} \geq 0, x_{1,2} \geq 0, x_2 \geq 0, s \in \{0, 1\}. & \quad \{\text{variable limits}\}
\end{align*}
\]

where \(M\) is a arbitrary, large number.
Example A.10

Consequences of the binary variable

• Assume $M = 100$.
• It is optimal to avoid using $x_{1,1}$ and to use $x_{1,2}$ as much as possible, because the cost of $x_{1,2}$ is lower!
• For $s = 0$ we get
  
  \[
  x_{1,1} \geq 5s \quad \iff \quad x_{1,1} \geq 0 \quad \implies \quad x_{1,1} \geq 0
  \]
  \[
  x_{1,2} \leq M \cdot s
  \]

• For $s = 1$ we get
  
  \[
  x_{1,1} \geq 5s \quad \iff \quad x_{1,1} \geq 5 \quad \implies \quad x_{1,1} \geq 5
  \]
  \[
  x_{1,2} \leq M \cdot s \quad \iff \quad x_{1,2} \leq 100 \quad \implies \quad x_{1,2} \leq 100
  \]
Example A.11

Limited offer

• For the first five pears are discounted and the price is 3 SEK/pear. For additional pears, Alice gets to pay the full price, i.e., 5 SEK/pear.
Example A.10

Solution

- Reformulate the problem with a piecewise linear function.

\[
\begin{align*}
\text{min} & \quad 3x_{1,1} + 5x_{1,2} + 5x_2 & \{\text{cost}\} \\
\text{s.t.} & \quad \frac{1}{6}x_{1,1} + \frac{1}{6}x_{1,2} + 0.3x_2 \geq 2, & \{\text{volume constraint}\} \\
& \quad x_{1,1} + x_{1,2} + x_2 \geq 10, & \{\text{quantity constraint}\} \\
& \quad x_{1,1} \leq 5, \\
& \quad x_{1,1} \geq 0, x_{1,2} \geq 0, x_2 \geq 0. & \{\text{variable limits}\}
\end{align*}
\]

- In this case, it is preferable to use \(x_{1,1}\) rather than \(x_{1,2}\); therefore, it is sufficient to have an upper limit for \(x_{1,1}\) \(\Rightarrow\) no need for a binary variable!
Nonlinear programming (NLP)

- Class of optimisation problems where at least one constraint or the objective function is nonlinear.
- Some well-behaved NLP problems can be solved reasonably fast.
- Other NLP problems can take long time to solve and we cannot be sure that we find the global optimum.

⇒ Avoid nonlinear problems if not necessary!