Pattern avoidance and Catalan numbers

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I will use EC1 to refer to Section 1.5 in the second edition of Enumerative Combinatorics vol 1, by Richard Stanley.

1 Catalan numbers in brief

The Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n} = \binom{2n}{n} - \binom{2n}{n-1} = 1, 1, 2, 5, 14, 42, 132, \ldots$ enumerates a large number of combinatorial objects.
Exercise 1 $[1^+]$**:** Check that the three formulas mentioned are indeed equal.

A recursion for the Catalan numbers is

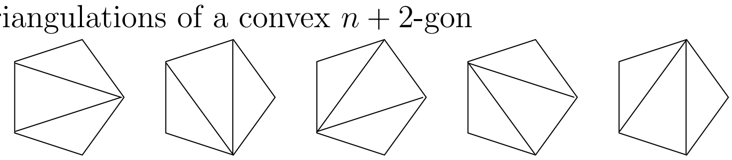
$$c_n = \sum_{i=0}^{n-1} c_i \cdot c_{n-i-1}.$$

Exercise 2 [2]**:** Check that the Catalan numbers satisfies this recursion.

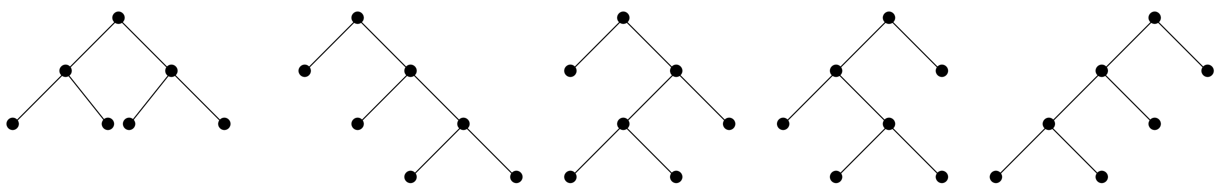
There are a very large number of mathematical objects enumerated by the Catalan numbers. Four basic examples are:
-well-matched expressions of n pairs of parentheses

((( )))   (( ) ( ))   (( )) ( )   ( ) (( ))   ( ) ( ) ( )

-triangulations of a convex $n+2$-gon



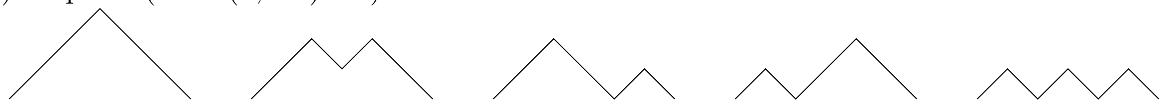
-binary trees (out degree 0 or 2) with $n+1$ unlabelled leafs



-permutations of length n avoiding a given pattern of length tree. See EC1 and Section 2 below.

An other important example are paths in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ using only steps $(1,1)$ and $(1,-1)$ that never go below the x-axis. Such paths are called Dyck paths. Tilting the Dyck paths $\pi/4$ clockwise we get the paths considered in EC1.

There is an easy bijection between Dyck paths and expressions of parentheses, where $(1,1)$ maps to (and $(1,-1)$ to).



((( )))   (( ) ( ))   (( )) ( )   ( ) (( ))   ( ) ( ) ( )

To see that the number of Dyck paths is counted by the Catalan numbers we denote with $(2(i+1),0)$ the first position where the Dyck paths returns to the x-axis. Then there are c_i ways to get there (first step has to be up, the last has to be down and in between there is a Dyck path of length $2i$) and there are c_{n-i-1} ways to continue the path to $(2n,0)$. Summing over i gives the mentioned recursion.

A more combinatorial proof is given by considering all paths (i.e. paths that also my go below the x-axis) from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ which is $\binom{2n}{n}$ (think Pascal's triangle). We now must subtract of the bad paths that go below the x-axis. Given such a path s, let $(2i+1,-1)$ be the first time s touches the line $x=-1$. Now we construct the path s' by reflecting the rest of the path in the line $x=-1$, i.e. s and s' are the same up until $(2i+1,-1)$ and thereafter s' does an up step when s does a down step and vice versa. This gives a bijection (check the details!) between the bad paths and all paths from $(0,0)$ to $(2n,-2)$. Hence the number of bad paths is $\binom{2n}{n-1}$. We can thus conclude that the number of Dyck paths is $\binom{2n}{n}-\binom{2n}{n-1}$ as desired.

We have now proved the following.

Theorem 1 *Dyck paths and well-matched parentheses are counted by the Catalan numbers.*

Exercise 3 [2-]: Prove that the number of triangulations of an $n+2$-gon and the number of binary trees is C_n using the recursion.
Exercise 4 [2+]: Prove the same thing by finding bijections to Dyck paths.
Exercise 5 [2+]: Enumerate the Dyck paths using Lindström's Lemma.

There are several interesting refinements of the Catalan numbers. We can get one refinement (called the α-refinement by the French combinatorialists) by the number of consecutive down steps at the end. We can get the numbers from the so called Catalan triangle:

1						
1	6					
1	5	20				
1	4	14	48			
1	3	9	28	90		
1	2	5	14	42	132	
1	1	2	5	14	42	132

Here we can see that the refinement is for example
$2=1+1, 5=2+2+1, 14=5+5+3+1, 42=14+14+9+4+1, \ldots$
Let $c_n(t)$ = number of paths from $(0,0)$ till $(2n-2-t,t)$ and we get the following theorem.

Theorem 2 *We have $c_n = \sum_{t=0}^{n-1} c_n(t)$, the recursion $c_n(t) = c_{n-1}(t-1) + c_n(t+1)$ and the exact formula $c_n(t) = \binom{2n-t-2}{n-t-1} - \binom{2n-t-2}{n-t-2}$.*

PROOF: **Exercise 6** [2]. □

2 Pattern avoidance

EC1 contains a short discussion of pattern avoidance for patterns of length three and bijections to the Dyck paths mentioned above. Read that. In this section we will make a slightly more general approach, but the main example is still patterns of length 3. With the previous section this will be proper proofs that they are counted by Catalan numbers in all cases.

DEFINITION: The permutation $\pi \in S_n$ is said to **contain the pattern** $\tau \in S_k$, $k \leq n$ if there exist $i_{\tau(1)} < i_{\tau(2)} < \cdots < i_{\tau(k)}$ such that $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$. □

One way to visualise the definition is to use the permutation matrix for π (or the geometric description of the permutation as Stanley calls it) which has a 1 in position $(i, \pi(i))$ for all i and zeros elsewhere.

Then we delete all rows except $i_{\tau(1)} < i_{\tau(2)} < \cdots < i_{\tau(k)}$ and all columns except $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$. What remains is then the permutation matrix for τ.

EXAMPLE: A permutation matrix $\pi \in S_n$ contains 231 if there exist $i < j < l$, such that $\pi(l) < \pi(i) < \pi(j)$. The permutation 426351 contains 231, since 4.6..1 forms the pattern 231.

$$\begin{array}{c} 4 \\ 2 \\ 6 \\ 3 \\ 5 \\ 1 \end{array} \begin{pmatrix} 0 & 0 & 0 & \bar{1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{1} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \bar{1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

DEFINITION: A permutation that does not contain the pattern τ is called τ-**avoiding**. □

EXAMPLE: 521346 is 231-avoiding.

$$\begin{array}{c} 5 \\ 2 \\ 1 \\ 3 \\ 4 \\ 6 \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

DEFINITION: Let $U_n(\tau) := \#\{\pi \in S_n : \pi \text{ is } \tau\text{-avoiding}\}$ □

The basic question we will discuss here is: What is $U_n(\tau)$? This question was the starting point for a large activity among several combinatorialists the last 20 years to understand patterns in permutations from various aspects. To compute $U_n(\tau)$ we are often forced to understand the structure of permutations avoiding τ.

The classical case $k = 3$ is the most well known case.

3 Patterns of length 3

In this section we will prove the following theorem.

Theorem 3 *For all patterns τ of length 3, $U_n(\tau) = \frac{1}{n+1}\binom{2n}{n}$, the Catalan numbers.*

Let us start with an easy lemma.

Lemma 1 *If the permutation matrix of τ^s is the reflection (or rotation) of the permutation matrix of τ, then*

$$U_n(\tau) = U_n(\tau^s).$$

PROOF: The same reflection (or rotation) defines a bijection between τ-avoiding and τ^s-avoiding permutations. $\square$

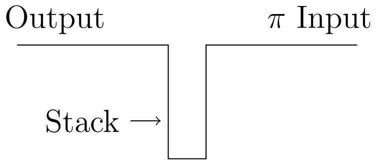
For patterns of length 3 there are thus two cases:

$$U_n(123) = U_n(321) \qquad \text{och}$$

$$U_n(132) = U_n(213) = U_n(231) = U_n(312).$$

3.1 Sorting and 231-avoiding permutations

To determine $U_n(231)$ we shall study sorting with one stack. This presentation is inspired by the approach of Donald Knuth, who was one of the first to study pattern avoidance.

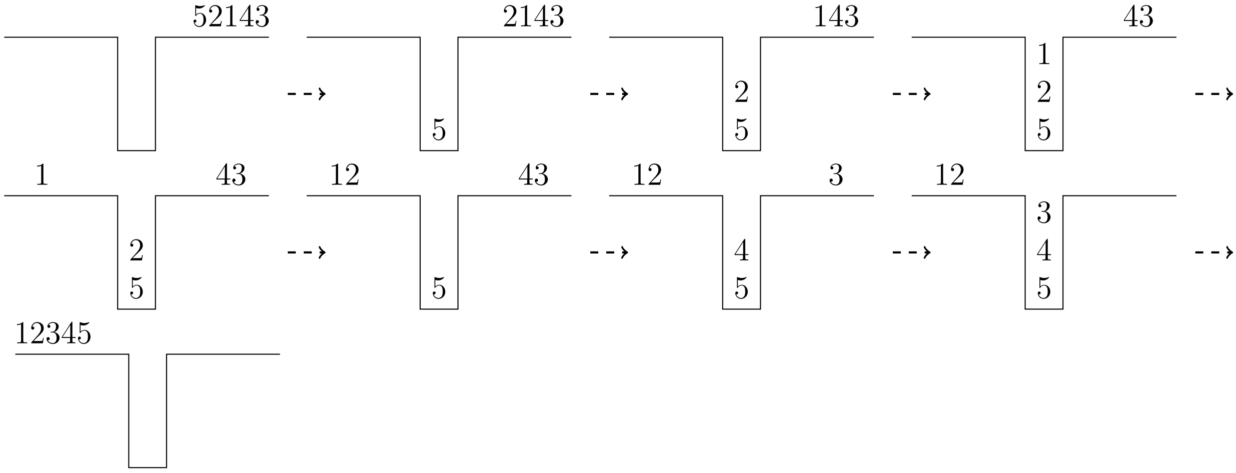


The rules for the stack are as follows. The permutation π is written in word form and the numbers are entering the stack in the order $\pi(1), \pi(2), \pi(3), \ldots, \pi(n)$. At every step the element about to enter the stack $\pi(i)$ is compared with the top element in the stack b.

- If $b > \pi(i)$ or the stack is empty, move $\pi(i)$ to the stack on top of b
- If $b < \pi(i)$, move b from the stack to the right end of the sorted word at the left, which is the output of the sorting procedure.
- When $\pi(n)$ has been put on the stack all the numbers in the stack are removed one by one and placed at the end of the sorted word.

Note that the stack is always completely ordered and that the process is deterministic so there is no choice in the sorting.

EXAMPLE: 52143 can be sorted with one stack.



Lemma 2 *π is sortable with one stack $\iff$ π is 231-avoiding.*

PROOF:
$\Longrightarrow$)

If π contains a 231 pattern, $i < j < l, \pi(l) < \pi(i) < \pi(j)$, then $\pi(i)$ (the "2") must leave the stack before $\pi(j)$ "3" enters the stack. Thus "2" and "1" will be in the wrong order in the output.

$\Longleftarrow$)

Assume π is not sortable with one stack. Then there exists $i < j$ such that j is before i after the sorting. Hence j must have been forced to leave the stack by some larger number l that in π must have been after j but before i. Then $..j..k..i..$ forms a 231 pattern in π. $\square$

Theorem 4 $U_n(231) = \frac{1}{n+1}\binom{2n}{n}$*, the Catalan numbers.*

PROOF: We define a map α from permutations of length n sortable with one stack to well-matched expressions of parentheses of length n.

For every step in the sorting procedure we write (when an element is put on the stack and) when an element is removed from the stack. Note that $\alpha(\pi)$ is a well-matched expression, since we can never remove more elements from the stack than has been put there.

Now, let β be the following map in the other direction. Given a well-matched expression of parentheses we number the) from the left with $1, 2, \ldots, n$. Then we number the (with the same number as the right parentheses it is matching. The numbers for the left parentheses gives us a permutation in S_n, which is the image of the map β.

It is not difficult to go through the details and see that β is the inverse of α and hence they are bijections between equinumerous sets. $\square$

EXAMPLE: $\alpha(51243) = (()()(()))$, (check the details). The numbering by β gives $(_5(_1)^1(_2)^2(_4(_3)^3)^4)^5$

You should compare this with the bijection given in EC1.

3.2 123-avoiding

Our last step will be to show that also 123-avoiding permutations are counted by the Catalan numbers

Theorem 5 $U_n(123) = C_n = \frac{1}{n+1}\binom{2n}{n}$, *the Catalan numbers.*

This can also be shown in many different ways. EC1 uses a bijection to Dyck paths.
PROOF: Here we will use a bijection to 132-avoiding permutation, originally due to Simion and Schmidt

$$f : \{\pi \in S_n : \pi \text{ 123-avoiding}\} \to \{\pi \in S_n : \pi \text{ 132-avoiding }\},$$

which also tells us something interesting about the structure of these permutations.

Assume π is written in word form. We define $\pi' = f(\pi)$ by first setting $\pi'(i) = \pi(i)$ for all the left-to-right minima $\pi(i)$. Define $LRmin(\pi)$ to be the set of left-to-right minima in π. For the remaining positions we define them successively from the left by

$$\pi'(i) = \min\{a : a > \pi'(i-1) \text{ and } a \neq \pi'(j), j = 1, \ldots, i-1 \text{ and } a \notin LRmin(\pi)\}.$$

It is easy to see that π' is then a 132-avoiding permutation since between every pair of left-to-right minima we are filling out in an increasing order with smallest possible elements larger than the last left-to-right minima. That is for each pair $i < j$ such that $\pi'(i) < \pi'(j)$, the $\pi'(j)$ cannot be in $LRmin(\pi)$ and there is for every x, $\pi'(i) < x < \pi'(j)$, some $k < j$ with $\pi'(k) = x$. There is hence no "2" that could create a 132 pattern. Note that once the left-to-right minima are fixed there is only one way to fill in the rest to avoid 132 patterns.

Example: $\pi = \overline{8}\ \overline{6}\ 10\ 9\ \overline{4}\ \overline{3}\ 7\ \overline{1}\ 5\ 2$ gives $\pi' = \overline{8}\ \overline{6}\ 7\ 9\ \overline{4}\ \overline{3}\ 5\ \overline{1}\ 2\ 10$, where left-to-right minima are indicated with a bar.

We get the inverse to f by again fixing the left-to-right minima π'. It is the same elements as in π. The remaining positions are defined successively from left to right by

$$\pi(i) = \max\{a : a \neq \pi(j), j = 1, \ldots, i-1 \text{ and } a \notin LRmin(\pi')\}.$$

With this definition we get that π consists of two decreasing sequences and can thus not contain a 123 pattern. Since left-to-right minima are not changed and the remaining elements can only be filled in in a unique way to avoid creating a 123 pattern this is the inverse of f and hence it is a bijection. □

Note that this property of 123-avoiding permutations having two separate decreasing sequences was essential to the proof in EC1 as well.

Exercise 7 [2]**:**

a) Find a refinement of 123-avoiding permutations in n subsets after some parameter t such that subset t has $c_n(t)$ elements.

b) Find a refinement of 231-avoiding permutations in n subsets after some parameter t such that subset t has $c_n(t)$ elements.