Lecture 1:
Properties and representations of electromagnetic fields

by Thomas Johnson

Outline

Fundamentals of electromagnetic theory (Ch. 1)

- Maxwell's equations
- Continuity equations
- Scalar and vector potentials

Tensor index notation (Ch. 2)

- Component representations of vectors and matrixes
- Scalar and vector products
- Symmetric/Antisymmetric and Hermitian/Antihermitian

Multipoles (Ch. 3)

- Multipole moments
- Multipole fields

Maxwell's equations

Gauss's law for **E**-fields $\nabla \bullet \mathbf{E} = \frac{\rho}{\varepsilon_0}$

Gauss's law for **E**-fields $\nabla \bullet \mathbf{B} = 0$

Ampere's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$

Faraday's law $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

Wave equation

Time derivative of Faraday's law: $\nabla \times \frac{\partial \mathbf{E}}{\partial t} = -\frac{\partial^2 \mathbf{B}}{\partial t^2}$

Curl of Ampere's law: $\nabla \times \nabla \times \mathbf{B} = \mu_0 \nabla \times \mathbf{J} + \frac{1}{c^2} \nabla \times \frac{\partial \mathbf{E}}{\partial t}$

The *wave equation* in $\mathbf{B}$:

$$\nabla \times \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = \mu_0 \nabla \times \mathbf{J}$$

Maxwell's equations: alternative formulation

Time evolution:

$$\begin{cases} \dfrac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \\[2ex] \dfrac{\partial \mathbf{E}}{\partial t} = c^2\left(\nabla \times \mathbf{B} - \boxed{\mu_0 \mathbf{J}}\right) \end{cases}$$

Constraints:

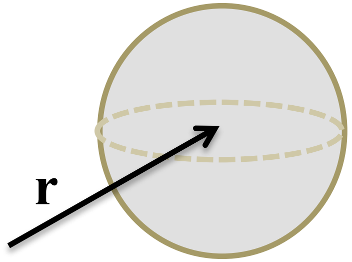
$$\begin{cases} \nabla \bullet \mathbf{B} = 0 \\[2ex] \nabla \bullet \mathbf{E} = \boxed{\dfrac{\rho}{\varepsilon_0}} \end{cases}$$

Source terms

Knowing $\rho(\mathbf{r},t)$ and $\mathbf{J}(\mathbf{r},t)$, Maxwell's equations tells us how to compute $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$…at least in principle…

What is ρ in Maxwell's equations?

To define the charge density, ρ, at a point $\mathbf{r}$ in space, consider a little ball around $\mathbf{r}$. Assume the ball is large enough to cover a *significant* number of charged particles.



- Let the volume of the ball be: V
- Let particle i have a charge: q_i
- Let the total charge inside the ball: $Q = \sum_i q_i$

Then the charge density is: $\rho = \dfrac{Q}{V} = \dfrac{1}{V} \sum_i q_i$

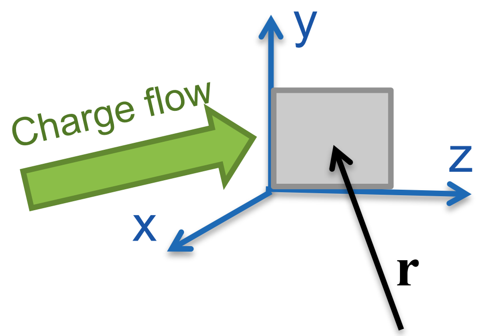
What is J in Maxwell's equations?

Define the current density vector, $\mathbf{J}$, at a point $\mathbf{r}$ by looking at one vector component at a time; start with J_x.

Introduce a small square surface with area A in the y-z plane.

Count the net charge, Q, that crosses the surface in a time t, while moving towards increasing values of x.

Define: $$J_x = \frac{Q}{At}$$

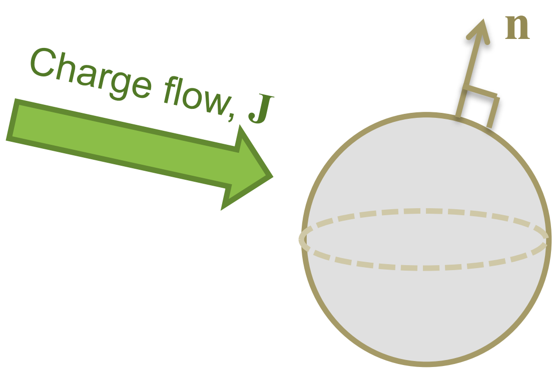


Question: Calculate the J_x in a metal when the electron density is $n=10^{25}$ m^{-3} and the mean electron velocity $v_x=10^{-6}$ m/s. Assume that the electron charge is $e=-10^{-19}$.

Charge conservation and charge continuity

How does the total charge inside a sphere change in time?

- The total charge: $Q = \iiint \rho dV$
- Charge conservation: Q changes only due to currents, I, into the sphere:



$$\Delta Q = \Delta t I = \Delta t \oint\!\!\oint \mathbf{J} \bullet \mathbf{n} dS$$

$$\Rightarrow \frac{\partial Q}{\partial t} = \oint\!\!\oint \mathbf{J} \bullet \mathbf{n} dS = \iiint \nabla \bullet \mathbf{J} dV$$

Gauss

$$\Rightarrow \iiint \left(\frac{\partial \rho}{\partial t} - \nabla \bullet \mathbf{J} \right) dV = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \rho}{\partial t} - \nabla \bullet \mathbf{J} = 0}$$

Continuity eq.

Energy

When a particle is accelerated by an electro-static field it gains both energy and momentum. Where do the energy and the momentum come from?

If the particle is accelerated by an electromagnetic wave; where do the energy and the momentum come from?

How to measure the energy density of electric fields

Take a charged capasitor.

- The electric field inside the capasitor will try to push the plates together.
- To push the plates apart will require work.
  - *Question*: If we invest energy pushing the plates apart; where do this energy go?
  - *Answer*: It increases the electric energy. The strength of the electric field is not changed, but the volume between the plates have changed.
- From this experiment we can get the energy density:

$$W_E = \frac{\varepsilon_0 \left| E^2 \right|}{2}$$

The energy density of magnetic fields

Using a solinoid, a similar experiment can be performed to identify the energy density of magnetic fields:

$$W_M = \frac{\left|B^2\right|}{2\mu_0}$$

Time evolution of electromagnetic energy

What processes can change the energy density at a point $\mathbf{r}$?

$$\frac{\partial}{\partial t}\left(W_E + W_M\right) = \frac{\partial}{\partial t}\left(\frac{\varepsilon_0 \left|E^2\right|}{2} + \frac{\left|B^2\right|}{2\mu_0}\right) = \left(\frac{\varepsilon_0}{2}\mathbf{E} \bullet \frac{\partial}{\partial t}\mathbf{E} + \frac{1}{2\mu_0}\mathbf{B} \bullet \frac{\partial}{\partial t}\mathbf{B}\right) = \ldots$$

Insert Maxwell's equations…and do some algebra…

$$\frac{\partial}{\partial t}\left(W_E + W_M\right) + \nabla \bullet \left(\mathbf{E} \times \mathbf{B}/\mu_0\right) = -\mathbf{J} \bullet \mathbf{E}$$

This is the *energy continuity equation*.

Compare charge/energy conservation equations

Charge continuity

$$\frac{\partial \rho}{\partial t} + \nabla \bullet \mathbf{J} = 0$$

Charge density , ρ
Charge flux , $\mathbf{J}$

+

Charge conservation , 0

Energy continuity

$$\frac{\partial}{\partial t}\left(W_E + W_B\right) + \nabla \bullet \left(\mathbf{E} \times \mathbf{B} / \mu_0\right) = -\mathbf{J} \bullet \mathbf{E}$$

$W_E + W_M$, Energy density
$\mathbf{E} \times \mathbf{B} / \mu_0$, Energy flux, the **Poynting flux**

+

$\mathbf{J} \bullet \mathbf{E}$, Work by $\mathbf{E}$ on charge flow $\mathbf{J}$

Continuity of the momentum density

The electro-magnetic momentum density is related to the pointing flux:

$$\mathbf{P}_{EM} = \mathbf{E} \times \mathbf{B} / \mu_0 c^2$$

A continuity equation for the momentum density can be calculated similarly to the energy density.

$$\frac{\partial}{\partial t}\left(\mathbf{P}_{EM}\right) = \frac{1}{\mu_0 c^2}\left(\mathbf{E} \times \frac{\partial}{\partial t}\mathbf{B} - \mathbf{B} \times \frac{\partial}{\partial t}\mathbf{E}\right) = \nabla \bullet \overline{\overline{T}}_{EM} - \rho\mathbf{E} - \mathbf{J} \times \mathbf{B}$$

Electric force

The momentum flux

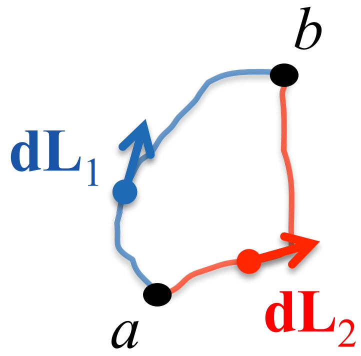
Magnetic force

Part of this first ***home assignments*** for next Tuesday concerns a similar relation for the angular momentum density.

Electrostatic potential

Faraday's law for static fields reads: $\nabla \times \mathbf{E} = 0$

$$\oint \mathbf{E} \bullet d\mathbf{L} = \iint (\nabla \times \mathbf{E}) \bullet \mathbf{n} dS = 0$$



Thus, $\mathbf{E}$ is a conservative field,
i.e. the line intergral is path independent

$$U \equiv \int_a^b \mathbf{E} \bullet d\mathbf{L}_1 = \int_a^b \mathbf{E} \bullet d\mathbf{L}_2$$

Here U is related to the electrostatic (scalar) potential $\phi(\mathbf{r})$

$$U = \phi(b) - \phi(a) \quad ; \quad \mathbf{E} = -\nabla\phi$$

Electro-static grounding

The electro-static potential is not uniqely defined, i.e. two different potentials may generate the same electric fields

$$\left.\begin{aligned}\phi_1(\mathbf{r}) &= f(\mathbf{r}) \\ \phi_2(\mathbf{r}) &= f(\mathbf{r}) + c\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}\mathbf{E}_1(\mathbf{r}) &= -\nabla\phi_1(\mathbf{r}) = -\nabla f(\mathbf{r}) \\ \mathbf{E}_2(\mathbf{r}) &= -\nabla\phi_2(\mathbf{r}) = -\nabla f(\mathbf{r}) - \nabla c = -\nabla f(\mathbf{r})\end{aligned}\right.$$

Using ϕ_1 or ϕ_2 gives the exact same model of the physics!

We say, when selecting ϕ we select a *gauge*.

Compare to grounding of electrical circuits:

- *An electrical circuit may be grounded at any one point. This does **not** change the behaviour of the circuit.*

Vector potential

Gauss law for the magnetic field: $\nabla \bullet \mathbf{B} = 0$

A general solution to this Gauss law is:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\Rightarrow \nabla \bullet \mathbf{B} = \nabla \bullet \left(\nabla \times \mathbf{A}\right) = 0$$

Here $\mathbf{A}$ is called the *vector potential*.

Like the electro-static potential, $\mathbf{A}$ is not unique. Different gauges can be selected according to:

$$\mathbf{A}_2 = \mathbf{A}_1 + \nabla\psi$$

$$\Rightarrow \mathbf{B}_2 = \nabla \times \mathbf{A}_2 = \nabla \times \mathbf{A}_1 + \nabla \times \nabla\psi = \nabla \times \mathbf{A}_1 = \mathbf{B}_1$$

Time dependent potential representations

For time-dependent problem $\mathbf{E}$ and $\mathbf{B}$ are coupled.

So are also the potentials:

$$\begin{cases} \mathbf{E} = \dfrac{\partial \mathbf{A}}{\partial t} - \nabla \phi \\ \mathbf{B} = \nabla \times \mathbf{A} \end{cases}$$

Using these combined potententials, the gauge can be selected through a function ψ:

$$\begin{cases} \phi_1 = \phi_2 + \dfrac{\partial \psi}{\partial t} \\ \mathbf{A}_1 = \mathbf{A}_2 - \nabla \psi \end{cases}$$

Common gauge conditions

There are three gauges that are particularly common:

Coulomb gauge

$$\nabla \bullet \mathbf{A} = 0$$

Simplifies Coulombs eq.

$$\frac{\rho}{\varepsilon_0} = \nabla \bullet \mathbf{E}$$

$$= \nabla \bullet \left(\nabla \phi + \frac{\partial \mathbf{A}}{\partial t} \right)$$

$$= \nabla \bullet \nabla \phi$$

Lorentz gauge

$$\frac{\partial \phi}{\partial t} + c^2 \nabla \bullet \mathbf{A} = 0$$

Simplifies the wave eqs:

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \phi = \frac{\rho}{\varepsilon_0}$$

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \mathbf{A} = \mu_0 \mathbf{J}$$

Temporal gauge

$$\phi = 0$$

No scalar potential!!

All fields given by $\mathbf{A}$.

Use: time-dependent electro-magnetic problems.

Outline

Fundamentals of electromagnetic theory (Ch. 1)

- Maxwell's equations
- Continuity equations
- Scalar and vector potentials

Tensor index notation (Ch. 2)

- Component representations of vectors and matrixes
- Scalar and vector products
- Symmetric/Antisymmetric and Hermitian/Antihermitian

Multipoles (Ch. 3)

- Multipole moments
- Multipole fields

Basis vectors and vector components

What is a vector?

- A vector on $\mathcal{R}^3$ is defined by a *length* and a *direction* in $\mathcal{R}^3$.

To represent different directions we need a *basis* for $\mathcal{R}^3$.

Use a set of three orthogonal vector $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of unit length.

An arbitrary vector in $\mathcal{R}^3$ can now be represented as a superposition of the basis:

$$\mathbf{E} = E_1\mathbf{e}_1 + E_2\mathbf{e}_2 + E_3\mathbf{e}_3 = \sum_{i=1}^{3} E_i\mathbf{e}_i$$

Here E_i is denoted the i:th *component* of $\mathbf{E}$.

Once the basis is fixed we may write: $\mathbf{E} = \begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix}$

Einstein's summation convention

Einstein got so tired of writing the summation symbol that he defined a more concise notation, simply:

$$\sum_{i=1}^{3} E_i \mathbf{e}_i \rightarrow E_i \mathbf{e}_i$$

If two factors carry the same vector-index, then an implicit summation is implied.

OR: Always sum over repeated indexes within a term.

These indexes are called *dummy index*, or *summation indexes*.

The process of summing over indexes is called a *contraction*.

Note: there's no summation if two *terms* have the same index, e.g. $a_i + b_i$

The Kronicker delta

What is the scalar product: $\mathbf{e}_i \bullet \mathbf{e}_j$?

If $i=j$, then the vectors have are in the same direction and have unit length, i.e. $\mathbf{e}_1 \bullet \mathbf{e}_1 = \mathbf{e}_2 \bullet \mathbf{e}_2 = \mathbf{e}_3 \bullet \mathbf{e}_3 = 1$

But if i and j are different, the vectors are orthogonal and the scalar product is zero.

This scalar product called the *Kroniker-delta*, δ_{ij}, and it forms a unit matrix:

$$\delta_{ij} = \mathbf{e}_i \bullet \mathbf{e}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question: What is δ_{kk}?

Scalar products

What is the scalar product of two vectors?

…in fact we already have the definitions we need!

$$\mathbf{A} \bullet \mathbf{B} = A_i \mathbf{e}_i \bullet B_j \mathbf{e}_j = A_i B_j \mathbf{e}_i \bullet \mathbf{e}_j = A_i B_j \delta_{ij}$$

Next consider $B_j \, \delta_{ij}$. After summation over j we have a vector component with index i. Look first at $i=1$:

$$[B_j][\delta_{1j}] = \begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = B_1$$

Repeating for i=2,3 gives: $B_j \delta_{ij} = B_i \quad \Rightarrow \quad \boxed{\mathbf{A} \bullet \mathbf{B} = A_i B_i}$

Tensors

The tensor concept is an extension to the vector concept; they are objects with *directions* and *lengths* on $\mathcal{R}^3$.

Given a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can describe tensors of different rank:

- 1-tensors are simply a vector, e.g. $\mathbf{E} = E_i \mathbf{e}_i$
- 2-tensors can be written on the form: $M = M_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ where $\otimes$ is the outer product.
- Similarly 3-tensors: $A = A_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$
- And n-tensors: $B = B_{i \ldots n} \mathbf{e}_i \otimes \ldots \otimes \mathbf{e}_n$
- There are also 0-tensor; these simply scalars!

Note: Einsteins summation convention apply to both i, j, k and n.

2-tensors as matrixes

The components of a 2-tensors, e.g. M_{ij}, have two indexes.
In fact, it forms a matrix:

$$\left[M_{ij}\right] = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

In most practical situations we use a fixed basis and then we can think of a 2-tensor as a *matrix of tensor component*.

Note: in this course we only deal with simple cartesian coordinates.

Scalar products between tensors and vectors

The scalar product between a 2-tensor and a vector is like a multiplication of a matrix and component-vector.

There alternatives; a *right-side* and a *left-side* product:

$$M \bullet \mathbf{v} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{e}_i M_{ij} v_j$$

$$\mathbf{v} \bullet M = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \bullet \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} = v_i M_{ij} \mathbf{e}_i$$

Vector products using indexes

To perform vector products using index notation we need an array call the Levi-civita symbol, ε_{ijk} .

Note: The Levi-civita symbol has 3 indexes, i.e. 27 components!

- $\varepsilon_{ijk}=1$ for even permutations $\{i,j,k\}=\{1,2,3\}$, $\{2,3,1\}$, $\{3,1,2\}$.
- $\varepsilon_{ijk}=-1$ for odd permutations $\{i,j,k\}=\{3,2,1\}$, $\{2,1,3\}$, $\{1,3,2\}$.
- $\varepsilon_{ijk}=0$ for all other index combinations.

The i:th component of a vector products is then:

$$(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k$$

Try yourself: $(\mathbf{e}_1 \times \mathbf{e}_2)_3 = ?$

$(\mathbf{e}_2 \times \mathbf{e}_1)_3 = ?$

Symmetric and anti-symmetric matrixes

Symmetric matrixes are symmetric around the diagonal.

Examples: $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$$M : M_{ij} = M_{ji}$$

Antisymmetric matrixes are anti-symmetric around the diagonal.

Examples: $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ *Note that the diagnoal is always zero!*

$$M : M_{ij} = -M_{ji}$$

The symmetric and anti-symmetric parts of a matrix

Every matrix M has a symmetric part, M^{sym}, and anti-symmetric part, M^{asym}; using T for transpose.

$$\left\{\begin{aligned} M^{sym} &= \frac{1}{2}\left(M + M^T\right) \\ M^{asym} &= \frac{1}{2}\left(M - M^T\right) \end{aligned}\right\} \text{ or } \left\{\begin{aligned} M_{ij}^{sym} &= \frac{1}{2}\left(M_{ij} + M_{ji}\right) \\ M_{ij}^{asym} &= \frac{1}{2}\left(M_{ij} - M_{ji}\right) \end{aligned}\right\}$$

Compare with complex numbers; they have a real and an imaginary part:

$$\Re(z) = \frac{1}{2}(z + z^*)$$

$$i\Im(z) = \frac{1}{2}(z - z^*)$$

Hermitian matrixes/tensors

In electromagnetic theory we often use Fourier transforms and then all tensors are complex, i.e. they have *matrixes of complex components*.

For such matrixes we will need a so called *Hermite conjugate*, which combine transpose and complex conjugate:

$$M^{\dagger} = (M^*)^T$$

The Hermitian and anti-hermitian parts of a matrix, M, are:

$$M^H = \frac{1}{2}\left(M + M^{\dagger}\right)$$

$$M^A = \frac{1}{2}\left(M - M^{\dagger}\right)$$

Outline

Fundamentals of electromagnetic theory (Ch. 1)

- Maxwell's equations
- Continuity equations
- Scalar and vector potentials

Tensor index notation (Ch. 2)

- Component representations of vectors and matrixes
- Scalar and vector products
- Symmetric/Antisymmetric and Hermitian/Antihermitian

Multipoles (Ch. 3)

- Multipole moments
- Multipole fields

The monopole

The first and simplest of the terms in the multipole expansion is the *monopole*.

The ideal monopole is a *point charge*.

The electro-static potential from a point-charge particle with charge q^K located at $\mathbf{r}^K$ is:

$$\phi(\mathbf{r}) = \frac{q^K}{4\pi\varepsilon_0 \left|\mathbf{r} - \mathbf{r}^K\right|}$$

The scalar potential from a distant object

Consider a charge object far away. If we know the exact location of all particle we can sum up the total potential:

$$\phi(\mathbf{r}) = \sum_K \frac{q^K}{4\pi\varepsilon_0 \left|\mathbf{r} - \mathbf{r}^K\right|}$$

Here K is the particle index.
Note: Could be 10^{25} terms in the sum!

Since the object is far a away the distance $\left|\mathbf{r} - \mathbf{r}^K\right|$ is almost the same for all particles, K. Thus:

$$\phi(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0 \left|\mathbf{r} - \mathbf{r}_{object}\right|} \quad , \quad Q = \sum_K q^K \rightarrow \iiint \rho(\mathbf{r}) dV$$

Charge density within the object

where Q is called the *monopole moment*.

Refined description of distant object

If the monopole description is not good enough, can we improve the description without describing every particle?

Macroscopic object have both positive and negative charge, making them almost neutral.

Neutral objects give off electro-static field if e.g. positive and negative particles are pushes apart.

Ideal dipole: Consider the field from a negative and a positive particle very close together:

$$\phi(\mathbf{r}) \approx \frac{\mathbf{d} \bullet \left(\mathbf{r} - \mathbf{r}_{object}\right)}{4\pi\varepsilon_0 \left|\mathbf{r} - \mathbf{r}_{object}\right|^2} \quad , \quad \mathbf{d} = \sum_K q^K \mathbf{r}^K \rightarrow \iiint \rho(\mathbf{r})\mathbf{r}\,dV$$

The field is called a *dipole field* and $\mathbf{d}$ is the *dipole moment*.

Refined description of distant object

The multipole expansion provides a series to describe fields from distant objects (when you move away from the object, the higher order terms gets less important).

The second term in the expansion is the *dipole*.

The **ideal dipole** includes two charged particles with charge $q^1 = -q$ and $q^2 = +q$ at a small distance $\Delta\mathbf{r} = \mathbf{r}^2 - \mathbf{r}^1$. When D approaches zero the potential takes the form

$$\phi(\mathbf{r}) \approx \frac{\mathbf{d} \bullet \left(\mathbf{r} - \mathbf{r}_{dipole}\right)}{4\pi\varepsilon_0 \left|\mathbf{r} - \mathbf{r}_{dipole}\right|^2} \quad , \quad \mathbf{d} = q\Delta\mathbf{r} = \sum_K q^K \mathbf{r}^K \rightarrow \iiint \rho(\mathbf{r})\mathbf{r}\, dV$$

The field is called a *dipole field* and $\mathbf{d}$ is the *dipole moment*.

Multipole expansions

The monopole and the dipole are the two first terms in the so called the *multipole expansion*.

Note that the monopole and dipole fields decayed like

$$\phi_{monopole} \sim \left|\mathbf{r} - \mathbf{r}_{object}\right|^{-1}$$

$$\phi_{dipole} \sim \left|\mathbf{r} - \mathbf{r}_{object}\right|^{-2}$$

Thus at large distances only the monopole is the most important one (unless the monopole moment is zero).

The other terms in the expansion are even smaller.

$$\phi_{n:th-pole} \sim \left|\mathbf{r} - \mathbf{r}_{object}\right|^{-n}$$

Electric multipole moments

The multipole moments are tensors.

- The monopole moment is a 0-tensor (a scalar)

$$Q = \iiint \rho(\mathbf{r})dV$$

- The dipole moment a 1-tensor (a vector)

$$\mathbf{d} = \iiint \rho(\mathbf{r})\mathbf{r}dV \Leftrightarrow d_i = \iiint \rho(\mathbf{r})x_i dV$$

- The next moment is a 2-tensor called the *quadropole moment*:

$$q = \iiint \rho(\mathbf{r})\mathbf{r}\otimes\mathbf{r}dV \Leftrightarrow q_{ij} = \iiint \rho(\mathbf{r})x_i x_j dV$$

The trace of q_{ij} gives no field; alternative: $d_{ij} = 3q_{ij} - q_{ss}\delta_{ij}$

Magnetic multipole moments

Non essential material

Similarly there is a multipole expansion for magnetic fields.

- Magnetic monopoles does *not exist!* (Well, it can't be proved…)
- Magnetic dipole moment:

$$\mathbf{m} = \iiint \mathbf{r} \times \mathbf{J}(\mathbf{r})dV$$

- In general the magnetic moment are *generated* from the series:

$$\begin{cases} \mu_i = \iiint J_i(\mathbf{r})dV \\ \mu_{ij} = \iiint x_i J_j(\mathbf{r})dV \\ \mu_{ijk} = \iiint x_i x_j J_k(\mathbf{r})dV \\ \ldots \\ \mu_{i\ldots nm} = \iiint x_i \ldots x_n J_m(\mathbf{r})dV \end{cases}$$

  - Example: $m_i = \frac{1}{2}\varepsilon_{ijk}\mu_{jk}$

Relation between electric and magnetic multipole moments

The charge and the current are related by the charge continuity equation. The first $\mathbf{r}$-moment of this equation reads:

$$\iiint x_i \left(\frac{\partial \rho(\mathbf{r})}{\partial t} + \frac{\partial}{\partial x_k} J_k(\mathbf{r}) \right) dV = 0$$

$$\frac{\partial}{\partial t} \iiint x_i \rho(\mathbf{r}) dV = -\iiint x_i \frac{\partial}{\partial x_k} J_k(\mathbf{r}) dV = \ldots = \iiint J_i(\mathbf{r}) dV$$

Identify the electric and magnetic moments:

$$\frac{\partial}{\partial t} d_i = \mu_i$$

Similar relations exists also for higher order moments.

The scalar potential from multipoles

The scalar potential from the multipoles is most easily described in the Coulomb gauge:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \iiint dV' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} =$$

$$= \frac{1}{4\pi\varepsilon_0} \iiint dV' \rho(\mathbf{r}') \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\mathbf{r}' \bullet \frac{\partial}{\partial \mathbf{r}} \right)^k \frac{1}{r}$$

$$= \frac{Q}{4\pi\varepsilon_0 r} - \mathbf{d} \bullet \nabla \frac{1}{4\pi\varepsilon_0 r} + \frac{1}{2} q_{ij} \frac{\partial^2}{\partial x_j \partial x_j} \frac{1}{4\pi\varepsilon_0 r} + \ldots$$

$$= \frac{Q}{4\pi\varepsilon_0 r} + \frac{\mathbf{d} \bullet \mathbf{r}}{4\pi\varepsilon_0 r^3} + \frac{q_{ij}\left(3x_j x_j - r^2 \delta_{ij}\right)}{4\pi\varepsilon_0 r^5} + \ldots$$

The vector potential from multipoles

Similarly for the vector potential:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint dV' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} =$$

$$= \frac{\mu_0}{4\pi} \iiint dV' \mathbf{J}(\mathbf{r}') \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\mathbf{r}' \bullet \frac{\partial}{\partial \mathbf{r}} \right)^k \frac{1}{r} =$$

$$= \frac{\mu_0}{4\pi} \sum_{k=0}^{\infty} \mu_{i \ldots k} \frac{1}{k!} \left(\frac{\partial^k}{\partial x_i \ldots \partial x_k} \right) \frac{1}{r}$$

Fields from higher order moments

For monopole and dipoles, the fields can fairly easily be described in cartesian coordinates. The higher order moments are usually simpler to handle in spherical coordinates.

In spherical coordinates the multipoles can be written using spherical harmonic functions $Y_l^m(\theta,\varphi)$; the same functions used to describe the orbitals of the hydrogen atom.

The full expansion for the scalar potential

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{d_l^m Y_l^m(\theta,\varphi)}{r^{l+1}}$$

Here d_l^m is given in the book, eq. (3.37)

Spherical harmonics