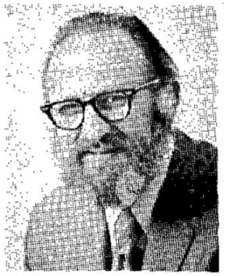
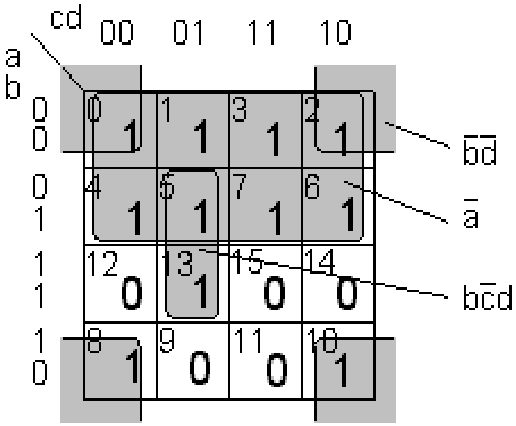
Maurice Karnaugh





The Karnaugh map makes it easy to minimize Boolean expressions!

A function of four variables a b c d.

The truthtable consists of 11 "1" and 5 "0". According to earlier, we know that the function can be expressed in the SoP form with 11 mintermer or PoS form with 5 maxtermer.

	abcd	f
0	0000	1
1	0001	1
2	0010	1
3	0011	1
4	0100	1
5	0101	1
6	0110	1
7	0111	1
8	1000	1
9	1001	0
10	1010	1
11	1011	0
12	1100	0
13	1101	1
14	1110	0
15	1111	0

$$f(a,b,c,d) = \sum(0,1,2,3,4,5,6,7,8,10,13)$$

$$f = \bar{a}\bar{b}\bar{c}\bar{d} + \bar{a}\bar{b}\bar{c}d + \bar{a}\bar{b}c\bar{d} + \bar{a}\bar{b}cd + \bar{a}b\bar{c}\bar{d} + \bar{a}b\bar{c}d + \bar{a}bc\bar{d} + \bar{a}bcd + a\bar{b}\bar{c}\bar{d} + a\bar{b}c\bar{d} + ab\bar{c}d$$

$$f(a,b,c,d) = \prod(9,11,12,14,15)$$

$$f = (\bar{a}+b+c+\bar{d})\cdot(\bar{a}+b+\bar{c}+\bar{d})\cdot(\bar{a}+\bar{b}+c+d)\cdot(\bar{a}+\bar{b}+\bar{c}+d)\cdot(\bar{a}+\bar{b}+\bar{c}+\bar{d})$$

ab \ cd	00	01	11	10
00	0: 1	1: 1	3: 1	2: 1
01	4: 1	5: 1	7: 1	6: 1
11	12: 0	13: 1	15: 0	14: 0
10	8: 1	9: 0	11: 0	10: 1

Anyone who used the Boolean algebra know that it then follows hard work to produce simpler expressions. Minterms could be combined in many different ways, which all result in different simplified expression - How do we know that we have found the minimal expression?

A map with frames at unit distance

The Karnaugh map is the Truth Table but with the minterms in a different order.

Note the numbering!

	abcd
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
14	1110
15	1111

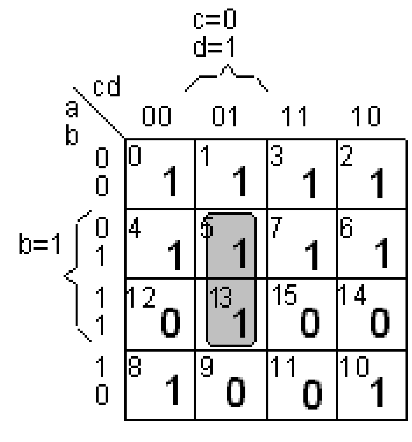
ab \ cd	00	01	11	10
00	0	1	3	2
01	4	5	7	6
11	12	13	15	14
10	8	9	11	10

The frames are ordered in such way that only one bit changes between two vertical frames or horisontal frames. This order is called Gray code.

Two "neighbors"

The frames "5" and "13" are "neighbors" in the Karnaugh map (but they are distant from each other in the truthtable). They correspond to *two* minterm with *four* variables, and the figure shows how, with Boolean algebra, they can be reduced to one term with *three* variables.

	abcd	f	
0	0000	1	
1	0001	1	
2	0010	1	
3	0011	1	
4	0100	1	
5	0101	1	$\bar{a}b\bar{c}d$
6	0110	1	
7	0111	1	
8	1000	1	
9	1001	0	
10	1010	1	
11	1011	0	
12	1100	0	
13	1101	1	$ab\bar{c}d$
14	1110	0	
15	1111	0	



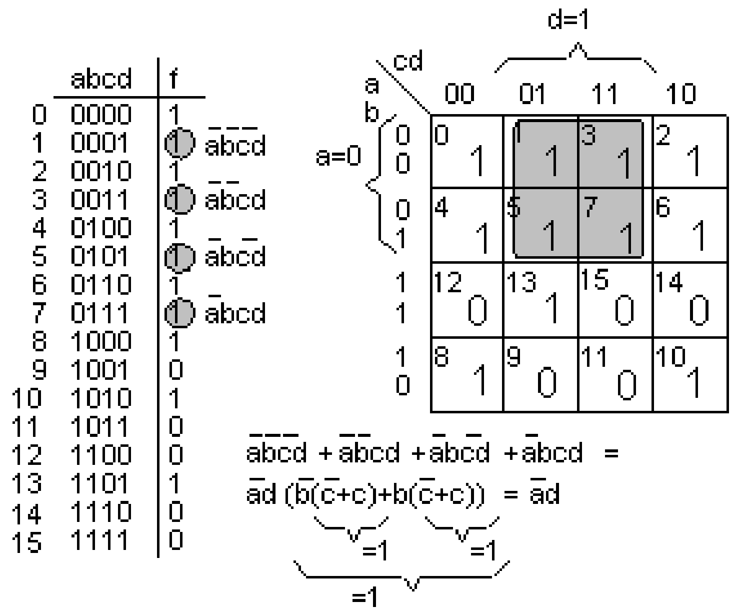
$$\bar{a}b\bar{c}d + ab\bar{c}d = b\bar{c}d(\bar{a}+a) = b\bar{c}d$$
$=1$

What the two frames have in common is that b = 1, c = 0 and d = 1; and the reduced term expresses just that.

Everywhere in the Karnaugh map where one can find two ones that are "neighbors" (vertically or horizontally) the minterms could be reduced to "what they have in common". This is called a **grouping**.

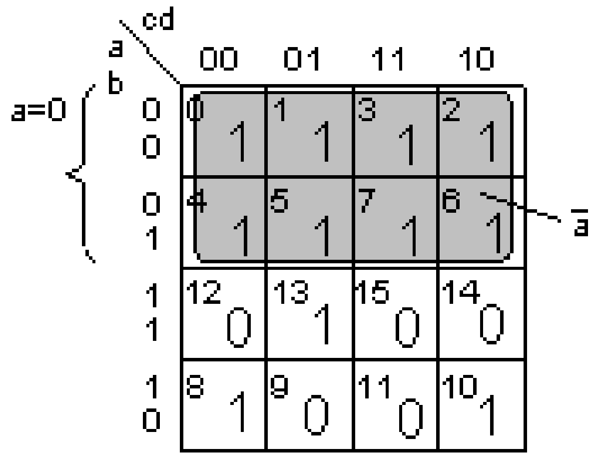
Four "neighbors"

Frames "1" "3" "5" "7" is a group of four frames with "1" that are "neighbors" to each other. Here too, the four minterms could be reduced to a term that expresses what is common for the frames, namely that a = 0 and d = 1.



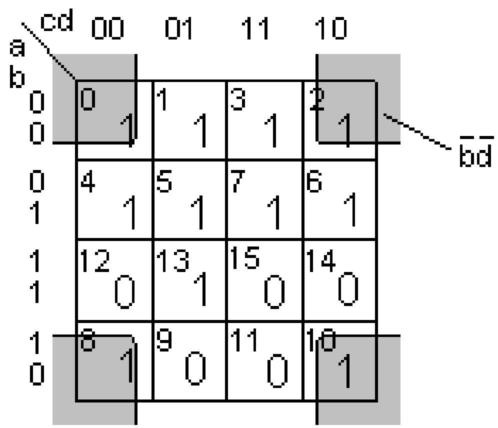
Everywhere in Karnaugh map where one can find such groups of four ones such simplifications can be done, **grouping of four**.

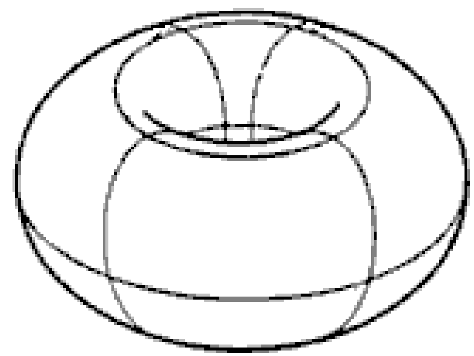
Eight "neighbors"



All groups of 2, 4, 8, (... 2^N ie. powers of 2) frames containing ones can be reduced to a term, with what they have in common, **grouping of n**.

Karnaugh - torus

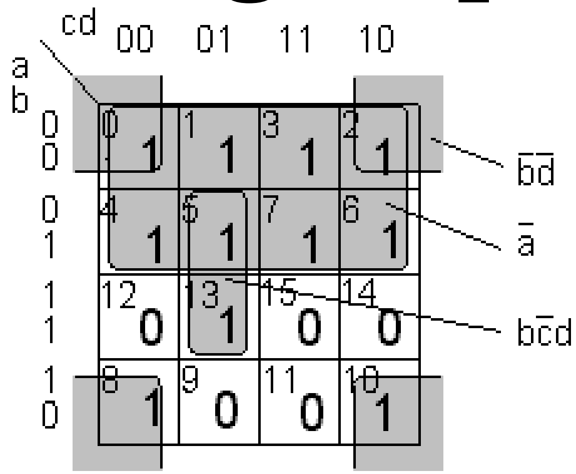




The Karnaugh map should be drawn on a torus (a donut). When we reach an edge, the graph continnues from the opposite side! Frame 0 is the "neighbor" with frame 2, but also the "neighbor" with frame 8 which is "neighbor" to frame 10.

The optimal groupings?

One is looking for the bigest grouping as possible. In the example, there is a grouping with eight ones (frames 0,1,3,2,4,5,7,6). Corners (0,2,8,10) is a group of four ones.



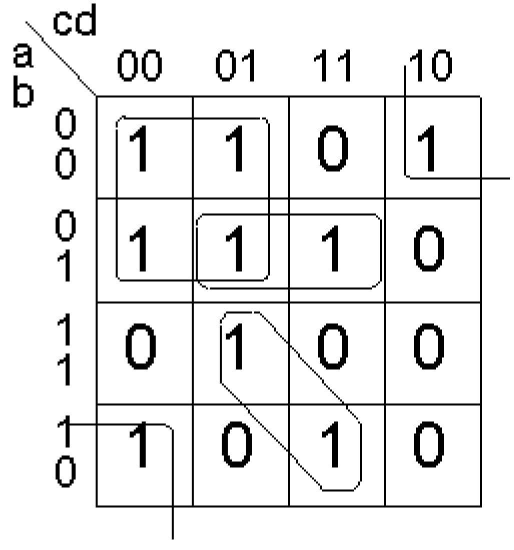
$f(a,b,c,d) = \bar{a} + \bar{b}\bar{d} + b\bar{c}d$

Two of the frames (0. 2) has already been included in the first group, but it does not matter if a minterm is included multiple times.
All ones in the logic function must either be in a grouping, or be included as a minterm. The "1" in frame 13 may form a group with "1" in frame 5, unfortunately there are no bigger grouping for this "1".

• The resulting function is apparently a major simplification compared to the orginal function with the 11 minterms.

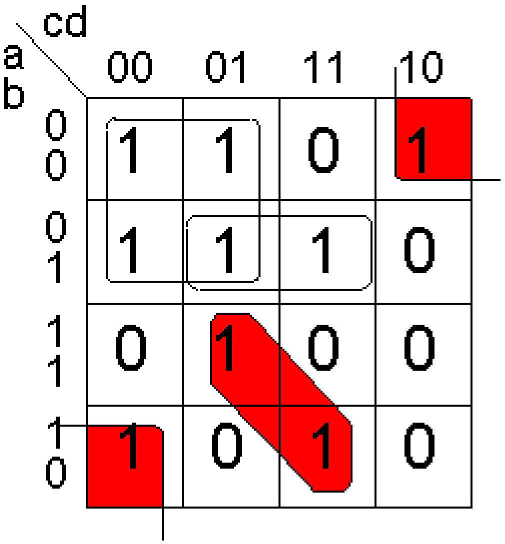
Incorrect groupings?

Is there any incorrect groupings in this Karnaugh diagrams?



Incorrect groupings?

Is there any incorrect groupings in this Karnaugh diagrams?



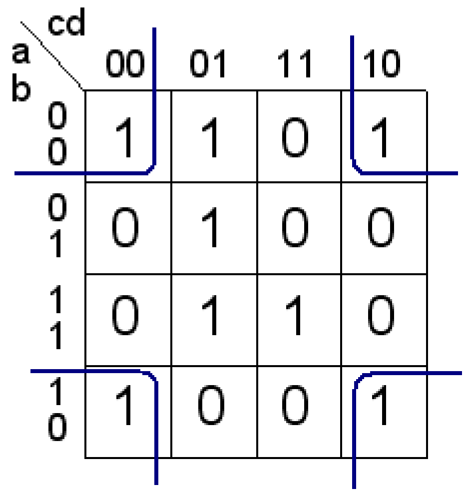
Groupings should be 2, 4, 8 (= power of two) "neighbors"   vertikal or horisontal, not diagonal.

Ex. 6.1 Karnaugh map

ab \ cd	00	01	11	10
00	1	1	0	1
01	0	1	0	0
11	0	1	1	0
10	1	0	0	1

Ex. 6.1 Karnaugh map

$\overline{b}\,\overline{d}$



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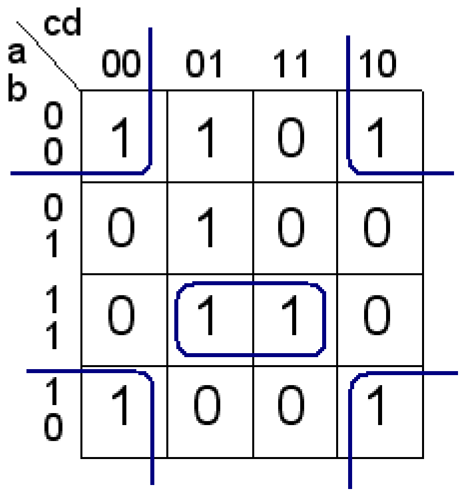
Ex. 6.1 Karnaugh map

$$\bar{b}\bar{d}$$

$$abd$$

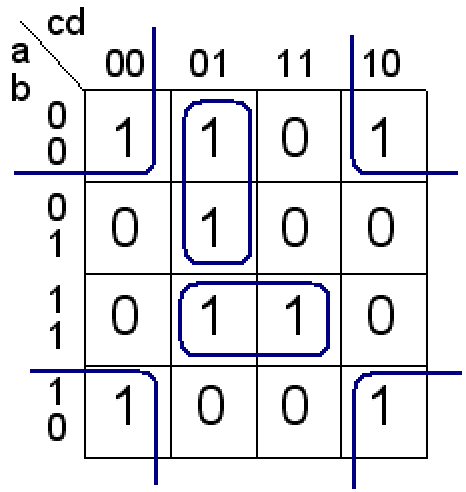


Ex. 6.1 Karnaugh map

$\bar{b}\bar{d}$

$\bar{a}\bar{c}d$

abd

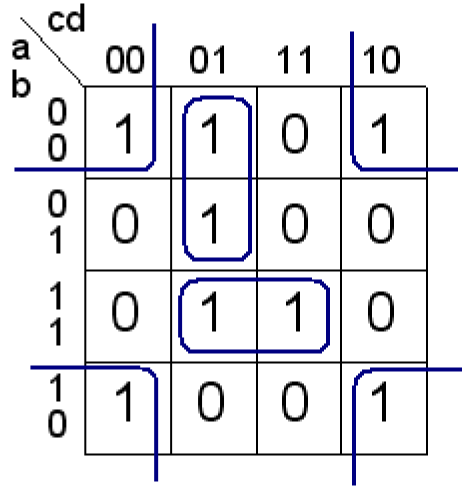


Ex. 6.1 Karnaugh map

$\overline{b}\,\overline{d}$

$\overline{a}\,\overline{c}d$

abd



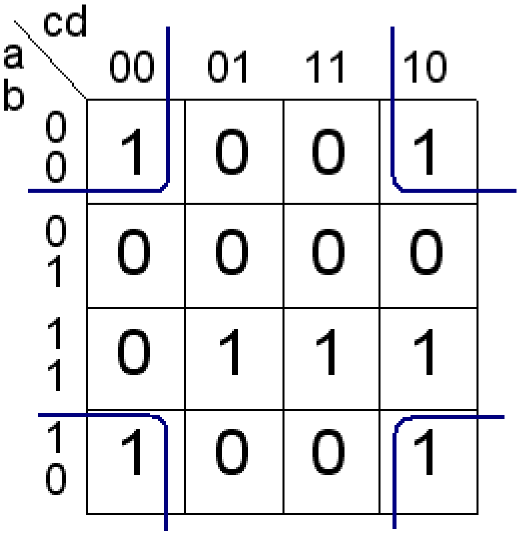
$$f = \overline{b}\,\overline{d} + \overline{a}\,\overline{c}d + abd$$

(Ex. 6.2 Karnaugh map)

ab \ cd	00	01	11	10
00	1	0	0	1
01	0	0	0	0
11	0	1	1	1
10	1	0	0	1

(Ex. 6.2 Karnaugh map)

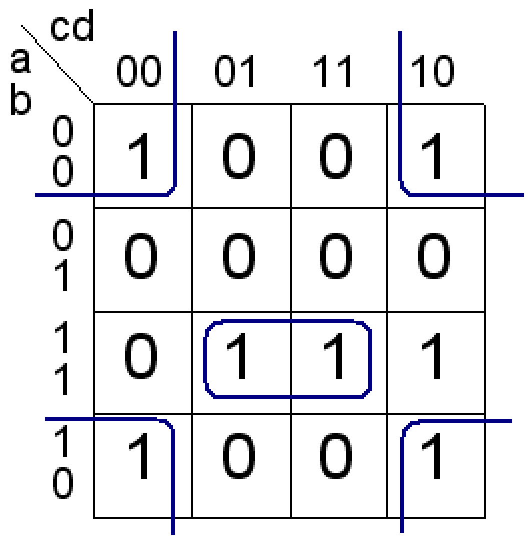
$\overline{b}\,\overline{d}$



(Ex. 6.2 Karnaugh map)

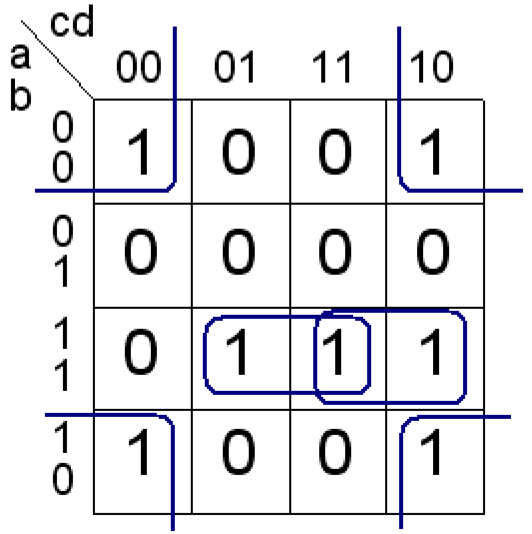
$\overline{b}\,\overline{d}$

abd



(Ex. 6.2 Karnaugh map)

$\overline{b}\,\overline{d}$



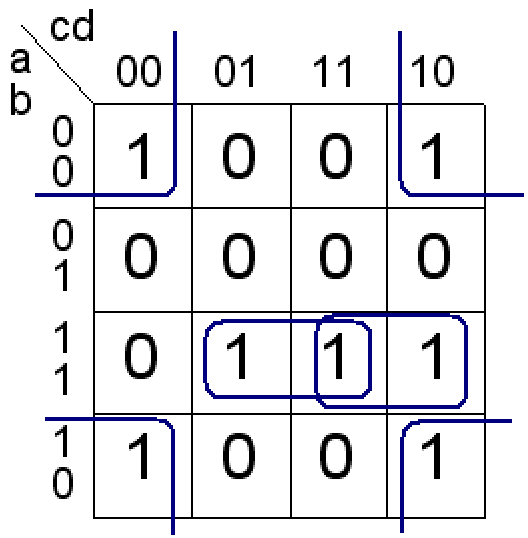
abd

abc

(Ex. 6.2 Karnaugh map)

$\overline{b}\,\overline{d}$

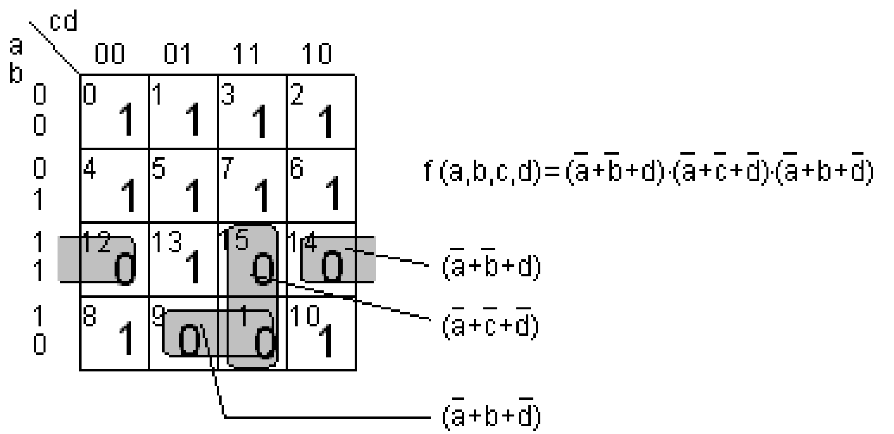
abd



abc

$$f = \overline{b}\,\overline{d} + abd + abc$$

Grouping of "0"

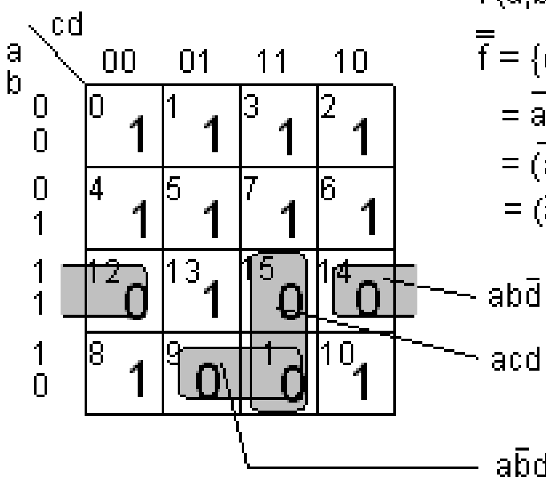


The Karnaugh map is also useful for groupings of 0's. The groupings may include the same number of frames as the case of groupings of 1's. In this example, 0: s are grouped together in pairs with their "neighbors". Maxterms are simplified to what is in common for the frames.

The simplified expression is the product of three sums it represents a very substantial simplification of the original functins's five maxterms.

De Morgan

- Tip!



$$\bar{f}(a,b,c,d) = a\bar{b}d + acd + ab\bar{d}$$

$$\bar{\bar{f}} = \{\text{deMorgan}\} =$$

$$= \overline{a\bar{b}d + acd + ab\bar{d}} =$$

$$= (\overline{a\bar{b}d})\cdot(\overline{acd})\cdot(\overline{ab\bar{d}}) =$$

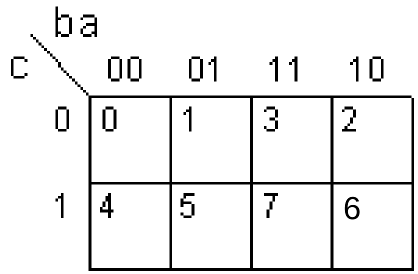
$$= (\bar{a}+b+\bar{d})\cdot(\bar{a}+\bar{c}+\bar{d})\cdot(\bar{a}+\bar{b}+d) = f$$

If you use ”0” as if they were ”1” you will get the function inverted! (totally wrong)

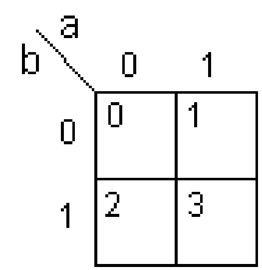
With De Morgans theorem you can invert the inverted function to get the result. (now correct)

Maps for other number of variables

	c	b	a
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1



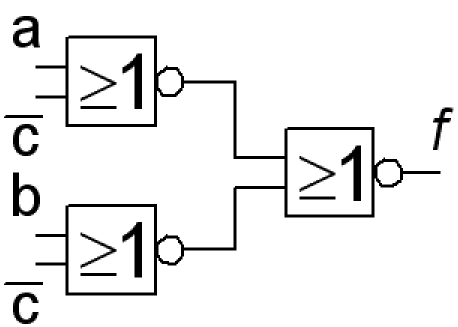
	b	a
0	0	0
1	0	1
2	1	0
3	1	1



Karnaugh maps with three and two variables are also useful.

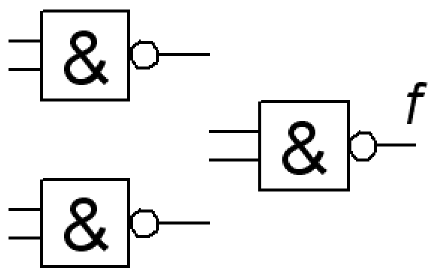
The Karnaugh map can conveniently be used for functions of up to four variables, and with a little practice up to six variables.

Ex 6.4 change NOR to NAND

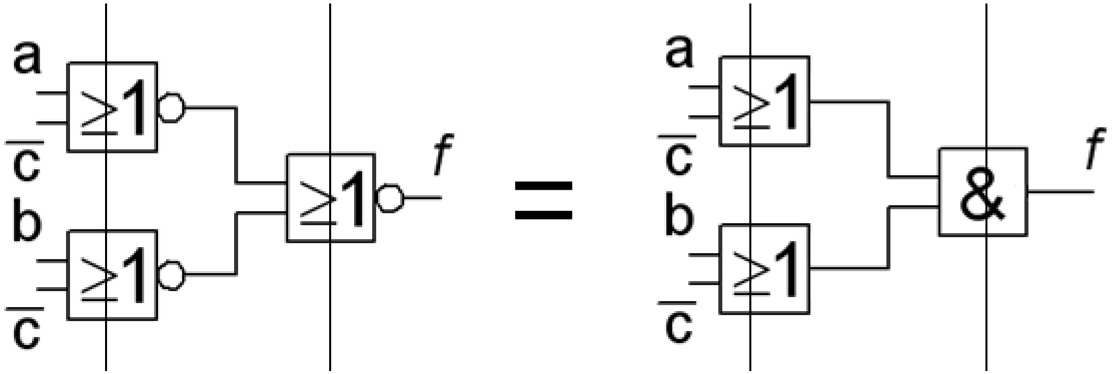


a \ bc	00	01	11	10
0	0	1	3	2
1	4	5	7	6

?

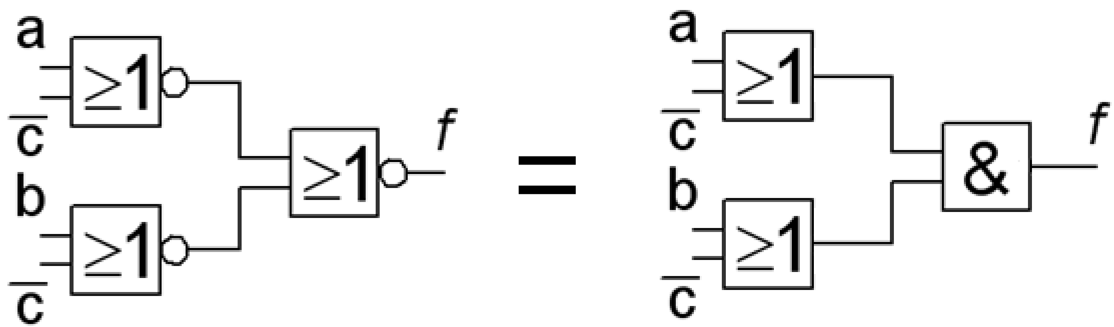


Ex. 6.4



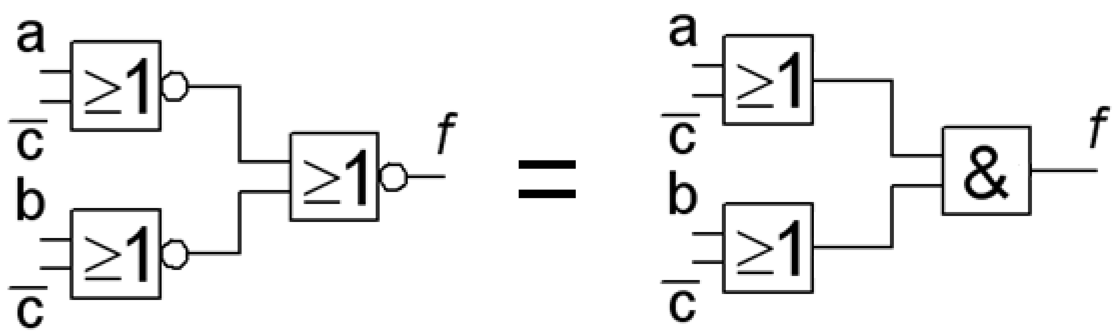
NOR-NOR to OR-AND change ”staight on!

Ex. 6.4

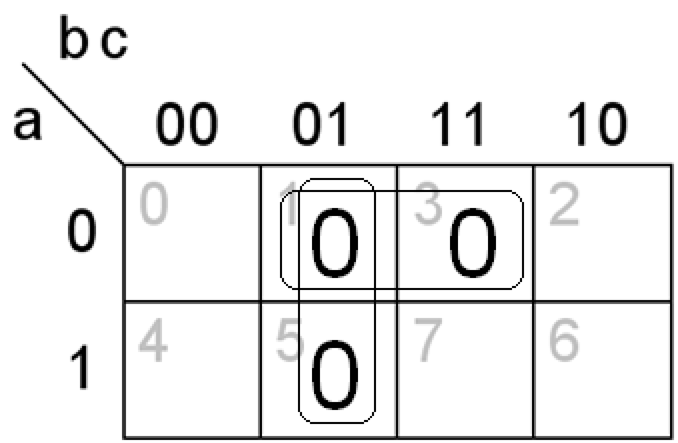


$$f(a,b,c) = (a+\overline{c})\cdot(b+\overline{c})$$

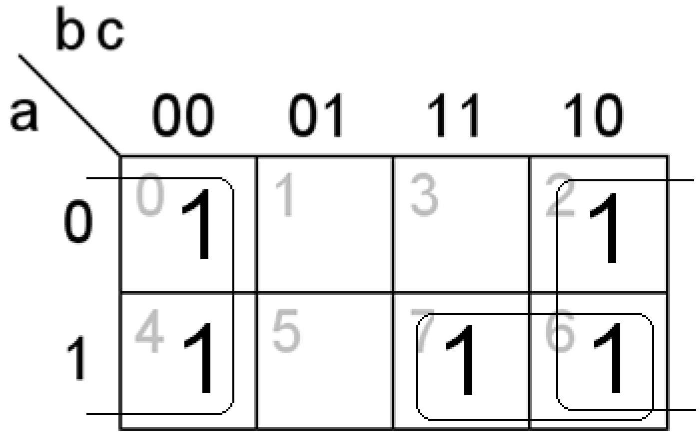
Ex. 6.4



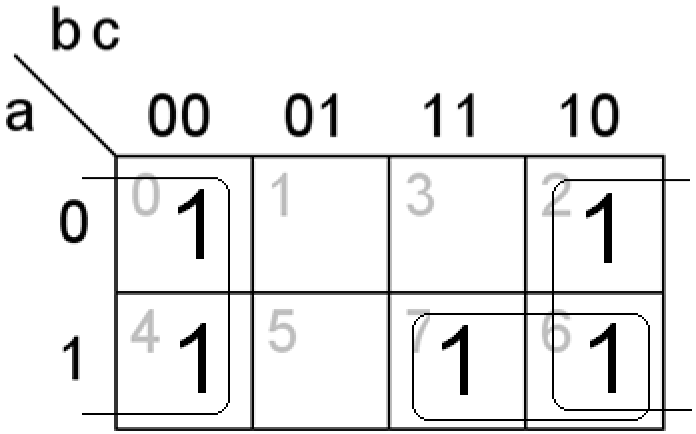
$$f(a,b,c) = (a+\overline{c}) \cdot (b+\overline{c})$$



Ex. 6.4

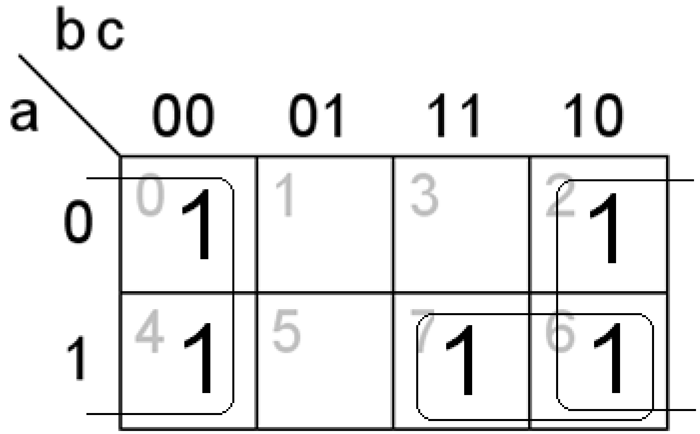


Ex. 6.4

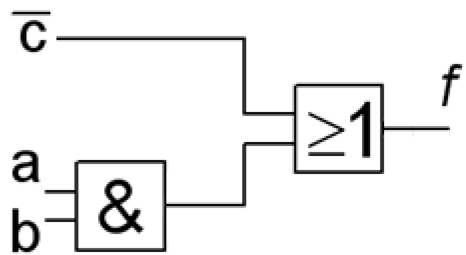


$$f(a,b,c) = \bar{c} + a \cdot b$$

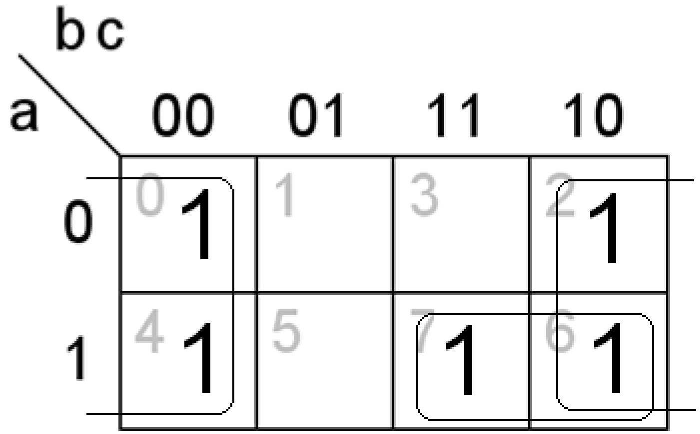
Ex. 6.4



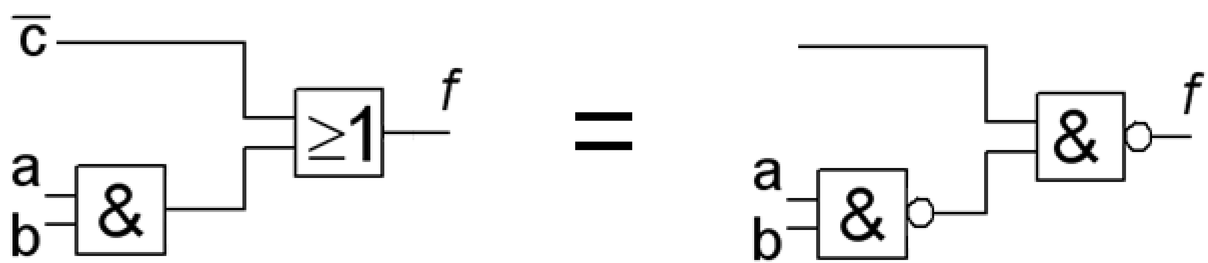
$$f(a,b,c) = \bar{c} + a \cdot b$$



Ex. 6.4



$$f(a,b,c) = \bar{c} + a \cdot b$$



AND-OR NAND-NAND change gates ”straight on”

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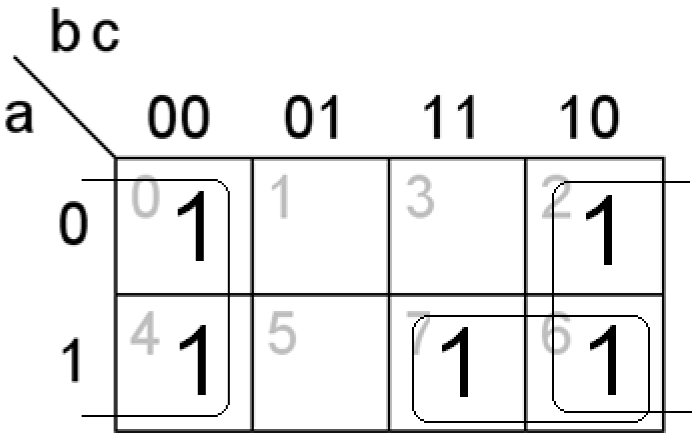
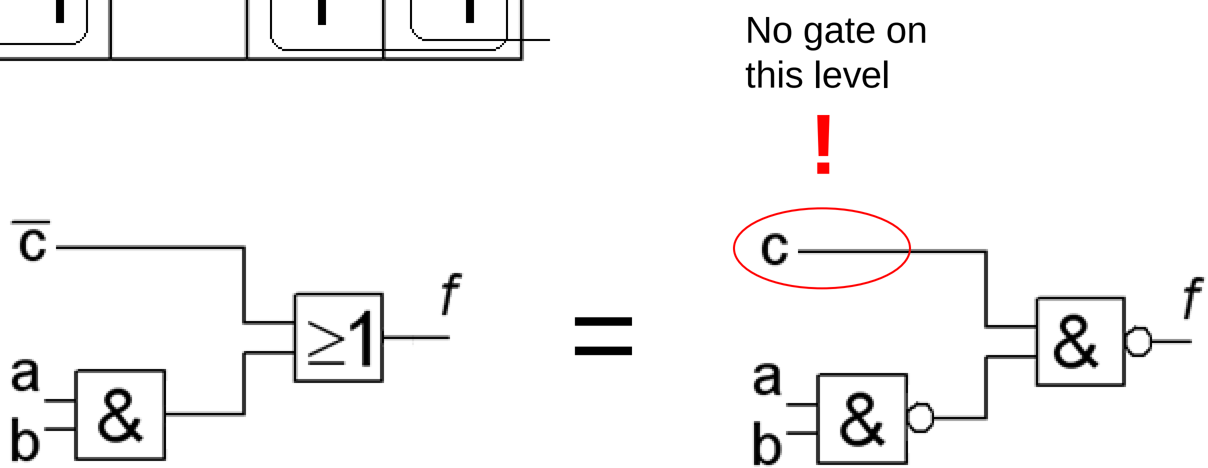
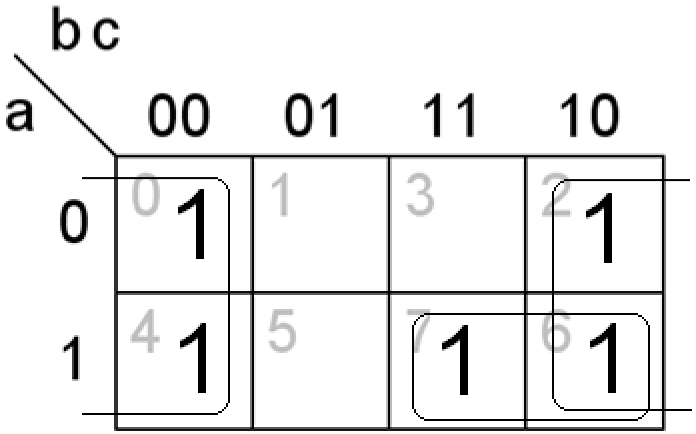
Ex. 6.4



$$f(a,b,c) = \bar{c} + a \cdot b$$



Ex. 6.4

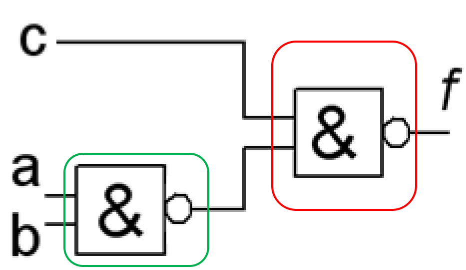


$$f(a,b,c) = \bar{c} + a \cdot b$$

Or algebraic:

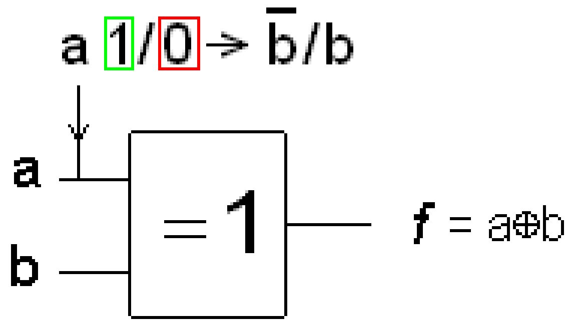
Double invert = standard trick ↓ De Morgan

$$\bar{c} + a \cdot b = \overline{\overline{\bar{c} + a \cdot b}} = \overline{c \cdot \overline{a \cdot b}}$$



PLD-chip has output inverters

PLD circuits often have an XOR gate at the output so that they shall be able to invert the function. One can then choose to bring together 0s or 1s after what is most advantageous.



a	b	f
0	0	0
0	1	1
1	0	1
1	1	0

When the control signal is a "1" the gates output is b's inverse, when a is "0", the output is equal to b

Ex. 6.5 Minimize with the K-map

$$f(x_3, x_2, x_1, x_0) = \sum m(0, 2, 4, 8, 10, 12) \quad f = ? \quad \overline{f} = ?$$

Ex. 6.5 Minimize with the K-map

$$f(x_3, x_2, x_1, x_0) = \sum m(0, 2, 4, 8, 10, 12) \quad f = ? \quad \overline{f} = ?$$

	x_3	x_2	x_1	x_0	f
0	0	0	0	0	1
1	0	0	0	1	0
2	0	0	1	0	1
3	0	0	1	1	0
4	0	1	0	0	1
5	0	1	0	1	0
6	0	1	1	0	0
7	0	1	1	1	0
8	1	0	0	0	1
9	1	0	0	1	0
10	1	0	1	0	1
11	1	0	1	1	0
12	1	1	0	0	1
13	1	1	0	1	0
14	1	1	1	0	0
15	1	1	1	1	0

Ex. 6.5 Minimize with the K-map

$$f(x_3, x_2, x_1, x_0) = \sum m(0, 2, 4, 8, 10, 12) \quad f = ? \quad \overline{f} = ?$$

	x_3	x_2	x_1	x_0	f
0	0	0	0	0	1
1	0	0	0	1	0
2	0	0	1	0	1
3	0	0	1	1	0
4	0	1	0	0	1
5	0	1	0	1	0
6	0	1	1	0	0
7	0	1	1	1	0
8	1	0	0	0	1
9	1	0	0	1	0
10	1	0	1	0	1
11	1	0	1	1	0
12	1	1	0	0	1
13	1	1	0	1	0
14	1	1	1	0	0
15	1	1	1	1	0

x_3x_2 \ x_1x_0	00	01	11	10
00	0	1	3	2
01	4	5	7	6
11	12	13	15	14
10	8	9	11	10

Ex. 6.5 Minimize with the K-map

$$f(x_3, x_2, x_1, x_0) = \sum m(0, 2, 4, 8, 10, 12) \quad f = ? \quad \overline{f} = ?$$

	x_3	x_2	x_1	x_0	f
0	0	0	0	0	1
1	0	0	0	1	0
2	0	0	1	0	1
3	0	0	1	1	0
4	0	1	0	0	1
5	0	1	0	1	0
6	0	1	1	0	0
7	0	1	1	1	0
8	1	0	0	0	1
9	1	0	0	1	0
10	1	0	1	0	1
11	1	0	1	1	0
12	1	1	0	0	1
13	1	1	0	1	0
14	1	1	1	0	0
15	1	1	1	1	0

x_3x_2 \ x_1x_0	00	01	11	10
00	^{0}1	^{1}0	^{3}0	^{2}1
01	^{4}1	^{5}0	^{7}0	^{6}0
11	^{12}1	^{13}0	^{15}0	^{14}0
10	^{8}1	^{9}0	^{11}0	^{10}1

Ex. 6.5 Minimize with the K-map

$$f(x_3, x_2, x_1, x_0) = \sum m(0, 2, 4, 8, 10, 12) \quad f = ? \quad \overline{f} = ?$$

Grouping of ”1”

x_3x_2 \ x_1x_0	00	01	11	10
00	(0) 1	(1) 0	(3) 0	(2) 1
01	(4) 1	(5) 0	(7) 0	(6) 0
11	(12) 1	(13) 0	(15) 0	(14) 0
10	(8) 1	(9) 0	(11) 0	(10) 1

Grouping of ”0”

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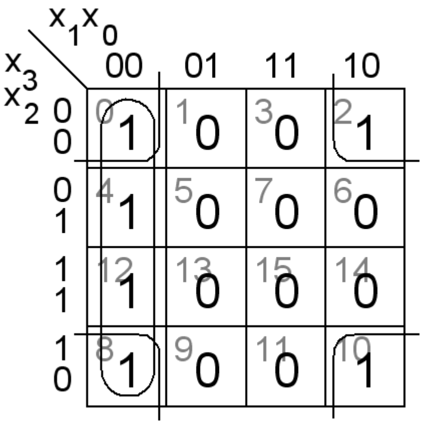
Ex. 6.5 Minimize with the K-map

$$f(x_3, x_2, x_1, x_0) = \sum m(0, 2, 4, 8, 10, 12) \quad f = ? \quad \overline{f} = ?$$

Grouping of ”1”

Grouping of ”0”



$$f = \overline{x}_1 \overline{x}_0 + \overline{x}_2 \overline{x}_0$$

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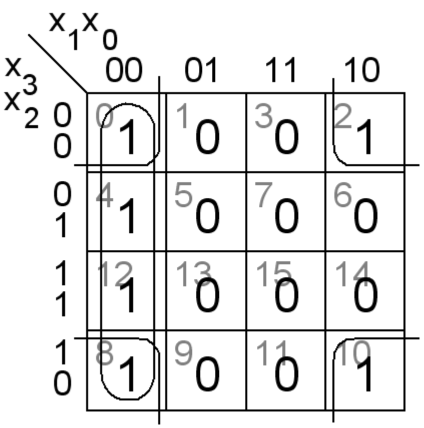
Ex. 6.5 Minimize with the K-map

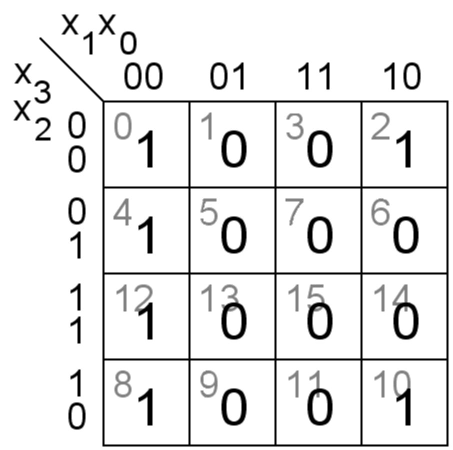
$$f(x_3, x_2, x_1, x_0) = \sum m(0, 2, 4, 8, 10, 12) \quad f = ? \quad \overline{f} = ?$$

Grouping of ”1”



$$f = \overline{x}_1\overline{x}_0 + \overline{x}_2\overline{x}_0$$

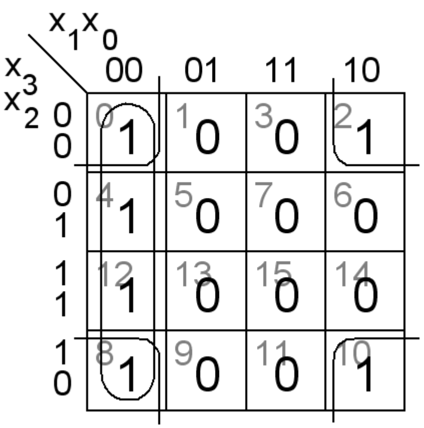
Grouping of ”0”



Ex. 6.5 Minimize with the K-map

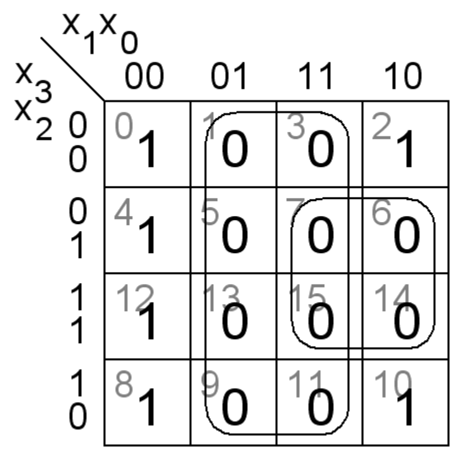
$$f(x_3, x_2, x_1, x_0) = \sum m(0, 2, 4, 8, 10, 12) \quad f = ? \quad \overline{f} = ?$$

Grouping of ”1”



$$f = \bar{x}_1 \bar{x}_0 + \bar{x}_2 \bar{x}_0$$

Grouping of ”0”

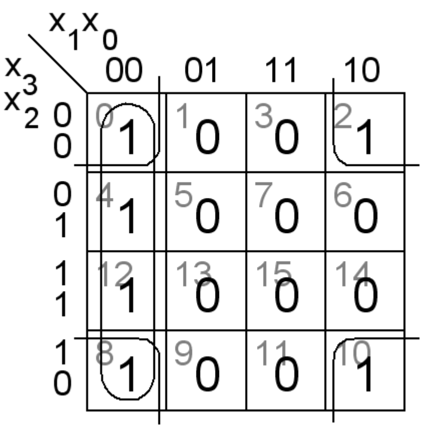


$$\overline{f} = \{\text{"0" as "1"}\} = x_0 + x_2 x_1$$

Ex. 6.5 Minimize with the K-map

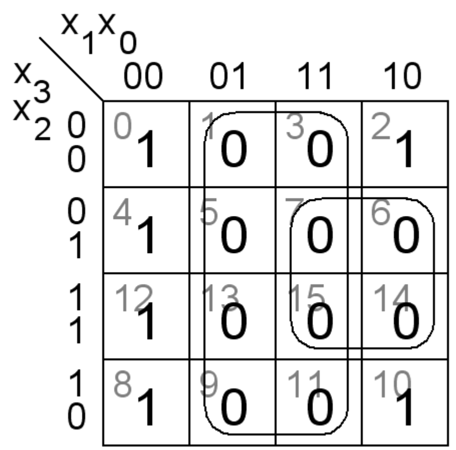
$$f(x_3, x_2, x_1, x_0) = \sum m(0, 2, 4, 8, 10, 12) \quad f = ? \quad \overline{f} = ?$$

Grouping of ”1”



$$f = \overline{x}_1 \overline{x}_0 + \overline{x}_2 \overline{x}_0$$

Grouping of ”0”



$$\overline{f} = \{\text{"0" as "1"}\} = x_0 + x_2 x_1$$

This time it was advantageous to group 0s and invert the output!

Ex. 6.8 Don’t Care

Sometimes, problem is such that certain input combinations is "impossible" and therefore can not occur. Such mintermer (or maxtermer) one denotes man d ("do not care") and use them as ones or zeros depending on what works best to get as big groupings as possible.

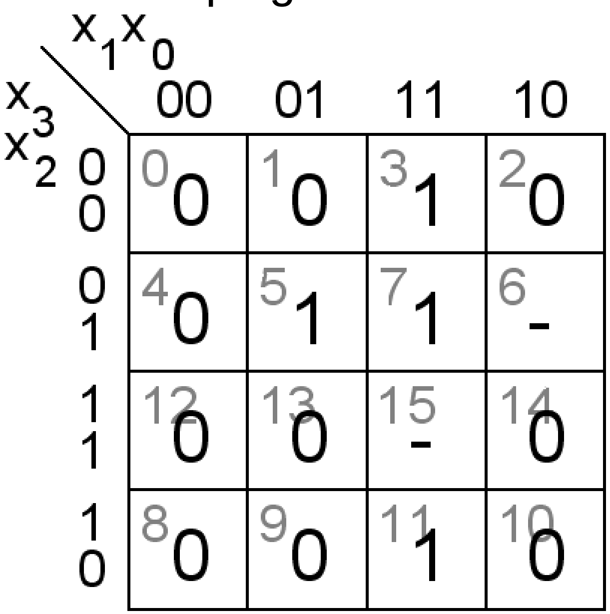
$$f(x_3, x_2, x_1, x_0) = \sum m(3, 5, 7, 11) + d(6, 15) \quad f = ? \quad \overline{f} = ?$$

(A risk may be that what is thought to be "impossible" still occurs!? Therefore, it may often be better to take care of all combinations.)

Ex. 6.8

$$f(x_3, x_2, x_1, x_0) = \sum m(3, 5, 7, 11) + d(6, 15) \quad f = ? \quad \overline{f} = ?$$

Grouping of ”1”



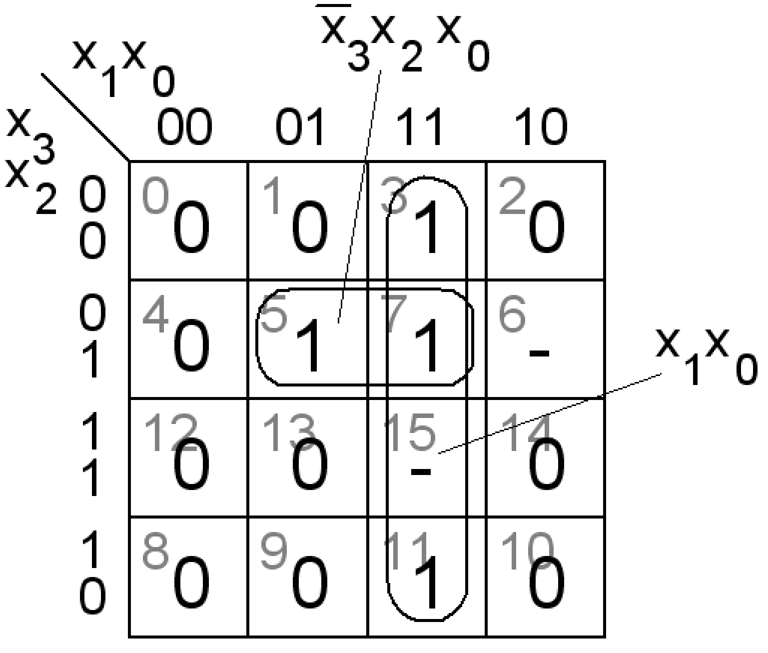
Grouping of ”0”

Ex. 6.8

$$f(x_3, x_2, x_1, x_0) = \sum m(3, 5, 7, 11) + d(6, 15) \quad f = ? \quad \overline{f} = ?$$

Grouping of ”1”

Grouping of ”0”

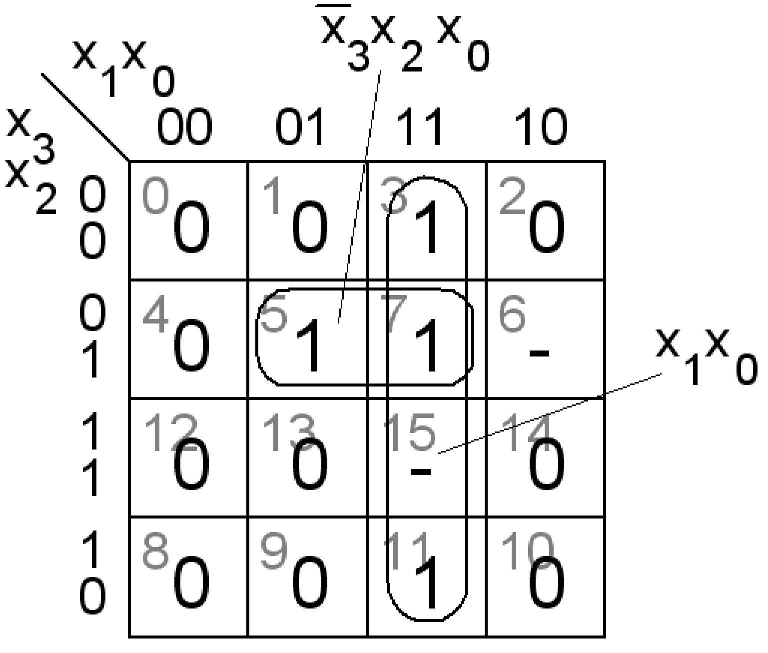


$$f = x_1 x_0 + \overline{x_3} x_2 x_0$$

Ex. 6.8

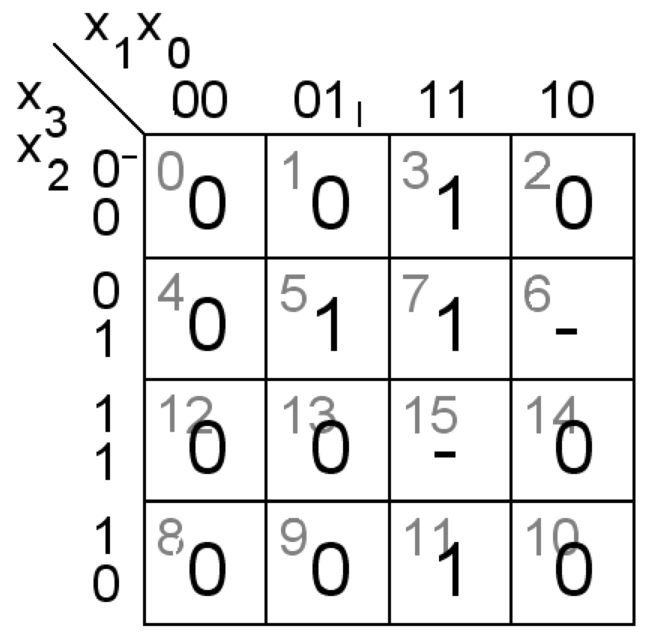
$$f(x_3, x_2, x_1, x_0) = \sum m(3, 5, 7, 11) + d(6, 15) \quad f = ? \quad \overline{f} = ?$$

Grouping of ”1”



$$f = x_1 x_0 + \overline{x_3} x_2 x_0$$

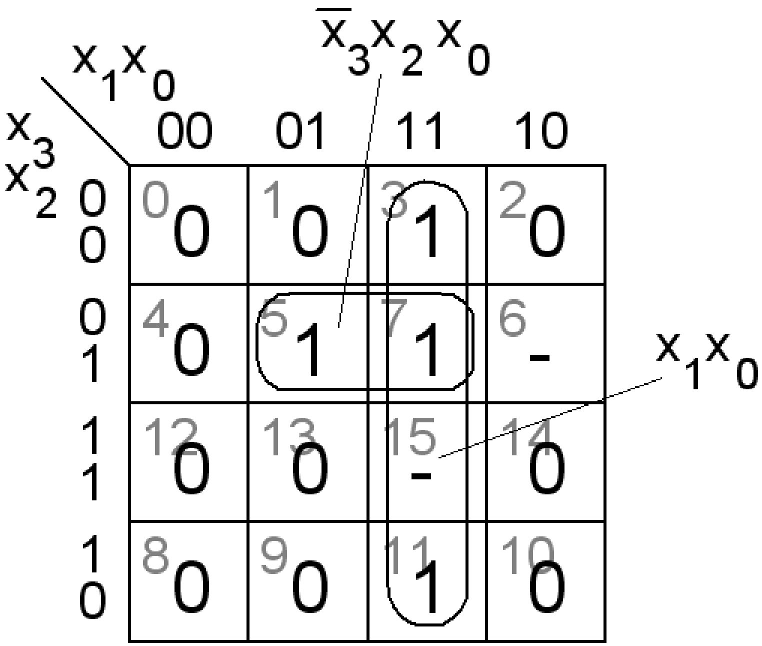
Grouping of ”0”



Ex. 6.8

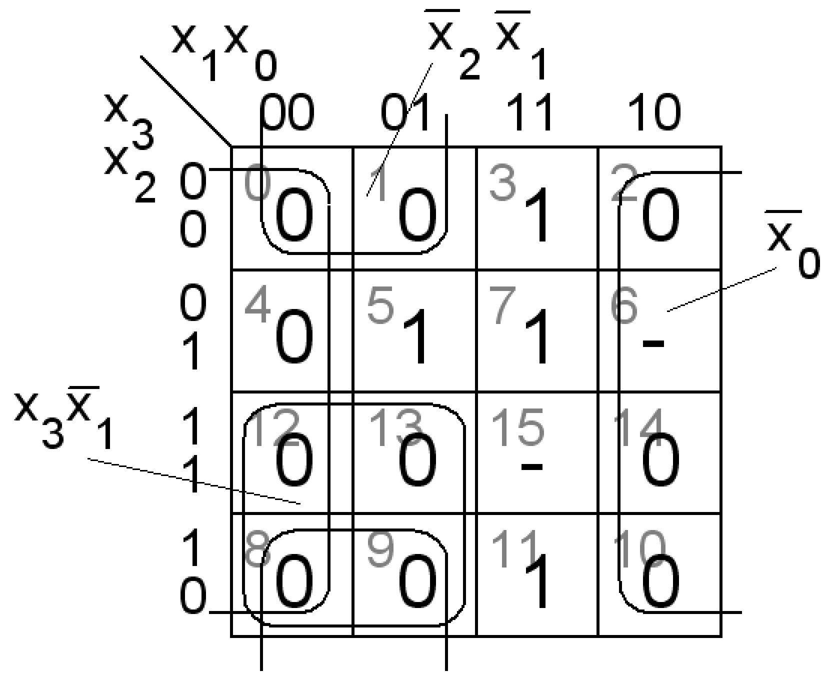
$$f(x_3, x_2, x_1, x_0) = \sum m(3, 5, 7, 11) + d(6, 15) \qquad f = ? \qquad \overline{f} = ?$$

Grouping of ”1”



$$f = x_1 x_0 + \overline{x_3} x_2 x_0$$

Grouping of ”0”



$$\overline{f} = \overline{x_0} + \overline{x_2}\,\overline{x_1} + x_3 \overline{x_1}$$

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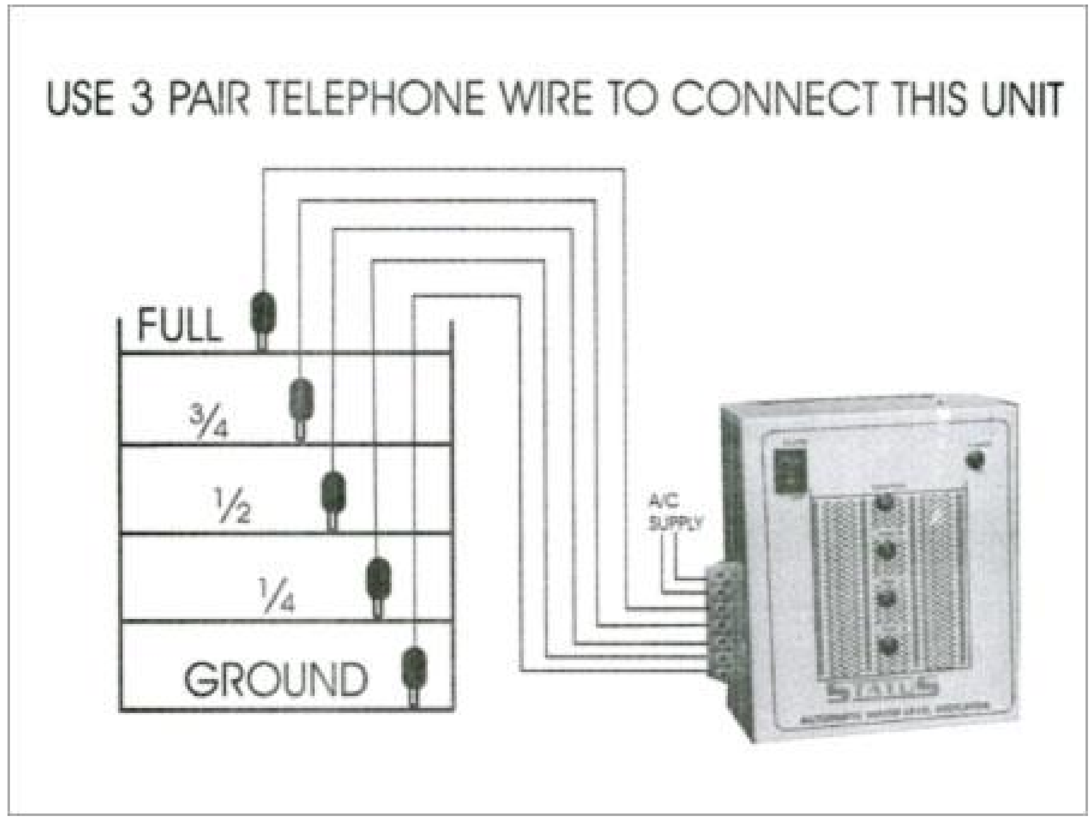
Alarm for water tank



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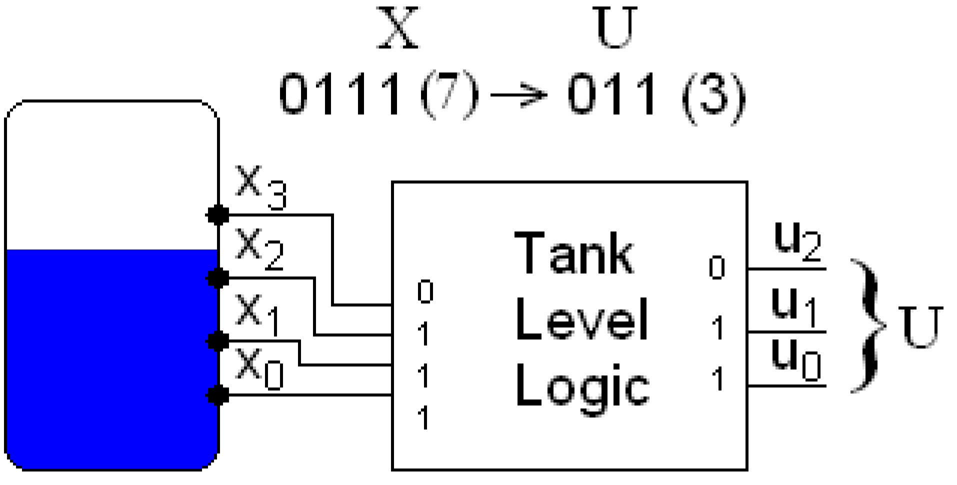
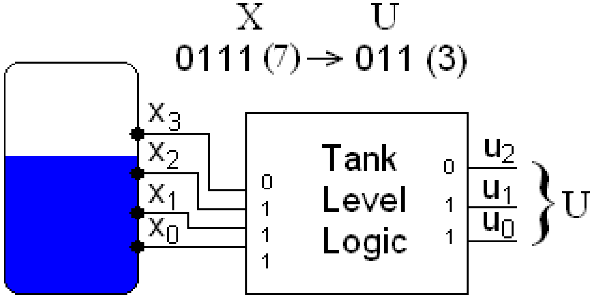
Ex. 8.2



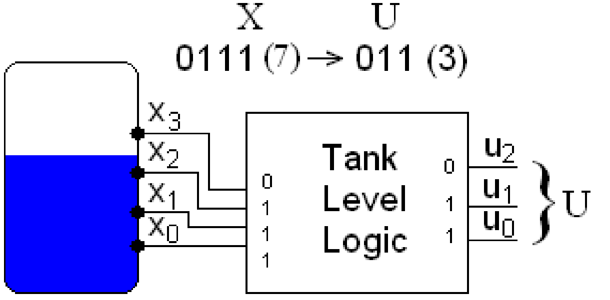
Ex. 8.2

X	x_3	x_2	x_1	x_0	U	u_2	u_1	u_0



Ex. 8.2

X	x_3	x_2	x_1	x_0	U	u_2	u_1	u_0
0	0	0	0	0	0	0	0	0
1	0	0	0	1	1	0	0	1
3	0	0	1	1	2	0	1	0
7	0	1	1	1	3	0	1	1
15	1	1	1	1	4	1	0	0



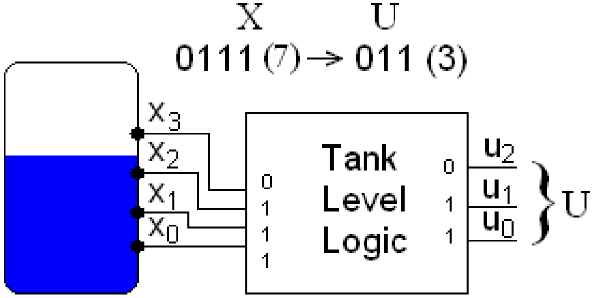
Only the in-combinations X 0, 1, 3, 7, 15 can occur. All other in-combinations can be used as ”don’t care”.

We can directly see from the table that u_2 and x_3 are same, u_2 can be directly connected to x_3. $u_2 = x_3$.

The other expressions are obtained by using their Karnaugh maps.

Ex. 8.2

X	x_3	x_2	x_1	x_0	U	u_2	u_1	u_0
0	0	0	0	0	0	0	0	0
1	0	0	0	1	1	0	0	1
3	0	0	1	1	2	0	1	0
7	0	1	1	1	3	0	1	1
15	1	1	1	1	4	1	0	0



Only the in-combinations X 0, 1, 3, 7, 15 can occur. All other in-combinations can be used as ”don’t care”.

We can directly see from the table that u_2 and x_3 are same, u_2 can be directly connected to x_3. $u_2 = x_3$.

The other expressions are obtained by using their Karnaugh maps.

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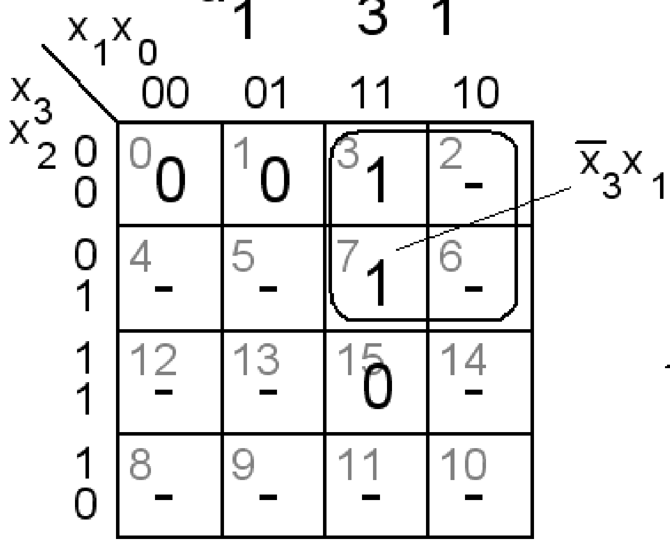
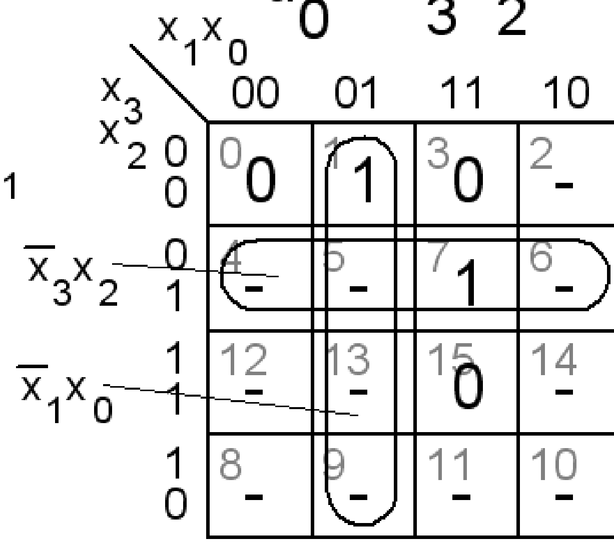
Ex. 8.2

X	x_3	x_2	x_1	x_0	U	u_2	u_1	u_0
0	0	0	0	0	0	0	0	0
1	0	0	0	1	1	0	0	1
3	0	0	1	1	2	0	1	0
7	0	1	1	1	3	0	1	1
15	1	1	1	1	4	1	0	0

$u_1 = \bar{x}_3 x_1$



$u_0 = \bar{x}_3 x_2 + \bar{x}_1 x_0$



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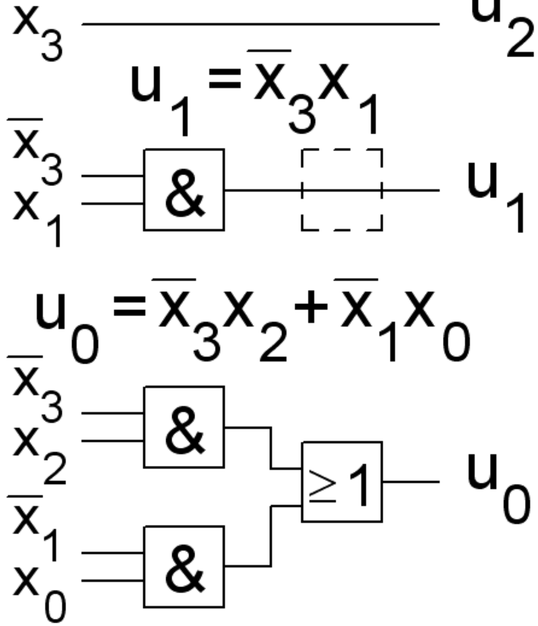
Ex. 8.2

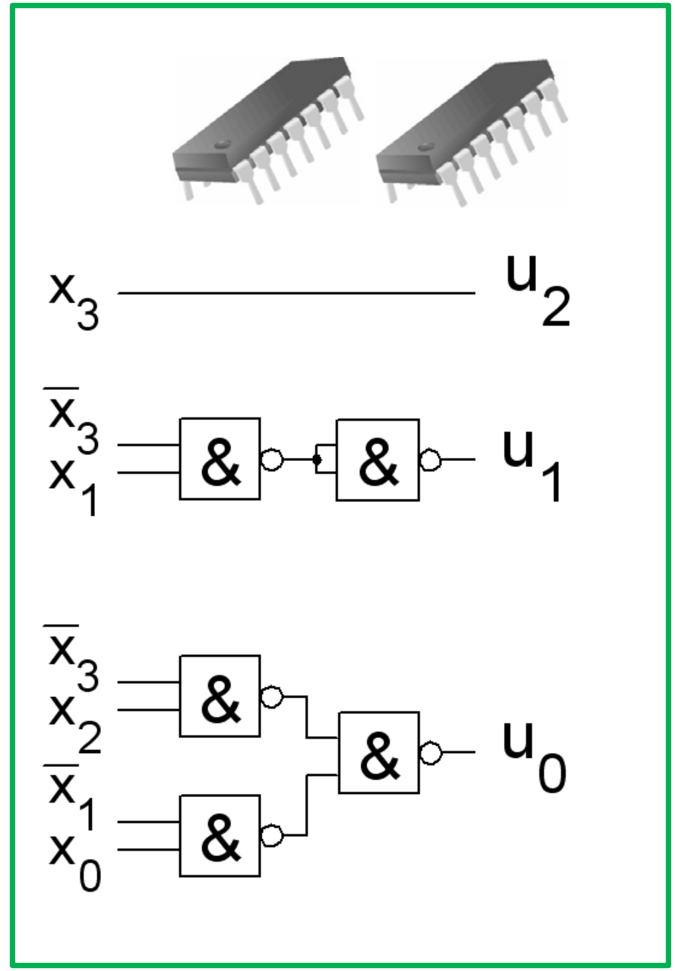
$u_2 = x_3$

$u_1 = \overline{x}_3 x_1$

$u_0 = \overline{x}_3 x_2 + \overline{x}_1 x_0$



=



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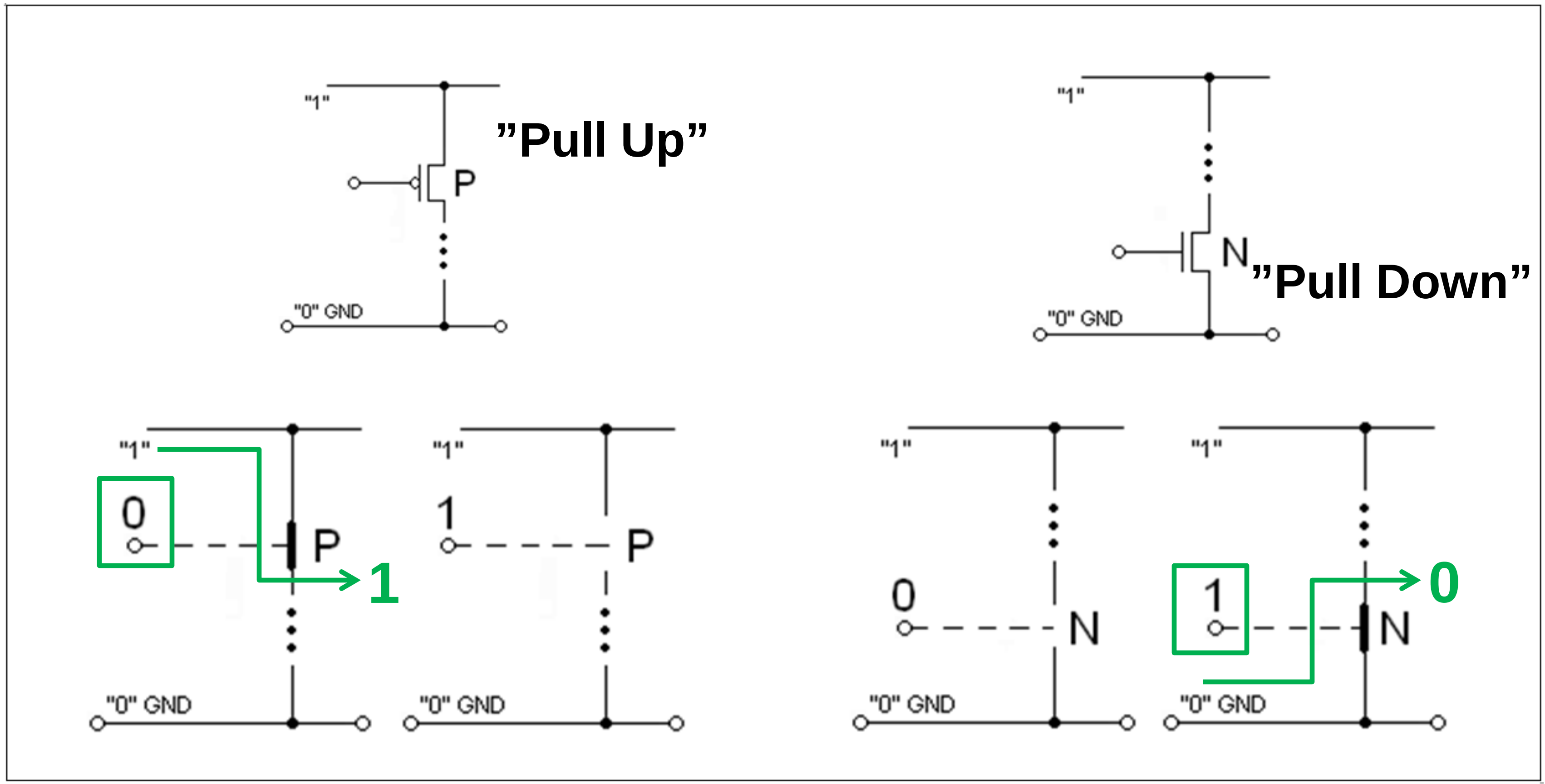
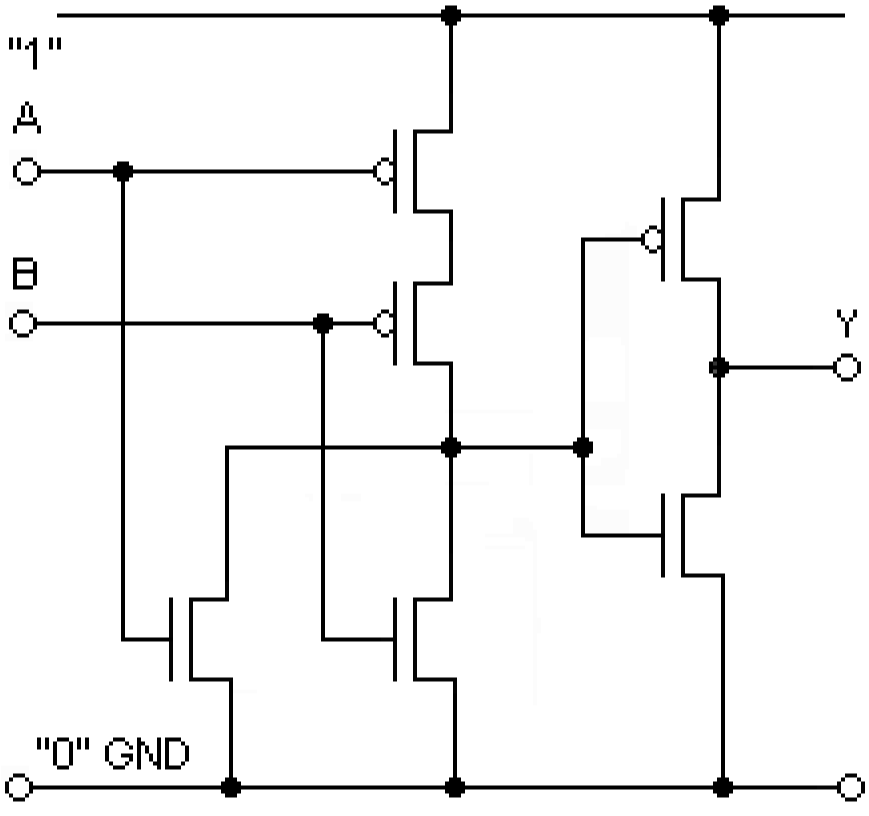
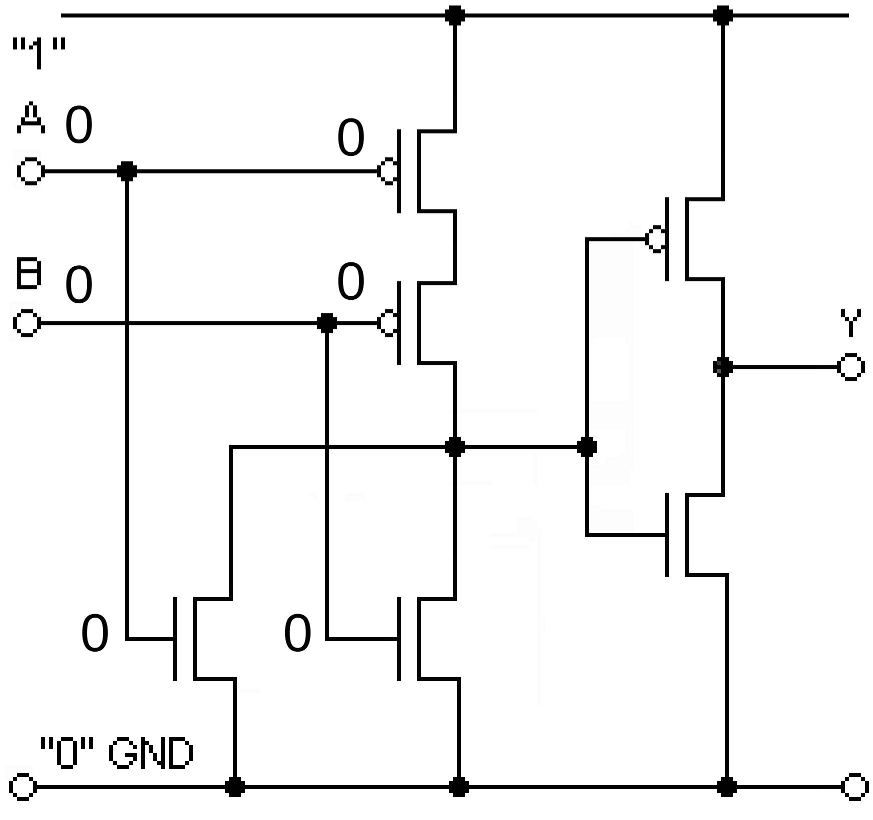
P and N MOS-transistors



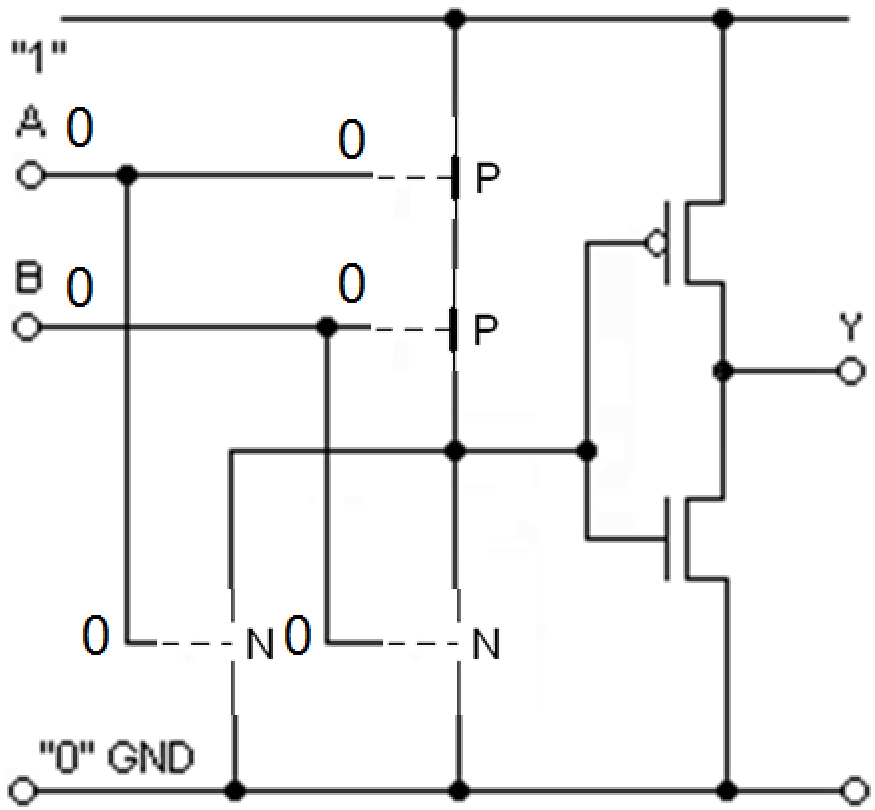
Ex. 7.3 CMOS-gate ?



Ex. 7.3   A=0 B=0



Ex. 7.3   A=0 B=0



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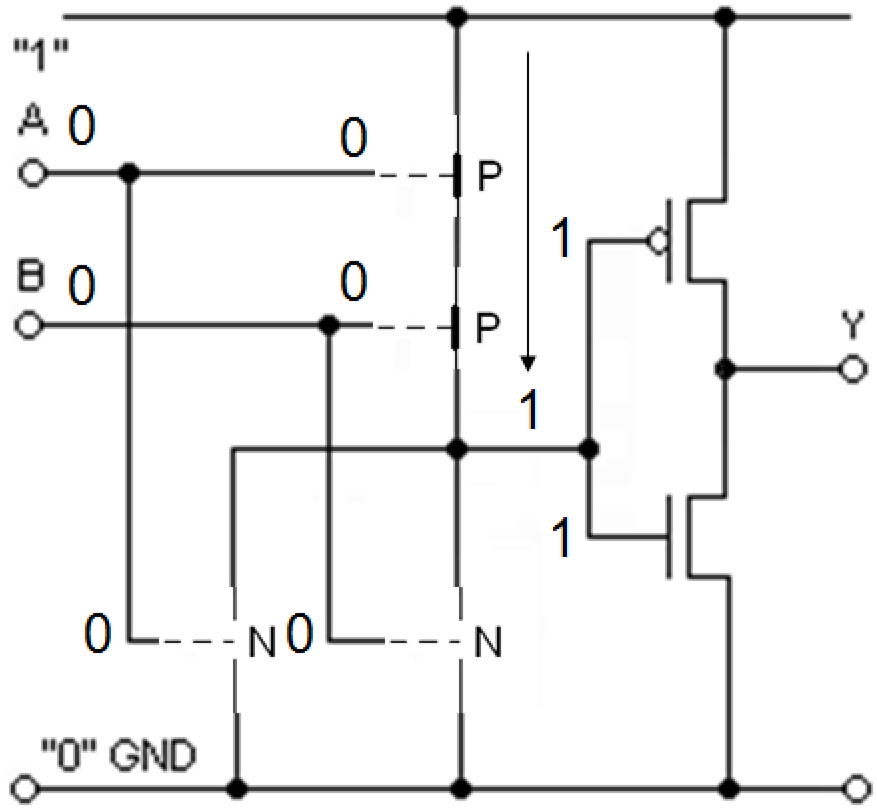
Ex. 7.3   A=0 B=0



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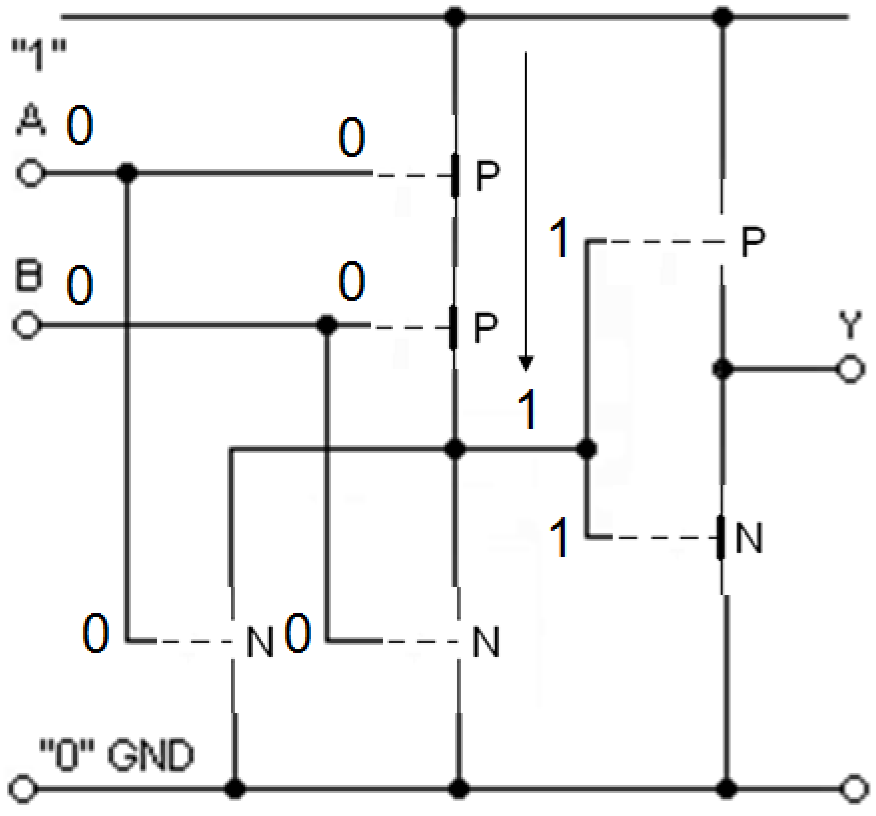
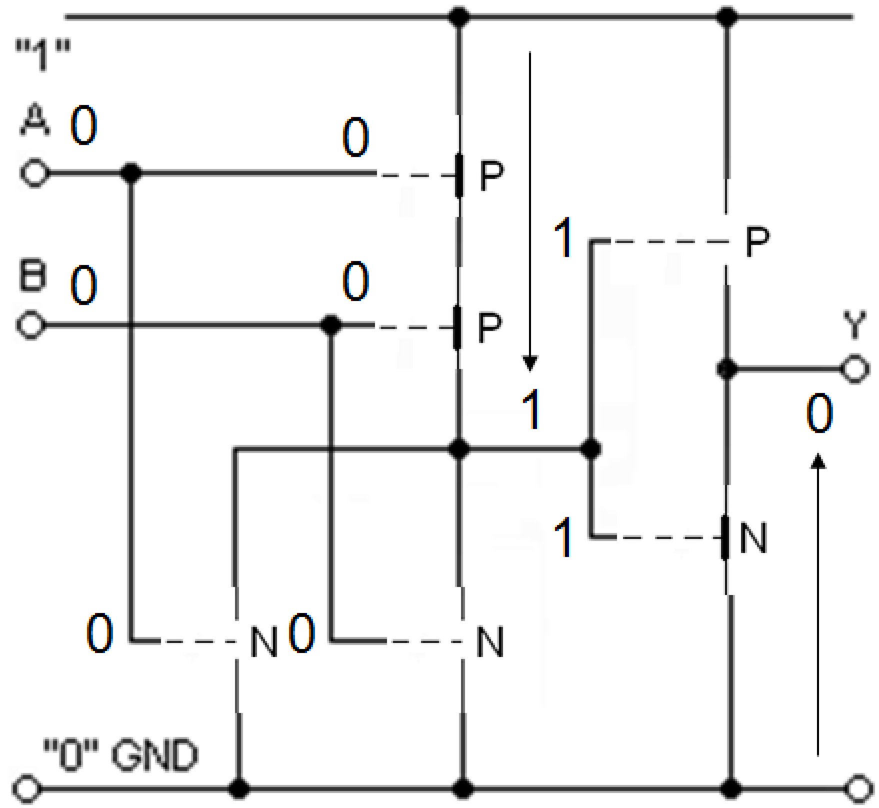
Ex. 7.3   A=0 B=0

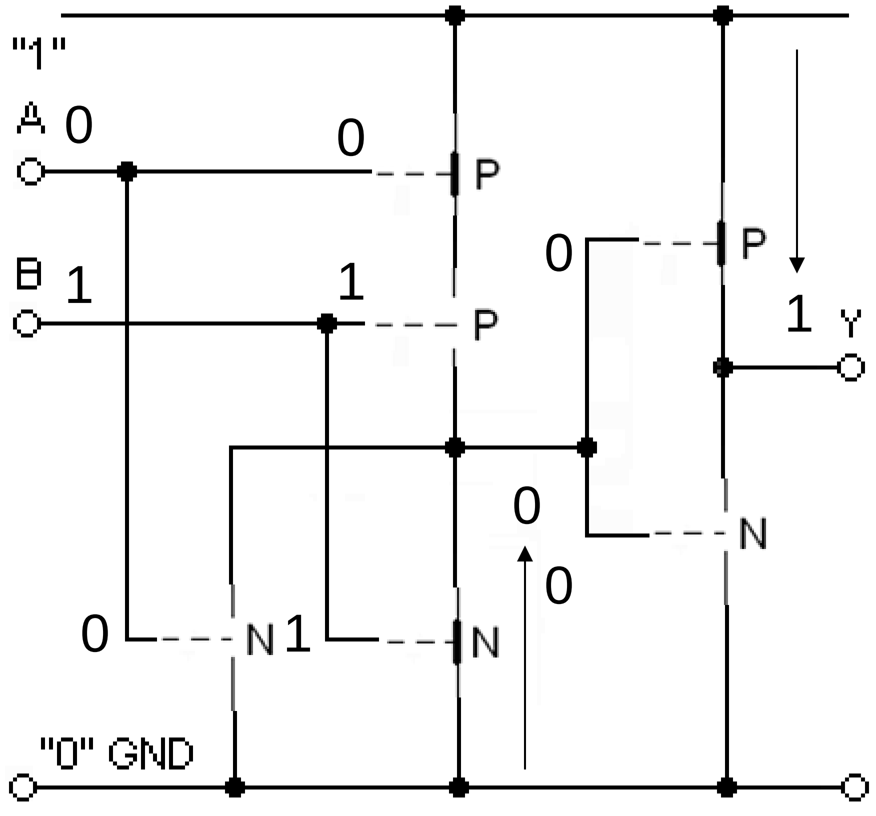


Ex. 7.3 A=0 B=0



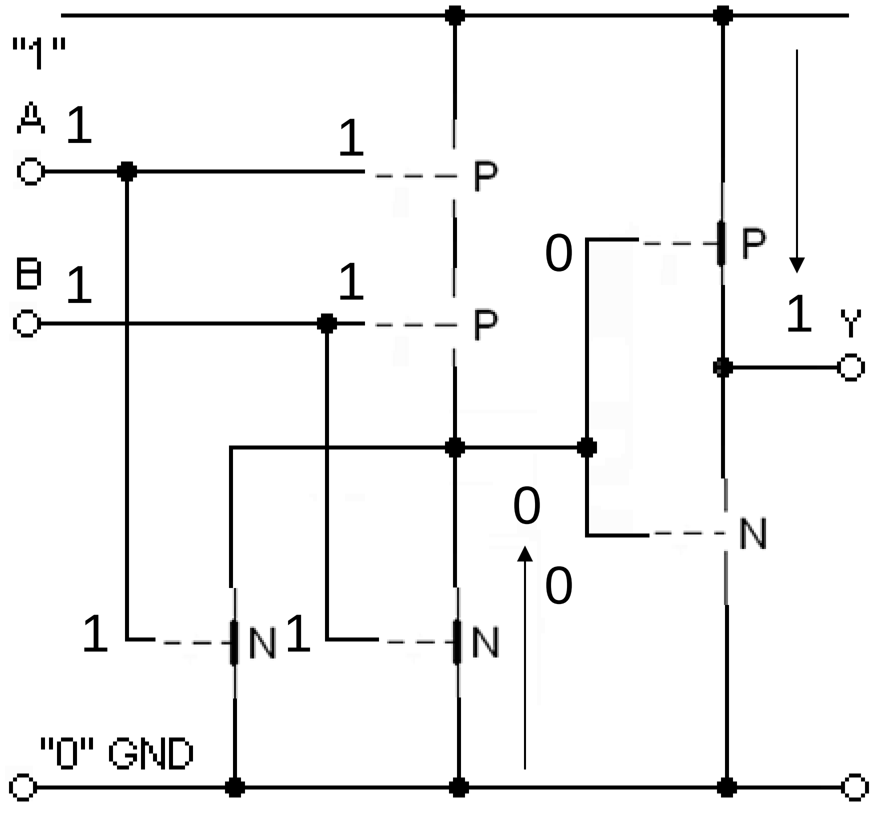
Y = 0

Ex. 7.3 A=0 B=1



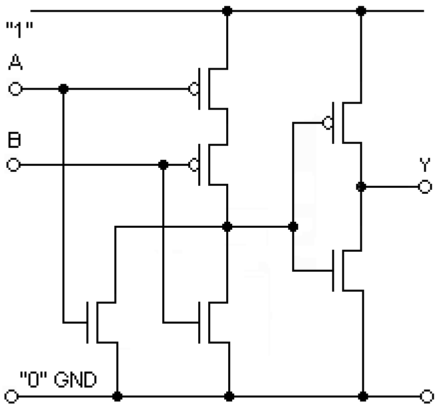
Y = 1

Ex. 7.3 A=1 B=1



Y = 1

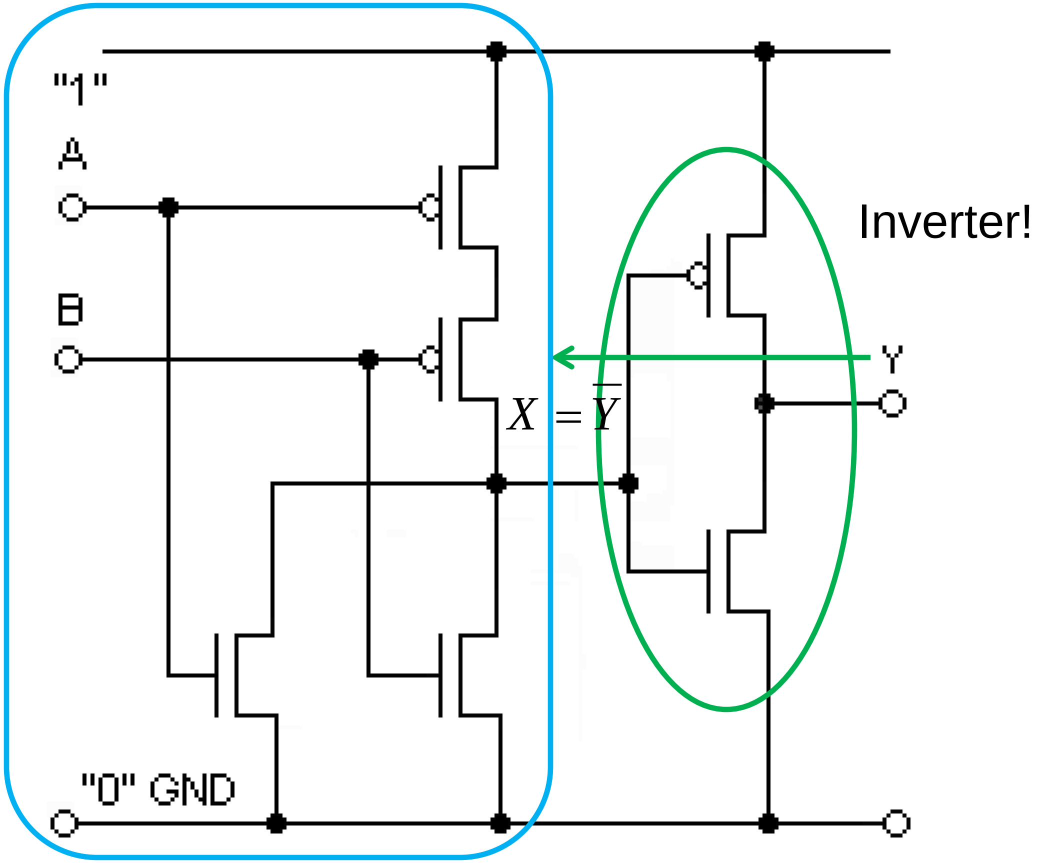
7.3



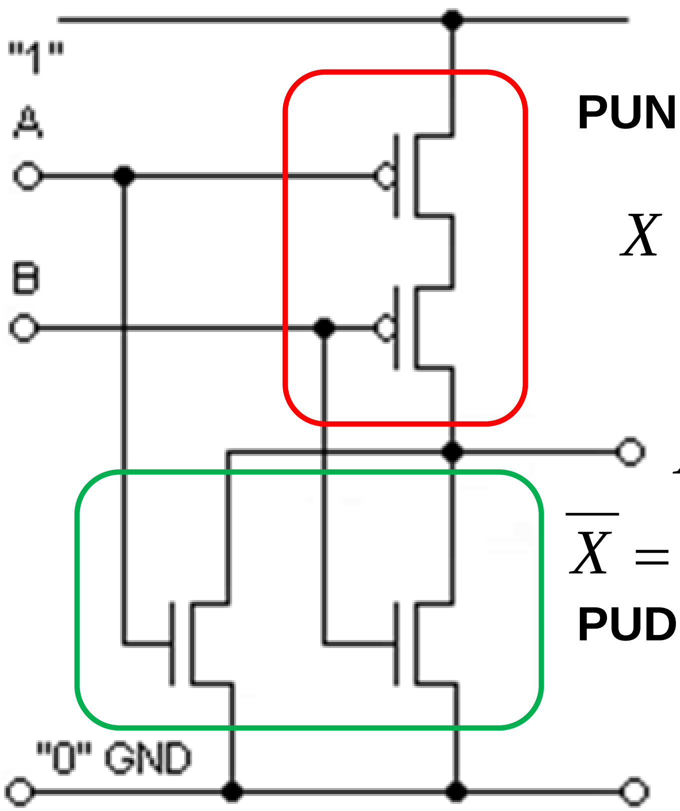
A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1

OR-gate

7.3 other solution



7.3

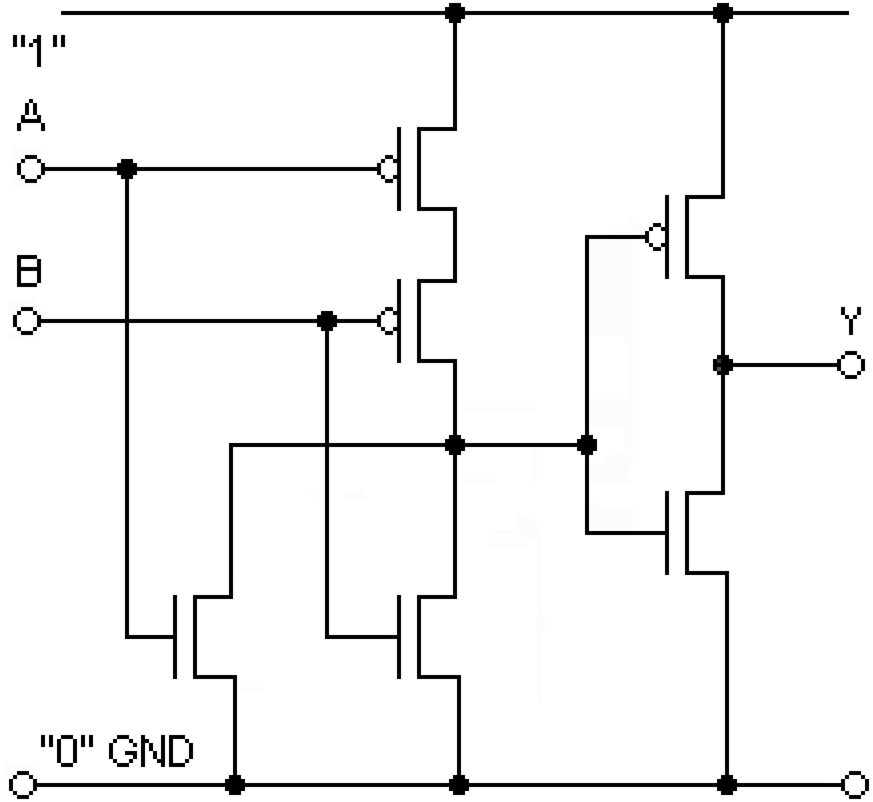


$$X = \overline{A} \cdot \overline{B} \quad \overline{X} = \overline{\overline{A} \cdot \overline{B}} = A + B$$

$$X = \overline{Y} \quad \Rightarrow \quad Y = \overline{X}$$

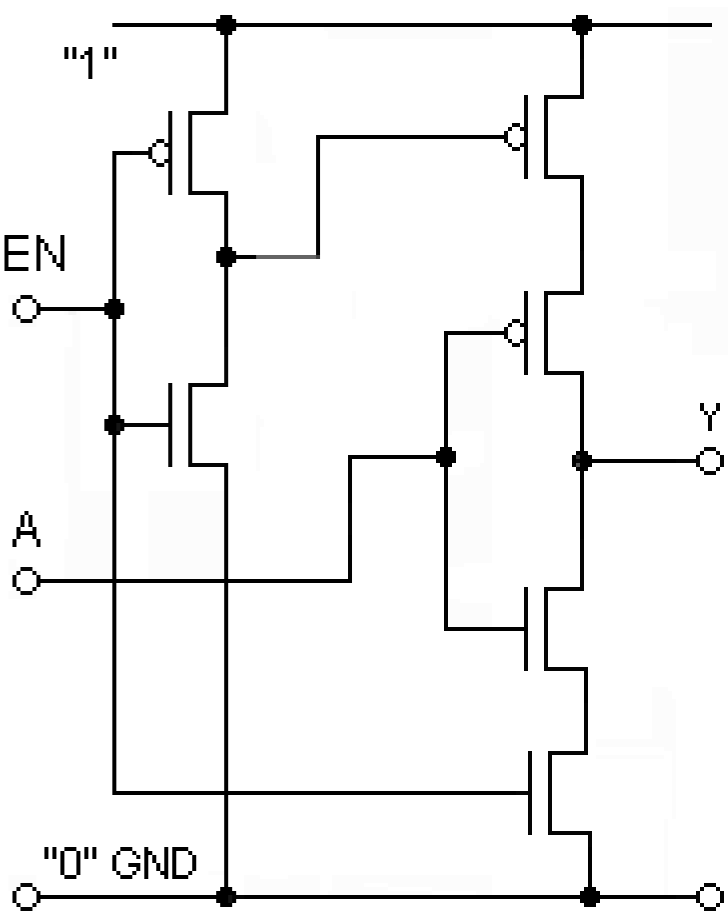
$$\overline{X} = A + B$$

7.3

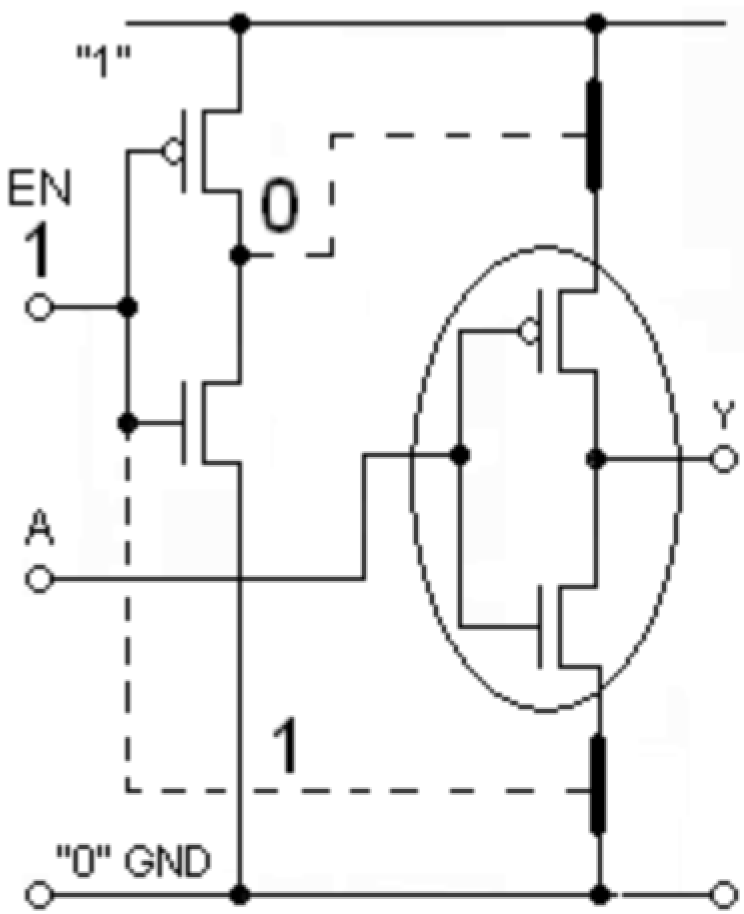


$$Y = \overline{A + B}$$

Ex. 7.4 CMOS-gate ?



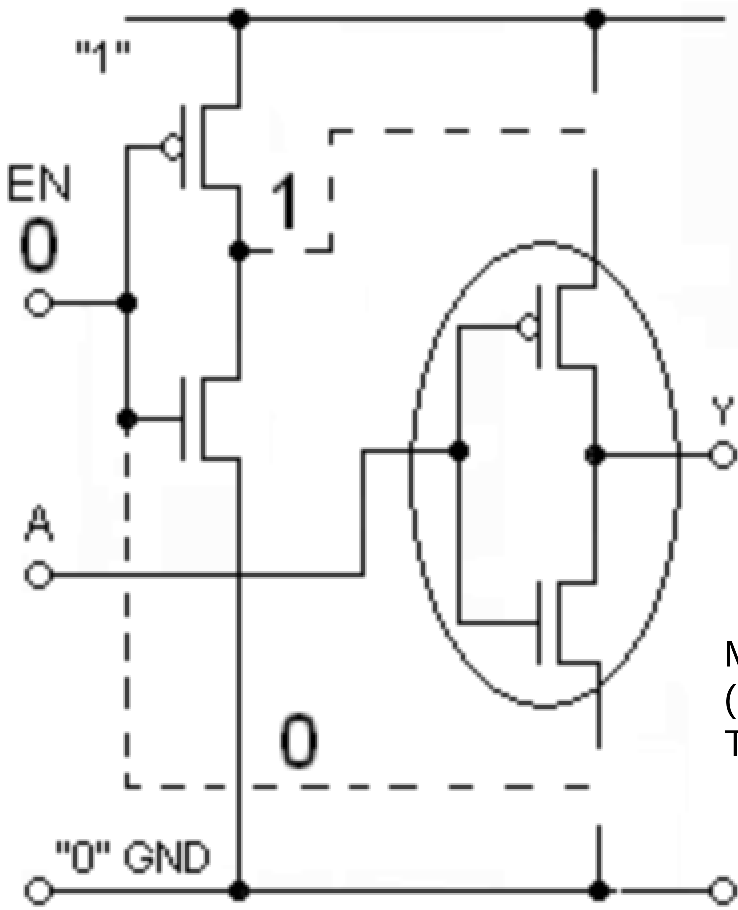
7.4 EN = 1



$$Y = \overline{A}$$

When EN = 1 we have an inverter.

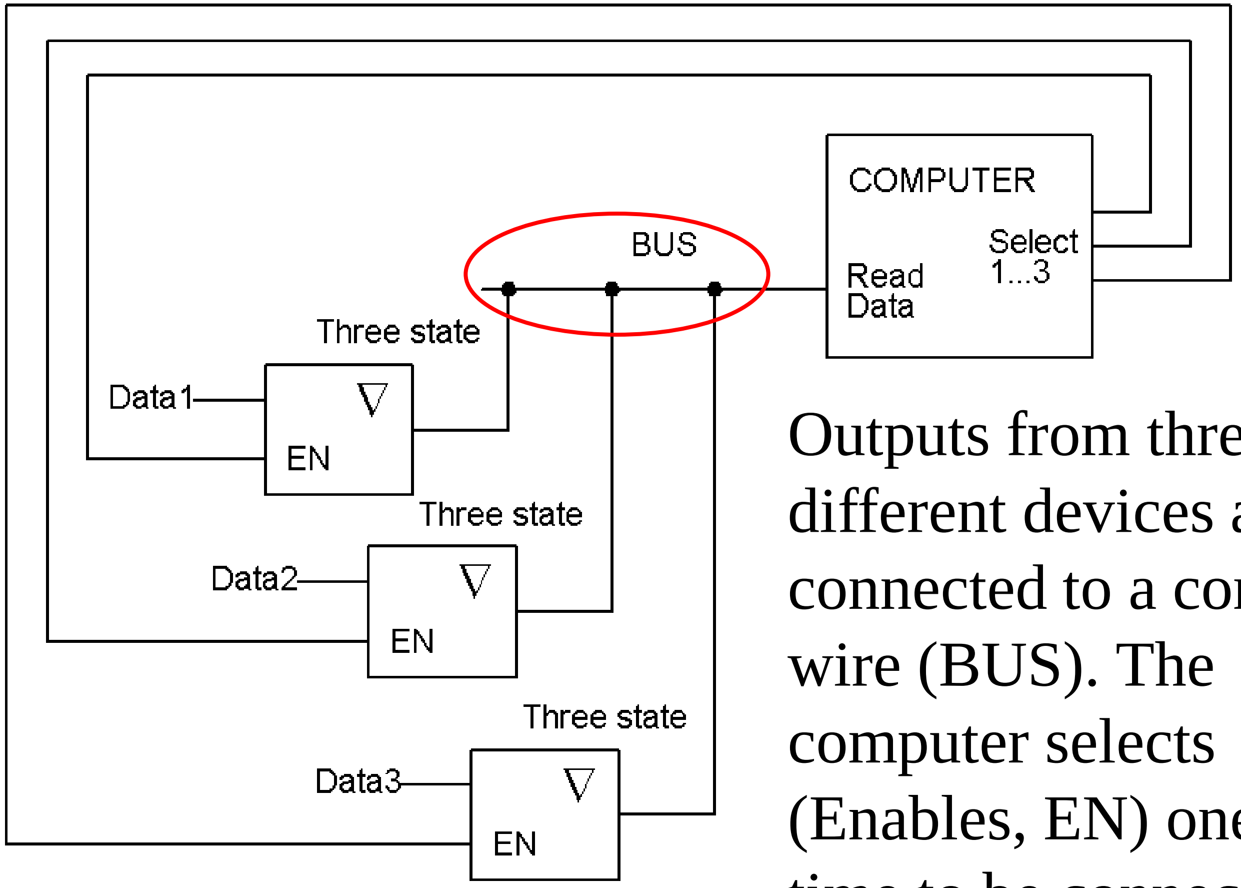
7.4 EN = 0



When EN = 0 the output is totaly disconnected from the supply voltage and ground. A can no longer influence the output value.

This is a third output state, "Three State".

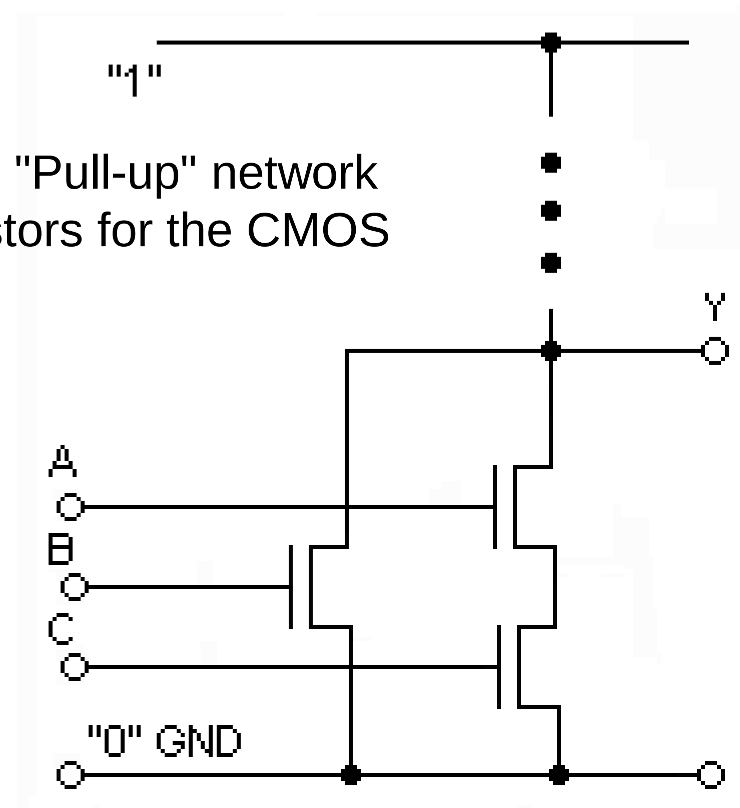
Many outputs could be connected to the same line ("bus"). One of the outputs at a time can be active. The otherare in their Three-state condition.



Outputs from three different devices are connected to a common wire (BUS). The computer selects (Enables, EN) one at a time to be connected to the bus. The other two remain disconnected, (Three state).

Ex. 7.5 CMOS-gate ?

Construct the "Pull-up" network with P-transistors for the CMOS gate.



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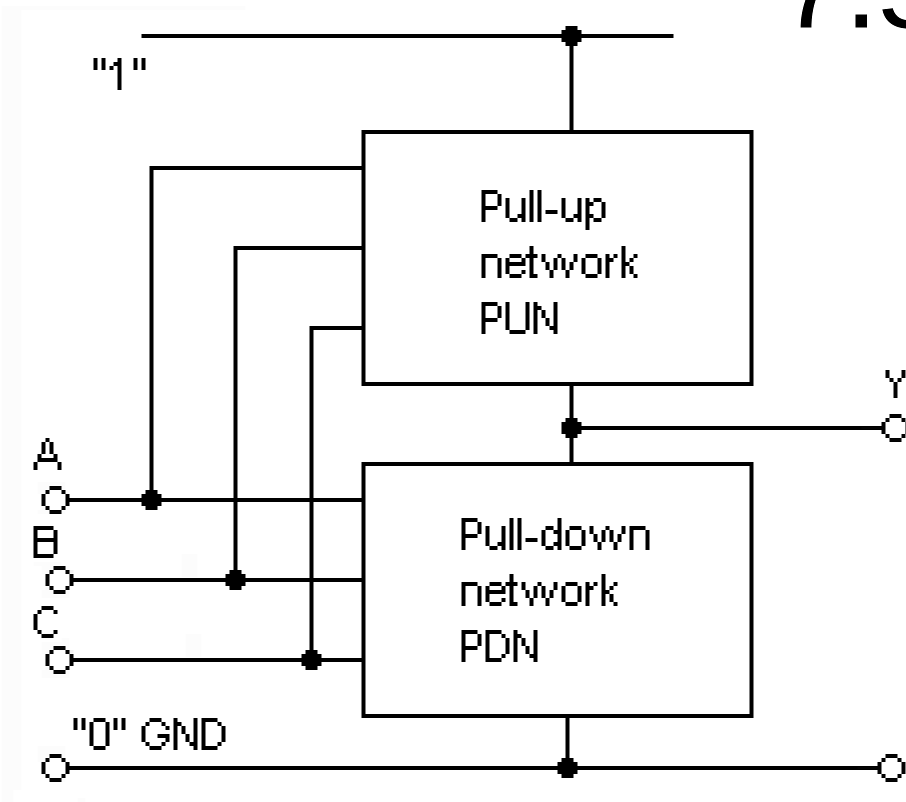
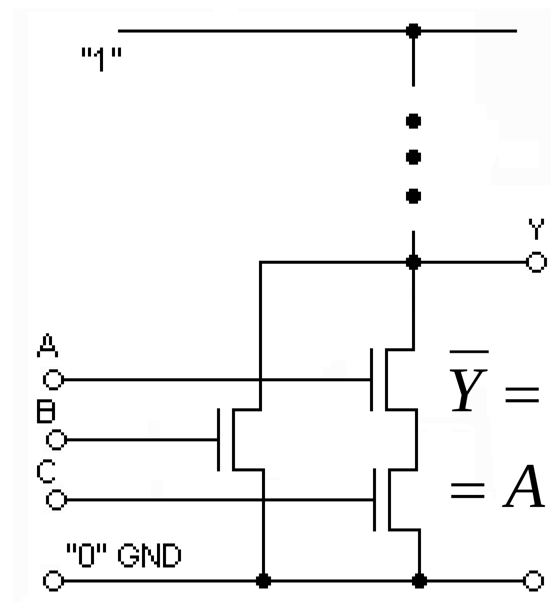
7.5





$$\overline{Y} = A \cdot C + B$$

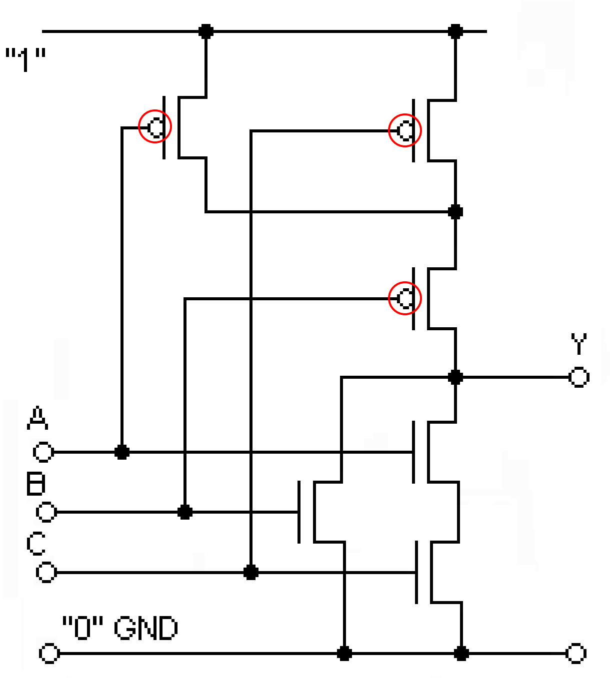
Pull-down circuit is the inverted function. The Pull-up circuit is the function noninverted:

$$\overline{Y} = A \cdot C + B \quad \Rightarrow \quad Y = \overline{A \cdot C + B} = \overline{A \cdot C} \cdot \overline{B} = \boxed{(\overline{A} + \overline{C}) \cdot \overline{B}}$$

7.5

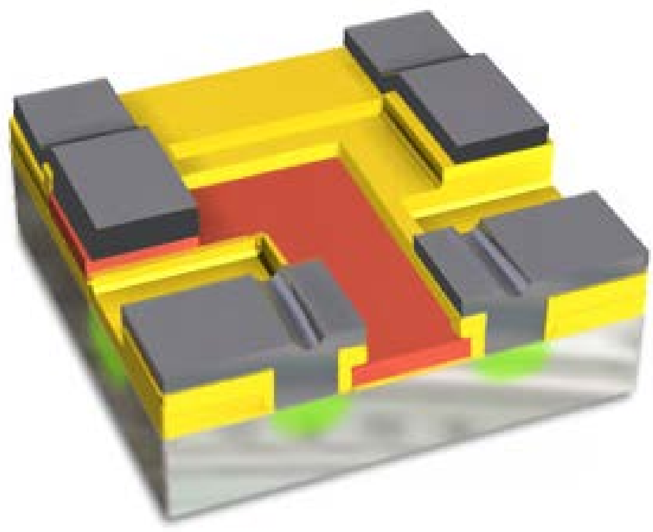
$$\overline{Y} = A \cdot C + B \quad \Rightarrow \quad Y = \overline{A \cdot C + B} = \overline{A \cdot C} \cdot \overline{B} = (\overline{A} + \overline{C}) \cdot \overline{B}$$

$$(\overline{A} + \overline{C})\overline{B}$$



The Pull-up net must therefore consist of A and C in parallel (+) then connected in series (·) with B. The use of PMOS transistors inverts the variables A, B and C.

A MOS-transistor ”on chip”



MOS-transistor step by step:

http://micro.magnet.fsu.edu/electromag/java/transistor/

Moore than 2.000.000.000 MOS-transistors/chip !

Pentium 4
has 50.000.000
MOS-transistors