Homework Set #9

The intention is that you do the exercises yourself. Oral discussion (without using pen/paper) between students is allowed, but the solution should be written down individually.
If you find the solution to any of the exercises do not simply copy it. Instead, reference the source and discuss the result and the method used for deriving it.
The homework must be submitted one day before each tutorial session either on paper (before 6 PM) or via email (before mid night).
A correctly solved problem gives either 1 or 2 points, depending on the problem. A partially correct solutions will be awarded half of the points (i.e. either 0.5 or 1 point). A mostly wrong or mostly incomplete solution gives 0 points.

Numbers below refer to problems in the text book: H. Van Trees *"Detection, Estimation, and Modulation Theory." (Part I)*.
NOTE: As opposed to the standard notation (e.g. Lapidoth), Van Trees uses lower case letters for random variables and upper case letters for the realizations. For example $X \sim x$ (instead of the more standard form $x \sim X$).

1. (1p) Exercise 2.2.1
2. (1p) Exercise 2.2.2
3. (1p) Exercise 2.2.4
4. (2p) Exercise 2.2.5
5. (1p) Exercise 2.2.6
6. (1p) Exercise 2.2.7
7. (1p) Exercise 2.2.10
8. (1p) Exercise 2.2.11
9. (1p) Exercise 2.3.3
10. (2p) Suppose that Y is a random variable that under hypothesis H_0 has pdf

$$f_{Y|H_0}(y|H_0) = \begin{cases} \frac{2}{3}(y+1) & 0 \leq y \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

Instructor: Tobias J. Oechtering, Email: oech@kth.se
Assistant: Ahmed Zaki, Email: zakiah@kth.se

and under hypothesis H_1 has pdf

$$f_{Y|H_1}(y|H_1) = \begin{cases} 1 & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the Bayes test and the minimum Bayes risk H_0 versus H_1 with uniform costs and equal priors.

(b) Find the minimax rule and minimax risk for uniform costs.

(c) Find the Neyman-Pearson test and the corresponding detection probability P_D for false-alarm probability P_F equal to $\alpha \in (0, 1)$.

11. (2p) Properties of the ROC (I). Consider a Neyman-Pearson binary detection problem with likelihood ratio L and threshold γ. That is, if $L \ge \gamma$ then choose hypothesis H_1; otherwise choose hypothesis H_0. The probabilities of detection P_D and false-alarm P_F are functions of the threshold value γ and are given by

$$P_D(\gamma) = \int_\gamma^\infty f_{L|H_1}(l|H_1)dl,$$
$$P_F(\gamma) = \int_\gamma^\infty f_{L|H_0}(l|H_0)dl,$$

respectively. Here $f_{L|H_0}(l|H_0)$ and $f_{L|H_1}(l|H_1)$ are the conditional density functions of L given H_0 and H_1, respectively. The receiver operating characteristic (ROC) is the curve defined by pairs $(P_F(\gamma), P_D(\gamma))$ for different values of γ. We want to show that the slope of this curve at a particular point $(P_F(\gamma), P_D(\gamma))$ is equal to the value of the threshold γ. That is,

$$\frac{dP_D}{dP_F} = \gamma \tag{1}$$

(You can assume that both P_D and P_F are smooth continuous functions of γ).

(a) Express

$$\frac{dP_D}{dP_F} = \frac{dP_D/d\gamma}{dP_F/d\gamma}.$$

in terms of $f_{L|H_0}$ and $f_{L|H_1}$.

(b) Express $P_D(\gamma)$ as an integral of L and $f_{L|H_0}(l|H_0)$ and show that

$$\frac{dP_D(\gamma)}{d\gamma} = -\gamma f_{L|H_0}(\gamma|H_0).$$

Hint:

$$\frac{d}{ds}\int_s^\infty g(t)dt = -g(s).$$

(Differentiation under the integral sign).

Solution due March 02, 2015

Instructor: Tobias J. Oechtering, Email: oech@kth.se
Assistant: Ahmed Zaki, Email: zakiah@kth.se

(c) Conclude the desired result (i.e. (1)) and verify it by drawing a typical ROC and considering $\gamma \to 0$ and $\gamma \to \infty$.

12. (1p) Properties of the ROC (II). Assume that the ROC is a continuous function. Argue that it must be concave downwards $\cap$.

13. (2p) Consider the following binary detection problem. The observation Y is the sum of the squares of two independent zero-mean Gaussian random variables whose variance depends on the hypothesis. That is,

$$Y = X_1^2 + X_2^2$$

with $X_1, X_2 \sim \mathcal{N}(0, \sigma_i^2)$ with

$$\sigma_i = \begin{cases} \sigma_0 & \mathrm{H}_0 \\ \sigma_1 & \mathrm{H}_1 \end{cases}$$

with $\sigma_1 > \sigma_0$.

(a) Compute $P_F(\gamma)$, $P_D(\gamma)$.
Hint: Use polar coordinates: $z = \sqrt{x_1^2 + x_2^2}, \theta = \arctan \frac{x_2}{x_1}$. *With this change of variables, Z is Rayleigh distributed and θ is uniform on* $[-\pi, \pi)$.

(b) Express P_D as a function of P_F (i.e. the explicit form of the ROC). Plot the resulting function for different values of $\frac{\sigma_1^2}{\sigma_0^2}$ and interpret the results.

14. (2p) Tests with unwanted parameters. Consider the following hypothesis testing problem. Under hypothesis H_0, Y is a $\mathcal{N}(0, \sigma^2)$ random variable. Under hypothesis H_1, Y is a $\mathcal{N}(M, \sigma^2)$ random variable. Thus,

$$\begin{aligned} H_0: \quad & f_{Y|H_0}(y|H_0) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}, \\ H_1: \quad & f_{Y|H_1}(y|H_1) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-M)^2}{2\sigma^2}}. \end{aligned}$$

We consider the case where σ^2 is known but M is an unwanted parameter. We look at two different approaches to model M:

(a) Bayesian approach. First assume that M is a random variable whose pdf under hypothesis H_1 is given by

$$f_{M|H_1}(m|H_1) = \frac{1}{\sqrt{2\pi}\sigma_m} e^{-\frac{m^2}{2\sigma_m^2}}.$$

(σ_m^2 known). Find the likelihood ratio test

$$\frac{f_{Y|H_1}(y|H_1)}{f_{Y|H_0}(y|H_0)}$$

and the threshold.
Hint: Integrate over the density of M.

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(b) Neyman-Pearson approach (I). Now assume that M is an unknown but deterministic positive parameter, i.e. $M > 0$. Write the integral defining P_F (i.e. the probability that H_1 is declared when H_0 is true) and sketch the probability distributions. Will the Neyman-Pearson test (i.e. likelihood ratio and threshold) depend on the value of M? Why?

(c) Neyman-Pearson approach (II). Finally, what happens if M is an unknown but deterministic parameter that can be positive or negative? Will the test depend on M?

Instructor: Tobias J. Oechtering, Email: oech@kth.se
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