P-PARTITIONS AND RELATED POLYNOMIALS

1. P-partitions

A *labelled poset* is a poset P whose ground set is a subset of the integers, see Fig. 1. We will denote by $<$ the usual order on the integers and $<_P$ the partial order defined by P. A function $\sigma : P \to \mathbb{Z}_+ = \{1, 2, 3, \dots\}$ is a *P-partition* if

(1) σ is order preserving, i.e., $\sigma(x) \le \sigma(y)$ whenever $x \le_P y$;
(2) If $x <_P y$ in P and $x > y$, then $\sigma(x) < \sigma(y)$.

Let $\mathcal{A}(P)$ be the set of all P-partitions and if $n \in \mathbb{Z}_+$ let

$$\mathcal{A}_n(P) = \{\sigma \in \mathcal{A}(P) : \sigma(x) \le n \text{ for all } x \in P\}.$$

The *order polynomial* of P is defined for positive integers n as

$$\Omega(P, n) = |\mathcal{A}_n(P)|.$$

A *linear extension* of P is a total order, L, on the same ground set as P satisfying

$$x <_P y \Longrightarrow x <_L y.$$

Hence, if L is a linear extension of P we may list the elements of P as $x_1 <_L x_2 <_L \cdots <_L x_p$, where p is the number of elements in P. The *Jordan–Hölder set* of P is the following set of permutations of P coming from linear extensions of P:

$$\mathcal{L}(P) = \{x_1 x_2 \cdots x_n : x_1 <_L \cdots <_L x_p \text{ is a linear extension of } P\}.$$

Proposition 1.1. *Let P be a finite labelled poset. Then $\mathcal{A}(P)$ is the disjoint union*

$$\mathcal{A}(P) = \bigsqcup_L \mathcal{A}(L), \tag{1}$$

where the union is over all linear extensions of P.

Proof. If L and L' are two different linear extensions of P then there are $x, y \in P$ for which x comes before y in L and y comes before x in L'. Suppose $x > y$ as integers. Then $\sigma(x) < \sigma(y)$ for all $\sigma \in \mathcal{A}(L)$ and $\sigma'(x) \ge \sigma'(y)$ for all $\sigma' \in \mathcal{A}(L')$.

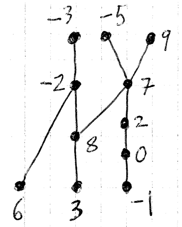

Figure 1. A labelled poset.

Clearly $\mathcal{A}(P) \supseteq \sqcup_L \mathcal{A}(L)$. I remains to prove the converse inclusion. If P is a total order then there is nothing to prove. Suppose that $x, y \in P$ are unrelated and $x > y$. Let P_{xy} be the poset with the same ground set as P defined by

$$z \leq_{P_{xy}} w \text{ if and only if } z \leq_P w, \text{ or } z \leq_P x \text{ and } y \leq_P w.$$

Clearly $\mathcal{A}(P) \supseteq \mathcal{A}(P_{xy}) \cup \mathcal{A}(P_{yx})$. If $\sigma \in \mathcal{A}(P)$, then either $\sigma(x) < \sigma(y)$ or $\sigma(x) \geq \sigma(y)$. In the first case $\sigma \in \mathcal{A}(P_{xy})$ and in the other $\sigma \in \mathcal{A}(P_{xy})$, and hence the union above is disjoint. Thus $\mathcal{A}(P) = \mathcal{A}(P_{xy}) \sqcup \mathcal{A}(P_{yx})$. If P_{xy} or P_{yx} is not a total order we may continue this process until we get a disjoint union over linear extensions. □

The following corollary proves that $\Omega(P, n)$ is indeed a polynomial, and may thus be defined for negative integers or for any complex number n.

Corollary 1.2. *Let P be a finite poset of cardinality p. Then*

$$\Omega(P, n) = \sum_{\pi \in \mathcal{L}(P)} \binom{n + p - \mathrm{des}(\pi) - 1}{p}.$$

In particular $\Omega(P, n)$ is a polynomial in n of degree p, with leading coefficient equal to $e(P)/p!$, where $e(P)$ is the number of linear extensions of P.

Proof. By Proposition 1.1

$$\Omega(P, n) = \sum_{\pi \in \mathcal{L}(P)} \Omega(P_\pi, n),$$

where P_π is the total order $\pi_1 <_L \cdots <_L \pi_p$. Now, $\Omega(P_\pi, n)$ is the number of sequences

$$1 \leq \sigma_1 \leq \cdots \leq \sigma_p \leq n, \text{ where } \sigma_j < \sigma_{j+1} \text{ whenever } \pi_j > \pi_{j+1}.$$

Let $\tau_j = \sigma_j + |\{i : i < j \text{ and } \pi_i < \pi_{i+1}\}|$. Then we see that $\Omega(P_\pi, n)$ is the number of sequences

$$1 \leq \tau_1 < \tau_2 < \cdots < \tau_p \leq n + p - 1 - \mathrm{des}(\pi),$$

which proves the corollary. □

Suppose that P is a labelled poset. Let $-P$ be the poset on $\{-x : x \in P\}$, with partial order defined by $x \leq_{-P} y$ if and only if $-x \leq_P -y$.

Theorem 1.3 (Reciprocity)**.** *Let P be a labelled poset of cardinality p and let n be a positive integer. Then*

$$\Omega(P, -n) = (-1)^p \Omega(-P, n).$$

Proof. For $\pi \in \mathcal{L}(P)$ let $-\pi$ be defined $-\pi = (-\pi_1)(-\pi_2) \cdots (-\pi_d)$. Then $\pi \in \mathcal{L}(P)$ if and only if $-\pi \in \mathcal{L}(-P)$. Since $\mathrm{des}(\pi) = p - 1 - \mathrm{des}(-\pi)$ we have by Corollary

1.2

$$\begin{aligned}\Omega(-P,n) &= \sum_{\pi\in\mathcal{L}(P)} \binom{n+p-\mathrm{des}(-\pi)-1}{p} \\ &= \sum_{\pi\in\mathcal{L}(P)} \binom{n+\mathrm{des}(\pi)}{p} \\ &= \sum_{\pi\in\mathcal{L}(P)} (-1)^p \binom{-n+p-\mathrm{des}(\pi)-1}{p} \\ &= (-1)^p\Omega(P,-n)\end{aligned}$$

□

The P-*Eulerian polynomial* is defined by

$$W(P,t) = \sum_{\pi\in\mathcal{L}(P)} t^{\mathrm{des}(\pi)+1}.$$

Theorem 1.4. *Let P be a finite labelled poset of cardinality p. Then*

$$\sum_{n\geq 1} \Omega(P,n)t^n = \frac{W(P,t)}{(1-t)^{p+1}}.$$

Proof. In view of Corollary 1.2 it suffices to prove that

$$\sum_{n\geq 1} \binom{n+p-k}{p} t^n = \frac{t^k}{(1-t)^{p+1}},$$

for all $k \geq 1$. This is a consequence of the binomial theorem for negative exponents. □

In terms of P-Eulerian polynomials, Theorem 1.3 translates as

$$W(-P,t) = t^p W(P,1/t) \tag{2}$$

Corollary 1.5. *Let d be a positive integer. Then*

$$\sum_{n\geq 1} n^d t^n = \frac{\sum_{\pi\in\mathfrak{S}_d} t^{\mathrm{des}(\pi)+1}}{(1-t)^{d+1}}.$$

Proof. Let A_d be the anti chain on $\{1,\ldots,d\}$. Then $\Omega(A_d,n) = n^d$ and $\mathcal{L}(A_d) = \mathfrak{S}_d$. Apply Theorem 1.4. □

2. Sign-graded posets

Let P be a finite labelled poset, and let $E(P)$ be the set of covering relations (Hasse-diagram) of P. Define a function $\epsilon_P : E(P) \to \{-1,1\}$ as follows.

$$\epsilon_P(x,y) = \begin{cases} 1 & \text{if } x<y, \\ -1 & \text{if } x>y. \end{cases}$$

Remark 1. Note that a function $\sigma : P \to \mathbb{Z}_+$ is a P-partition if and only if

$$\sigma(y) - \sigma(x) \geq -\epsilon_P(x,y)/2$$

for all $(x,y) \in E(P)$.

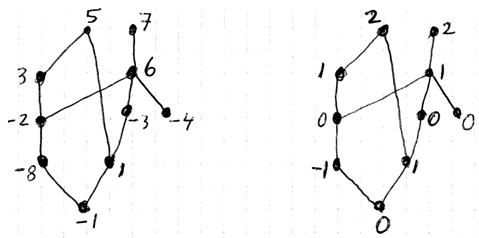

FIGURE 2. A sign graded poset poset and its rank function.

The labelled poset P is *sign-graded* of rank $r(P)$ if

$$\sum_{k=1}^{m} \epsilon_P(x_{k-1}, x_k) = r(P)$$

for all maximal chains $x_0 < x_1 < \cdots < x_m$ in P. If P is sign graded we define a *rank function* $\rho_P : P \to \mathbb{Z}$ by

$$\rho_P(x) = \sum_{k=1}^{\ell} \epsilon_P(x_{k-1}, x_k),$$

where $x_0 < x_1 < \cdots < x_\ell = x$ is any unrefinable chain from a bottom element to x. Note that $\rho_P(y) - \rho_P(x) = \epsilon_P(x, y)$ if y covers x. Note also that

$$\rho_P(x) \equiv \ell \mod 2. \tag{3}$$

Remark 2. A labelled poset, P, is called *natural* if $x \leq_P y$ implies $x \leq y$. Hence for natural labelled posets $\epsilon_P(x, y) = 1$ for all (x, y). Thus a natural labelled poset is sign graded if and only it is graded in the usual sense.

Theorem 2.1. *Let P and Q be sign graded posets of rank $r(P)$ and $r(Q)$, respectively. If P and Q are isomorphic as posets, then*

$$\Omega(P, n) = \Omega\left(Q, n + \frac{r(P) - r(Q)}{2}\right).$$

Proof. Suppose that $x \mapsto x'$ defines an isomorphism between P and Q. By (3) it follows that $\rho_P(x) \equiv \rho_Q(x') \mod 2$ for all $x \in P$. Define a function $\xi : \mathcal{A}(P) \to \mathbb{Z}^Q$ by

$$\xi(\sigma)(x') = \sigma(x) + (\rho_P(x) - \rho_Q(x'))/2.$$

We will show that ξ is a bijection between $\mathcal{A}_n(P)$ and $\mathcal{A}_{n+\mu}(Q)$, where $\mu = (r(P) - r(Q))/2$.

Suppose that y' covers x' in Q. Then by Remark 1

$$\begin{aligned}
\xi(\sigma)(y') - \xi(\sigma)(x') &= \sigma(y) - \sigma(x) + (\rho_P(y) - \rho_Q(y'))/2 - (\rho_P(x) - \rho_Q(x'))/2 \\
&= \sigma(y) - \sigma(x) + (\rho_P(y) - \rho_P(x))/2 - (\rho_Q(y') - \rho_Q(x'))/2 \\
&= \sigma(y) - \sigma(x) + \epsilon_P(x, y)/2 - \epsilon_Q(x', y')/2 \\
&\geq -\epsilon_Q(x', y')/2.
\end{aligned}$$

In particular, $\xi(\sigma)$ is order preserving. Hence $\xi(\sigma) \in \mathcal{A}(Q)$ provided that $\xi(\sigma)(x') \geq 1$ for all minimal elements $x' \in Q$. However, $\xi(\sigma)(x') = \sigma(x) \geq 1$ if x is minimal.

Moreover, if x is maximal then $\xi(\sigma)(x') = \sigma(x) + \mu$. Thus $\xi : \mathcal{A}_n(P) \to \mathcal{A}_{n+\mu}(Q)$. Clearly ξ is invertible with inverse given by

$$\xi^{-1}(\sigma)(x) = \sigma(x') + (\rho_Q(x') - \rho_P(x))/2.$$

□

Combining Theorems 1.3 and 2.1 we obtain:

Corollary 2.2. *Let P be a sign graded posets of rank r. Then*

$$\Omega(P, n) = (-1)^p \Omega(P, -n - r).$$

Proposition 2.3. *Let P and Q be sign graded posets of rank $r(P)$ and $r(Q)$, respectively. If P and Q are isomorphic as posets, then*

$$W(Q, t) = t^{(r(P)-r(Q))/2} W(P, t).$$

Proof. By the proof of Theorem 1.4 we see that if

$$\Omega(P, n) = \sum_{k \geq 1} w_k(P) \binom{n + p - k}{p},$$

then

$$W(P, t) = \sum_{k \geq 0} w_k(P) t^k.$$

Let $\mu = (r(P) - r(Q))/2$. By Theorem 2.1

$$\begin{aligned} \Omega(P, n) &= \Omega(Q, n + \mu) \\ &= \sum_k w_k(Q) \binom{n + \mu + p - k}{p} \\ &= \sum_j w_{j+\mu}(Q) \binom{n + p - j}{p}. \end{aligned}$$

Hence $w_j(P) = w_{j+\mu}(Q)$ for all j and the proposition follows. □

Combining (2) and Corollary 2.3 we see that the P-Eulerian polynomial of a sign-graded poset is palindromic:

$$t^{p-r(P)} W(P, 1/t) = W(P, t).$$

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