COMPLEX NUMBERS

FUNDAMENTAL THEOREM OF ALGEBRA

* EVERY NON-ZERO, SINGLE VARIABLE n DEGREE POLYNOMIAL HAS EXACTLY n ROOTS.

$(x+1)^2 = -9$ (NO REAL SOLUTIONS)

$x+1 = \pm\sqrt{-9}$

$x+1 = \pm 3\sqrt{-1}$

$x = -1 \pm 3\sqrt{-1}$

* IMAGINARY UNIT

$$\boxed{j^2 = -1}$$

$\Rightarrow \quad x = -1 \pm 3j$

* COMPLEX NUMBER

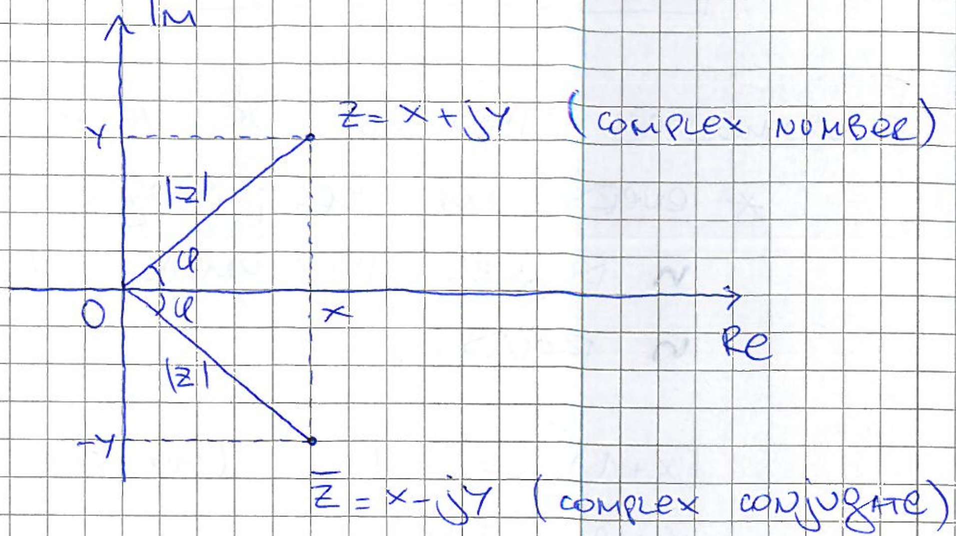
$$\boxed{z = x + jy}$$

$Re\{z\} = x$ – REAL PART, $x \in \mathbb{R}$

$Im\{z\} = y$ – IMAGINARY PART, $y \in \mathbb{R}$

CARDANO – 16^{TH} CENTURY

* COMPLEX PLANE



$X = |Z| \cos \varphi$

$Y = |Z| \sin \varphi$

* MAGNITUDE

$$|Z| = \sqrt{X^2 + Y^2}$$

* PHASE

$$\varphi = \tan^{-1}\left(\frac{Y}{X}\right)$$

$Z = X + jY$ - CARTESIAN FORM

$Z = |Z|(\cos \varphi + j \sin \varphi)$ - COMPLEX FORM

EULER'S FORMULA

* FOR ANY REAL NUMBER x,

$$\boxed{e^{jx} = \cos x + j \sin x}$$

WHERE e IS THE BASE OF THE NATURAL LOGARITHM

①

a) SHOW THAT $z.\bar{z} = |z|^2$

LET: $z = x + jy$

$\bar{z} = x - jy$

$\Rightarrow z.\bar{z} = (x+jy)(x-jy) = x^2 + y^2$ (1)

$\Rightarrow |z|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2 + y^2$ (2)

(1) AND (2) $\Rightarrow$ $\underline{\underline{z.\bar{z} = |z|^2}}$

b) SHOW THAT $\overline{z+w} = \bar{z} + \bar{w}$

LET: $z = a + jb$

$w = c + jd$

$\Rightarrow \overline{z+w} = \overline{(a+jb)+(c+jd)} = \overline{(a+c)+j(b+d)} =$

$= (a+c) - j(b+d)$ (1)

$\Rightarrow \bar{z} + \bar{w} = \overline{(a+jb)} + \overline{(c+jd)} = a - jb + c - jd =$

$= (a+c) - j(b+d)$ (2)

(1) AND (2) $\Rightarrow$ $\underline{\underline{\overline{z+w} = \bar{z} + \bar{w}}}$

C) SHOW THAT $\overline{z.w} = \bar{z}.\bar{w}$

LET: $z = a + jb$

$w = c + jd$

$\Rightarrow \overline{z.w} = \overline{(a+jb)(c+jd)} = \overline{(ac + ajd + cjb + \underset{-1}{j^2}bd)} =$

$= \overline{\underbrace{(ac-bd)}_{Re} + j\underbrace{(ad+cb)}_{Im}} = (ac-bd) - j(ad+cb)$ (1)

$\Rightarrow \quad \bar{z}\cdot\bar{w} = (a-jb)(c-jd) = ac - ajd - cjb + \underbrace{j^2}_{-1} bd =$

$= (ac-bd) - j(ad+cb) \qquad (2)$

(1) AND (2) $\Rightarrow \quad \overline{z\cdot w} = \bar{z}\cdot\bar{w}$

② WRITE EACH OF THE FOLLOWING IN CARTESIAN FORM $a+jb$.

a) $e^{-\frac{2\pi}{3}j}$

$\cos\left(-\frac{2\pi}{3}\right) + j\sin\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$

b) $12e^{\frac{\pi}{6}j}$

$12\left(\cos\left(\frac{\pi}{6}\right) + j\sin\left(\frac{\pi}{6}\right)\right) = 12\left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right) = 6\sqrt{3} + 6j$

c) $e^{\frac{2\pi}{3}j} + e^{\frac{4\pi}{3}j} + e^{\frac{6\pi}{3}j}$

$\cos\left(\frac{2\pi}{3}\right) + j\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + j\frac{\sqrt{3}}{2} \qquad (1)$

$\cos\left(\frac{4\pi}{3}\right) + j\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - j\frac{\sqrt{3}}{2} \qquad (2)$

$\cos(2\pi) + j\sin(2\pi) = 1 + 0\cdot j = 1 \qquad (3)$

$(1) + (2) + (3) = \cancel{-\frac{1}{2}} + \cancel{j\frac{\sqrt{3}}{2}} - \cancel{\frac{1}{2}} - \cancel{j\frac{\sqrt{3}}{2}} + \cancel{1} = 0$

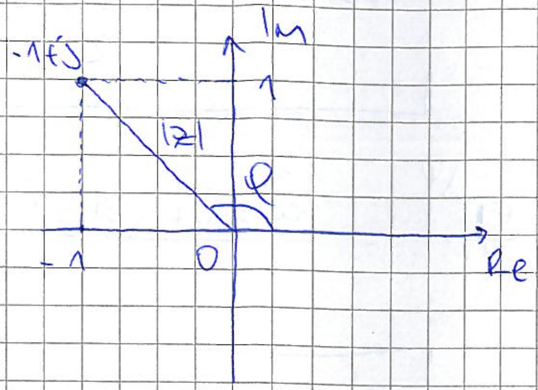
③ SHOW THAT $(-1+j)^7 = -8(1+j)$

POLAR FORM OF $-1+j$ (z)

$|z| = \sqrt{(-1)^2+1^2} = \sqrt{2}$

$\cos\varphi = \frac{x}{|z|} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$ (1)

$\sin\varphi = \frac{y}{|z|} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ (2)



(1) AND (2) $\Rightarrow \varphi = \frac{3\pi}{4}$ $(135°)$

$z = |z|(\cos\varphi + j\cdot\sin\varphi)$

$e^{j\varphi} = \cos\varphi + j\sin\varphi$

$\Rightarrow z = |z|e^{j\varphi} = \sqrt{2}e^{j\frac{3\pi}{4}}$

$\Rightarrow z^7 = \left(\sqrt{2}e^{j\frac{3\pi}{4}}\right)^7 = \sqrt{2}^7 e^{j\frac{21\pi}{4}} = \sqrt{2}^7 \cdot e^{j4\pi + j\frac{5\pi}{4}} =$

$= \sqrt{2}^7 e^{j4\pi} \cdot e^{j\frac{5\pi}{4}}$

$e^{j4\pi} = \underset{1}{\underset{\shortparallel}{\cos 4\pi}} + j\underset{0}{\underset{\shortparallel}{\sin 4\pi}} = 1$

$\Rightarrow z^7 = \sqrt{2}^7 \cdot e^{j\frac{5\pi}{4}} = \sqrt{2}^7\left(\underset{-\frac{\sqrt{2}}{2}}{\underset{\shortparallel}{\cos\frac{5\pi}{4}}} + j\underset{-\frac{\sqrt{2}}{2}}{\underset{\shortparallel}{\sin\frac{5\pi}{4}}}\right) =$

$= \sqrt{2}^7\left(-\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\right) = \sqrt{2}^7\left(-\frac{\sqrt{2}}{2}(1+j)\right) = -\frac{\sqrt{2}^8}{2}(1+j) =$

$= -\frac{16}{2}(1+j) = -8(1+j)$

$=$

④ SHOW THAT

a) $$\boxed{\cos\varphi = \frac{e^{j\varphi} + e^{-j\varphi}}{2}}$$

$$\frac{e^{j\varphi} + e^{-j\varphi}}{2} = \frac{\cos\varphi + \cancel{j\sin\varphi} + \cos\varphi - \cancel{j\sin\varphi}}{2} = \frac{2\cos\varphi}{2} = \cos\varphi$$

b) $$\boxed{\sin\varphi = \frac{e^{j\varphi} - e^{-j\varphi}}{2j}}$$

$$\frac{e^{j\varphi} - e^{-j\varphi}}{2j} = \frac{\cos\varphi + j\sin\varphi - (\cos\varphi - j\sin\varphi)}{2j} = \frac{\cancel{2j}\sin\varphi}{\cancel{2j}} = \sin\varphi$$

SAMPLING

REDUCTION OF A CONTINUOUS SIGNAL TO A DISCRETE SIGNAL

CONTINUOUS		DISCRETE
$x(t)$	$\longrightarrow$	$x(nT)$, $n \in \mathbb{Z}$,

$T = \frac{1}{f_s}$ - SAMPLING PERIOD (1 sec)

$f_s = \frac{1}{T}$ - SAMPLING FREQUENCY (SAMPLES PER SEC)

$\vartheta_0 = \frac{f_0}{f_s} = f_0 T$ - NORMALISED FREQUENCY

① SIGNAL $x(t) = \sin(2\pi f_0 t)$ WITH FREQUENCY $f_0 = 2000\,Hz$ IS SAMPLED WITH $f_S = 8000\,Hz$.

a) CALCULATE THE NORMALISED FREQUENCY

$$\nu_0 = \frac{f_0}{f_S} = \frac{2000}{8000} = \frac{1}{4}$$ //

b) WRITE AN EXPRESSION FOR THE SAMPLED SIGNAL $y(n)$.

$$y(n) = x(nT) = \sin(2\pi f_0 n T) = \sin(2\pi \nu_0 n) =$$

$$= \sin \frac{\pi}{2} n$$ //