MATEMATISKA INSTITUTIONEN
KTH
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HT2014

Third set of hand-in-problems for SF 2741, Enumerative Combinatorics.

The problems are due **2015-01-12**. You may discuss the problems with other students in the class, but with no one else. You may not copy solutions you have found elsewhere. If you discuss with other student(s) in the class you should mention the name(s) for each problem. Each and every one should write down your own solution in your own words.

Write your name in the upper right corner of each paper. Staple in the upper left corner.

Maximal credit will be given only to complete and clear solutions.
Problem 2 is worth 20 points and the other problems 10 points each.

1. Let m and n be two positive integers. Let $P = P_{m,n}$ be the labelled poset which is a disjoint union of two chains $1 <_P 2 <_P \cdots <_P m$ and $m+1 <_P 2 <_P \cdots <_P m+n$.

   (a). Compute the order polynomial $\Omega(P_{m,n}, x)$, and the P-Eulerian polynomial $W(P_{m,n}, x)$.

   (b). Let $Q = Q_{m,n}$ be the labelled poset obtained from $P_{m,n}$ by adding the relation $m+1 <_Q m$. Compute $W(Q_{m,n}, x)$.

2. Let P be a finite naturally labelled poset.

   (a). Prove that if x and y are independent variables, then

   $$\Omega(P, x+y) = \sum_{I \in J(P)} \Omega(I, x)\Omega(P \setminus I, y),$$

   (We define $\Omega(\emptyset, x) = 1$). *Hint:* It suffices to prove it for positive integers x, y, since both sides of the equation are polynomials.

   (b). For $x \in \mathbb{C}$, let ξ_x the element of the incidence algebra $I(J(P), \mathbb{C})$ defined by $\xi_x(I, J) = \Omega(J \setminus I, x)$, where J/I is the induced sub-poset of P, and note that $\xi_0 = \delta$ and $\xi_1 = \zeta$. Prove $\xi_x \xi_y = \xi_{x+y}$ for all $x, y \in \mathbb{C}$.

   (c). Deduce that the Möbius function of $J(P)$ is

   $$\mu(I, J) = \xi_{-1}(I, J) = \Omega(J \setminus I, -1) = \begin{cases} (-1)^{|J \setminus I|} & \text{if } J \setminus I \text{ is an antichain,} \\ 0 & \text{otherwise.} \end{cases}$$

   Hint: Use reciprocity to evaluate $\Omega(J \setminus I, -1)$.

   (d). Prove the recursion

   $$\Omega(P, x) = \sum_{S \subseteq M(P)} (-1)^{|S|} \Omega(P \setminus S, x+1),$$

   where $M(P)$ is the set of minimal elements of P.

3. Problem 3.89 of the book.

4. Problem 3.114 a) and b) of the book. Note that the formula in a) is wrong in the book. It should be $\chi_{\mathcal{A}}(x) = \prod_{i=1}^{n}(x - i)$.

5. We say that π contains the pattern $1\overline{23}$ if it contains the pattern 123 and the numbers corresponding to "2" and "3" are adjacent in the wordform of π. We generalize further by for a set X of patterns letting $U_n(X)$ be the permutations of length n that avoids all the patterns in X. Prove that $|U_n(\{1\overline{23}, 2\overline{13}\})|$ is equal to the number of involutions in S_n (i.e. π s.t. $\pi^2 =$id).

6. The *cross polytope* C_n is a simplicial complex on the set of vertices $V = x_1, x_2, ..., x_n, y_1, ..., y_n$, where a subset $F \subset V$ is in C_n, if F does not contain $\{x_i, y_i\}$ for any $i = 1, ..., n$.

   a) Compute $\tilde{\chi}(C_n)$ directly from the definitionen by $\tilde{\chi}$ and C_n.

   b) Compute $\tilde{\chi}(C_n)$ using Proposition 3.8.6 (You have to find a poset with C_n as order complex).

Lycka till!
Petter och Svante