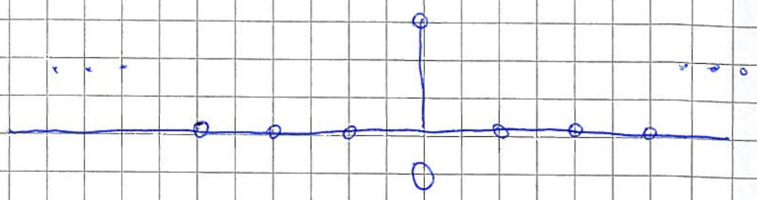
Z - TRANSFORM

* DISCRETE UNIT IMPULSE FUNCTION



$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

* IMPULSE RESPONSE OF LTI

  - $h(n)$ IS THE IMPULSE RESPONSE TO $\delta(n)$
  - $h(n)$ COMPLETELY DESCRIBES THE LTI SYSTEM

* CONVOLUTION

  - IN TIME DOMAIN

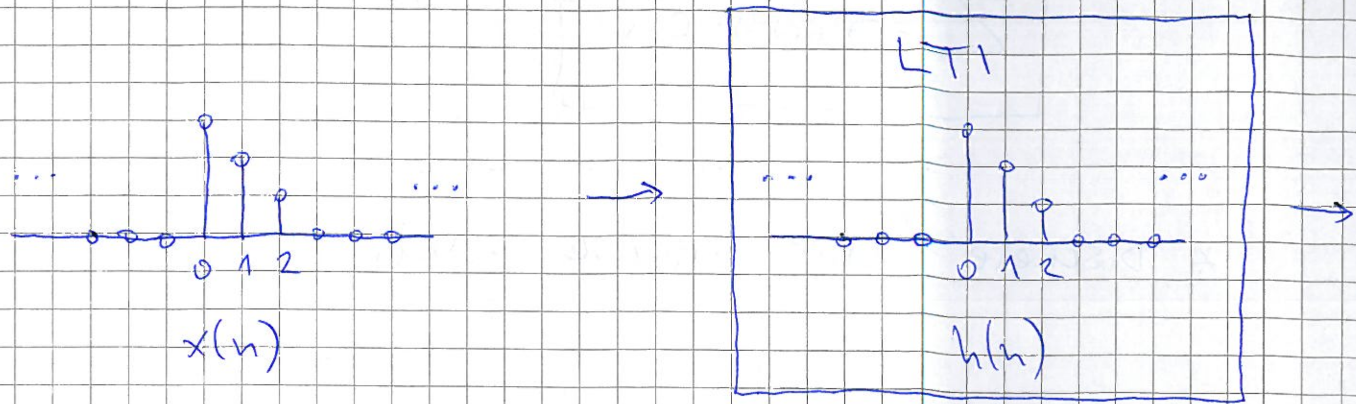
$$\boxed{\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \\ y(n) &= x(n) * h(n) \end{aligned}}$$

  - EXAMPLE

$x(n) = [3, 2, 1]$ - n

$h(n) = [3, 2, 1]$ - M

$y(n) = x(n) * h(n) = [9, 12, 10, 4, 1]$ - $n+m-1$



$$y(n) = x(n) * h(n)$$

* Z-TRANSFORM

$$z^n \longrightarrow \text{LTI} \longrightarrow \boxed{\sum_{k=-\infty}^{\infty} x(k)\,h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)\,h(k)}$$

COMMUTATIVITY

$$z^n \longrightarrow \text{LTI} \longrightarrow \sum_{k=-\infty}^{\infty} h(k)\,z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h(k)\,z^{-k}$$

$$H(z) = Z\{h(k)\} = \sum_{k=-\infty}^{\infty} h(k)\,z^{-k}$$ - TRANSFER FUNCTION

$$z^n \longrightarrow \text{LTI} \longrightarrow z^n H(z)$$

- CONVERTS DISCRETE-TIME DOMAIN SIGNAL INTO Z DOMAIN REPRESENTATION

$$\boxed{X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$$

$z = Ae^{jw} = A(\cos w + j\sin w)$ - COMPLEX NUMBER

A - MAGNITUDE OF z

j - IMAGINARY UNIT

w - PHASE

$$\boxed{Y(z) = H(z) . X(z)}$$

$Y(z)$ - Z-TRANSFORM OF $y(n)$

$X(z)$ - Z-TRANSFORM OF $x(n)$

$H(z)$ - Z-TRANSFORM OF $h(n)$

CONVOLUTION IN TIME DOMAIN BECOMES MULTIPLICATION IN Z DOMAIN.

- PROPERTY

$$y(n) = x(n) * h(n) \Leftrightarrow Y(z) = H(z) . X(z)$$

- PROPERTY

$$y(n) = y_1(n) + y_2(n) \Leftrightarrow Y(z) = Y_1(z) + Y_2(z)$$

- TRANSFORM PAIRS

$x(n)$	$X(z)$	ROC
$\delta(n)$	1	$\forall z$
$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert>1$
a^n	$\frac{z}{z-a}$	$\lvert z\rvert>1$

(1) FIND THE Z-TRANSFORM OF $x(n)=\delta(n)$

$$\delta(n)=\begin{cases}1, & n=0\\ 0, & n\neq 0\end{cases}$$

$$X(z)=\sum_{n=-\infty}^{\infty}\delta(n)z^{-n}\overset{n=0}{=\!=}1.z^{0}=1,\ \forall z$$

(2) FIND THE Z-TRANSFORM OF $x(n)=u(n)$

$$u(n)=\begin{cases}1, & n\geq 0\\ 0, & n<0\end{cases}$$

$$X(z)=\sum_{n=-\infty}^{\infty}u(n)z^{-n}\overset{n\geq 0}{=\!=}\sum_{n=0}^{\infty}z^{-n}=\sum_{n=0}^{\infty}(z^{-1})^{n}$$

$$\text{ROC: } |z^{-1}|<1$$

$$S=\sum_{n=0}^{\infty}(z^{-1})^{n}=1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\quad\left(\cdot\frac{1}{z}\right)$$

$$\frac{S}{z}=\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots$$

$$\Rightarrow S-\frac{S}{z}=1\Rightarrow S\left(1-\frac{1}{z}\right)=1\Rightarrow S=\frac{1}{1-z^{-1}},\ |z^{-1}|<1$$

③ FIND THE Z-TRANSFORM OF

$$x(n) = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \cos\left(\frac{\pi n}{4}\right) \cdot u(n)$$

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

LET,

$$a = \frac{1}{\sqrt{2}} \quad \text{AND} \quad \omega = \frac{\pi}{4}$$

THEN,

$$x(n) = a^n \cdot \cos(\omega n) \cdot u(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n \cos(\omega n) u(n) z^{-n}$$

$$X(z) \underset{=}{\overset{n\geq 0}{=}} \sum_{n=0}^{\infty} a^n \cos(\omega n) z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} \frac{a^n}{z^n}\left(\frac{e^{j\omega n} + e^{-j\omega n}}{2}\right)$$

$$X(z) = \frac{1}{2}\sum_{n=0}^{\infty}\left[\left(\frac{ae^{j\omega}}{z}\right)^n + \left(\frac{ae^{-j\omega}}{z}\right)^n\right]$$

$$X(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(\frac{ae^{j\omega}}{z}\right)^n + \sum_{n=0}^{\infty}\left(\frac{ae^{-j\omega}}{z}\right)^n\right]$$

$$\boxed{\text{FROM ex. ② } \quad \sum_{n=0}^{\infty} (z^{-1})^n = \frac{1}{1-z^{-1}}}$$

$$\text{ROC: } \left|\frac{ae^{\pm j\omega}}{z}\right| < 1$$

$$|ae^{\pm j\omega}| < |z| \Rightarrow |a| \cdot \underbrace{|e^{\pm j\omega}|}_{1} < |z| \Rightarrow |z| > |a| \Rightarrow |z| > 1/\sqrt{2}$$

$$\Rightarrow X(z) = \frac{1}{2}\left[\frac{1}{1-\frac{ae^{jw}}{z}} + \frac{1}{1-\frac{ae^{-jw}}{z}}\right]$$

$$X(z) = \frac{1}{2}\left(\frac{z}{z-ae^{jw}} + \frac{z}{z-ae^{-jw}}\right)$$

$$X(z) = \frac{z}{2}\left(\frac{z-ae^{-jw}+z-ae^{jw}}{(z-ae^{jw})(z-ae^{-jw})}\right)$$

$$X(z) = \frac{z}{2}\left(\frac{2z-a(e^{jw}+e^{-jw})}{z^2-aze^{-jw}-aze^{jw}+a^2}\right)$$

$$X(z) = \frac{z}{\cancel{2}}\left(\frac{\cancel{2}z-\cancel{2}a\cos w}{z^2-2az\cos w+a^2}\right)$$

$$X(z) = \frac{z^2-az\cos w}{z^2-2az\cos w+a^2},$$

$$\cos w = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$a = \frac{1}{\sqrt{2}}$$

$$\Rightarrow X(z) = \frac{z^2-\frac{1}{\cancel{\sqrt{2}}}\cdot\frac{\cancel{\sqrt{2}}}{2}\cdot z}{z^2-\frac{\cancel{2}}{\cancel{\sqrt{2}}}\cdot\frac{\sqrt{2}}{\cancel{2}}\cdot z+\frac{1}{2}}$$

$$X(z) = \frac{z^2-\frac{z}{2}}{z^2-z+\frac{1}{2}}, \quad |z| > \frac{1}{\sqrt{2}}$$

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Poles and Zeros

$$y(n) = 2y(n-1) - y(n-2) + x(n-1) + x(n-2)$$

$$y(n) - 2y(n-1) + y(n-2) = x(n-1) + x(n-2)$$

$$Z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$Z\{y(n-1)\} = \sum_{n=-\infty}^{\infty} y(n-1) z^{-n}$$

let,

$$n-1 = k \Rightarrow n = k+1$$

$$\Rightarrow Z\{y(n-1)\} = \sum_{k=-\infty}^{\infty} y(k) z^{-k-1}$$

$$\Rightarrow Z\{y(n-1)\} = z^{-1} \underbrace{\sum_{k=-\infty}^{\infty} y(k) z^{-k}}_{= Z\{y(n)\}}$$

$$\Rightarrow Y(z)(1 - 2z^{-1} + z^{-2}) = X(z)(z^{-1} + z^{-2})$$

* TRANSFER FUNCTION

$$\boxed{H(z) = \frac{Y(z)}{X(z)}} = \frac{z^{-1} + z^{-2}}{1 - 2z^{-1} + z^{-2}}$$

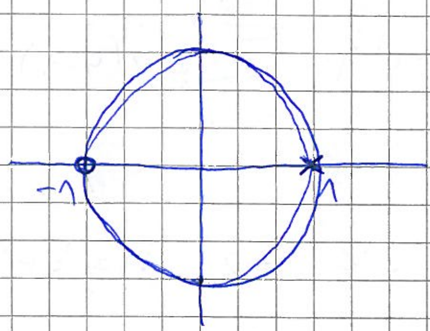
$$H(z) = \frac{z+1}{z^2 - 2z + 1}$$

- THE ROOTS OF THE NUMERATOR ARE THE **ZEROS**

$$z+1=0 \Rightarrow z=-1 \quad (o)$$

- THE ROOTS OF THE DENOMINATOR ARE THE **POLES**

$$(z-1)^2=0 \Rightarrow z_{1,2}=1 \quad (\times)$$



④ FIND THE ZEROS OF THE TRANSFER FUNCTION OF

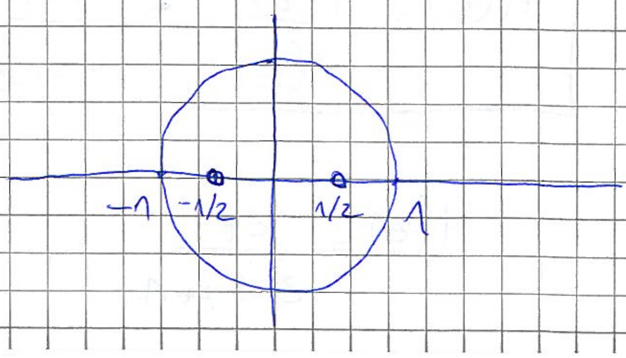
$$y(n)=x(n)-\frac{1}{4}x(n-2)$$

$$Y(z)=X(z)\left(1-\frac{1}{4}z^{-2}\right)$$

$$H(z)=\frac{Y(z)}{X(z)}=1-\frac{z^{-2}}{4}$$

THE ROOTS ARE,

$$1-\frac{z^{-2}}{4}=0 \Rightarrow z^{-2}=4 \Rightarrow z_{1,2}=\pm\frac{1}{2}$$



(5) DETERMINE THE SIGNAL $x(n)$ WHEN,

$$X(z) = \frac{z^2}{z^2 - z + 1}, \quad |z| > 1$$

- USE INVERSE Z-TRANSFORM ?
- USE TRANSFER PAIR

$$\boxed{\frac{z}{z-a} \longleftrightarrow a^n}$$

- PARTIAL FRACTIONS

$$\frac{z}{(z-p_1)(z-p_2)} = \frac{A}{z-p_1} + \frac{B}{z-p_2} \quad (\cdot z)$$

$$\frac{z^2}{\underbrace{(z-p_1)(z-p_2)}_{?}} = \frac{Az}{z-p_1} + \frac{Bz}{z-p_2}$$

- COMPLETING THE SQUARE

$$z^2 - z + 1 = 0$$

$$\left(z - \frac{1}{2}\right)^2 + 1 - \frac{1}{4} = 0$$

$$\left(z - \frac{1}{2}\right)^2 = -\frac{3}{4}$$

$$\Rightarrow z = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

$$\left.\begin{array}{l} \cos\omega = \frac{1}{2} \\ \sin\omega = \frac{\sqrt{3}}{2} \end{array}\right\} \omega = \frac{\pi}{3} \Rightarrow \boxed{z = e^{\pm j\frac{\pi}{3}}}$$

$$\Rightarrow \quad X(z) = \frac{z^2}{\left(z - e^{j\frac{\pi}{3}}\right)\left(pz - e^{-j\frac{\pi}{3}}\right)}$$

$$X(z) = \frac{Az}{z - e^{j\frac{\pi}{3}}} + \frac{Bz}{z - e^{-j\frac{\pi}{3}}}$$

$$\Rightarrow \quad z^2 = Az\left(z - e^{-j\frac{\pi}{3}}\right) + Bz\left(z - e^{j\frac{\pi}{3}}\right)$$

$$z^2 = Az^2 - Aze^{-j\frac{\pi}{3}} + Bz^2 - Bze^{j\frac{\pi}{3}}$$

$$z^2 = z^2(A+B) + z\left(-Ae^{-j\frac{\pi}{3}} - Be^{j\frac{\pi}{3}}\right)$$

COEFFICIENTS FOR z^2: $\quad 1 = A + B \quad (1)$

COEFFICIENTS FOR z^1: $\quad 0 = -Ae^{-j\frac{\pi}{3}} - Be^{j\frac{\pi}{3}}$

$$Be^{j\frac{\pi}{3}} = -Ae^{-j\frac{\pi}{3}}$$

$$B = -\frac{Ae^{-j\frac{\pi}{3}}}{e^{j\frac{\pi}{3}}} \quad (2)$$

COEFFICIENTS FOR z^0: $\quad 0 = 0$

THEN,

$$(2) \rightarrow (1) \qquad 1 = A - A\frac{e^{-j\frac{\pi}{3}}}{e^{j\frac{\pi}{3}}}$$

$$A\left(1 - \frac{e^{-j\frac{\pi}{3}}}{e^{j\frac{\pi}{3}}}\right) = 1$$

$$\Rightarrow \quad A = \frac{1}{1 - \frac{e^{-j\frac{\pi}{3}}}{e^{j\frac{\pi}{3}}}} = \boxed{\frac{e^{j\frac{\pi}{3}}}{e^{j\frac{\pi}{3}} - e^{-j\frac{\pi}{3}}}}$$

① $\longrightarrow$ $B = 1 - A = \dfrac{\cancel{e^{j\frac{\pi}{3}}} - e^{-j\frac{\pi}{3}} - \cancel{e^{j\frac{\pi}{3}}}}{e^{j\frac{\pi}{3}} - e^{-j\frac{\pi}{3}}} = \boxed{\dfrac{e^{-j\frac{\pi}{3}}}{e^{j\frac{\pi}{3}} - e^{-j\frac{\pi}{3}}}}$

$$X(z) = \frac{Az}{z - e^{j\frac{\pi}{3}}} + \frac{Bz}{z - e^{-j\frac{\pi}{3}}}$$

$$\frac{z}{z-a} \longleftrightarrow a^n$$

$$\Rightarrow \boxed{x(n) = A \cdot e^{\frac{j\pi n}{3}} + B \cdot e^{-\frac{j\pi n}{3}}}$$

$$e^{\pm j\frac{\pi}{3}} = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

$$\Rightarrow A = \frac{\frac{1}{2} + j\frac{\sqrt{3}}{2}}{\cancel{\frac{1}{2}} + j\frac{\sqrt{3}}{2} - \cancel{\frac{1}{2}} + j\frac{\sqrt{3}}{2}} = \frac{1 + j\sqrt{3}}{j\sqrt{3}} = \frac{1}{j\sqrt{3}} + 1 = \boxed{1 - \frac{j}{\sqrt{3}}}$$

$$B = \frac{\frac{1}{2} - j\frac{\sqrt{3}}{2}}{\cancel{\frac{1}{2}} - j\frac{\sqrt{3}}{2} - \cancel{\frac{1}{2}} - j\frac{\sqrt{3}}{2}} = \frac{1 - j\sqrt{3}}{-j\sqrt{3}} = 1 - \frac{1}{j\sqrt{3}} = \boxed{1 + \frac{j}{\sqrt{3}}}$$

$$\Rightarrow x(n) = \left(1 - \frac{j}{\sqrt{3}}\right) e^{\frac{j\pi n}{3}} + \left(1 + \frac{j}{\sqrt{3}}\right) e^{-\frac{j\pi n}{3}}$$

$$x(n) = e^{\frac{j\pi n}{3}} - \frac{j e^{\frac{j\pi n}{3}}}{\sqrt{3}} + e^{-\frac{j\pi n}{3}} + \frac{j e^{-\frac{j\pi n}{3}}}{\sqrt{3}}$$

$$x(n) = \underbrace{\left(e^{\frac{j\pi n}{3}} + e^{-\frac{j\pi n}{3}}\right)}_{2\cos\left(\frac{\pi n}{3}\right)} - \frac{j}{\sqrt{3}} \underbrace{\left(e^{\frac{j\pi n}{3}} - e^{-\frac{j\pi n}{3}}\right)}_{2j\sin\left(\frac{\pi n}{3}\right)}$$

$$\Rightarrow x(n) = 2\cos\left(\frac{\pi n}{3}\right) + \frac{2}{\sqrt{3}} \sin\left(\frac{\pi n}{3}\right) \quad //$$