Fourier transforms, Generalised functions and Greens functions

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Motivation

- A big part of this course concerns waves, oscillators, damping/growth
  - Plane waves:

$$f(\mathbf{x},t) = \hat{f} \exp\left[i\mathbf{k}\cdot \mathbf{x} - i\omega t\right]$$

  - Example for a growing/damped wave:

$$f(\mathbf{x},t) = \hat{f} \exp\left[i\mathbf{k}\cdot \mathbf{x} - i\omega t \pm \gamma t\right]$$

  - NOTE: growing and damped waves become infinity when $t \to \pm\infty$
- Fourier transforms may be used to describing plane waves
  - But it require special care (explained later)!
- For damped and growing waves Fourier transforms may not exist!
  - Instead Laplace transforms can sometimes be used
- In this lecture we will…
  - Study Fourier and Laplace transforms; focus on waves and oscillators

Outline

- Fourier transforms
  - Fourier's integral theorem
  - Truncations and generalised functions
  - Plemej formula
- Laplace transforms and complex frequencies
  - Theorem of residues
  - Causal functions
  - Relations between Laplace and Fourier transforms
- Greens functions
  - Poisson equation
  - d' Alemberts equation
  - Wave equations in temporal gauge
- Self-study: linear algebra and tensors

What functions can be Fourier transformed?

- The Fourier integral theorem:
  - *f(t)* is sectionally continuous over $-\infty < t < \infty$
  - *f(t)* is defined as $f(t) = \lim_{\delta \to 0} \frac{1}{2}\left[f(t+\delta) + f(t-\delta)\right]$
  - *f(t)* is *amplitude integrable*, that is, $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$

  Then the following identity holds:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z) e^{\pm iy(z-t)} dz dy$$

- For which of the following functions does the above theorem hold?

$$f(t) = 1$$

$$f(t) = \cos(t)$$

$$f(t) = \begin{cases} 0 \ , & t < 0 \\ \exp(-t) \ , & t \geq 0 \end{cases}$$

$$f(t) = \begin{cases} 0 & , t \text{ is a rational number} \\ \exp(-t^2) & , t \text{ is an irrational number} \end{cases}$$

What functions can be Fourier transformed?

Many commonly used functions are not amplitude integrable, e.g. $f(t)=cos(t)$, $f(t)=exp(it)$ and $f(t)=1$.

Solution: Use approximations of $cos(t)$ that converge asymptotically to $cos(t)$ – details comes later on

- The asymptotic limits of functions like $cos(t)$ will be used to define generalised functions, e.g. Dirac δ-function.

Dirac δ-function

- Dirac's generalised function can be defined as:

$$\delta(x) = \begin{cases} 0\,, & x \neq 0 \\ \infty\,, & x = 0 \end{cases} \qquad \& \qquad \int_{-\infty}^{\infty} \delta(x) = 1$$

Alternative definitions, as limits of well behaving functions, are shown shortly

- Important example:

$$\int_{-\infty}^{\infty} \delta\big(f(t)\big)dt = \sum_{i:f(t_i)=0} \frac{1}{f'(t_i)}$$

Proof: Whenever $|f(t)|>0$ the contribution is zero. For each $t = t_i$ where $f(t_i)=0$, perform the integral over a small region $t_i - \varepsilon < t < t_i + \varepsilon$ (where ε is small such $f(t) \approx (t - t_i)\, f'(t_i)$). Next, use variable substitution to perform the integration in $x = f(t)$, then $dt = dx\,/\,f'(t_i)$:

$$\int_{-\infty}^{\infty} \delta\big(f(t)\big)dt = \sum_{i:f(t_i)=0} \int_{-\infty}^{\infty} \frac{1}{f'(t_i)} \delta(x)dx = \sum_{i:f(t_i)=0} \frac{1}{f'(t_i)}$$

Truncations and Generalised functions

- To approximate the Fourier transform of $f(t)=1$, use *truncation*.

Truncation of a function $f(t)$:

$$f_T(t) = \begin{cases} f(t)\ , & |t| < T \\ 0\ , & |t| > T \end{cases} \text{, such that } f(t) = \lim_{T\to\infty} f_T(t)$$

- Then for $f(t)=1$

$$\mathbf{F}\{f_T(t)\} = \int_{-\infty}^{\infty} f_T(t)e^{-i\omega t}dt = \int_{-T}^{T} 1e^{-i\omega t}dt = \frac{\sin(\omega t/2)}{\omega/2}$$

  - When $T\to\infty$ then this function is zero everywhere except at $\omega=0$ and its integral is 2π, i.e.

$$\mathbf{F}\{1\} = \lim_{T\to\infty} \frac{\sin(\omega T/2)}{\omega/2} = 2\pi\delta(\omega)$$

  - Note: **F**{1} exists *only* as an asymptotic of an ordinary function, i.e. a *generalised function*.

More generalised function

- An alternative to truncation is *exponential decay*

$$f_\eta(t) = f(t)e^{-\eta|t|} \quad \text{, such that} \quad f(t) = \lim_{\eta\to 0} f_\eta(t)$$

- Three important examples:
  - *f(t)=1* (alternative definition of δ-function)

$$\mathbf{F}\{f_\eta(t)\} = \frac{2\pi\eta}{\omega^2+\eta^2} \quad \Longrightarrow \quad \mathbf{F}\{1\} = \lim_{\eta\to 0}\frac{2\pi\eta}{\omega^2+\eta^2} = 2\pi\delta(\omega)$$

  - The sign function sgn*(t)*: $\mathbf{F}\{\mathrm{sgn}(t)\} = \lim_{\eta\to 0}\mathbf{F}\{e^{-\eta|t|}\,\mathrm{sgn}(t)\} = \lim_{\eta\to 0}\frac{2i\omega}{\omega^2+\eta^2} = 2i\wp\left[\frac{1}{\omega}\right]$

    The generalised function is the *Cauchy principal value function:*

$$\wp\frac{1}{\omega} := \lim_{\eta\to 0}\frac{\omega}{\omega^2+\eta^2} = \begin{cases} 1/\omega\ , & \text{for } \omega \neq 0 \\ 0\ , & \text{for } \omega = 0 \end{cases}$$

  - Heaviside function *f(t)=H(t)* : $\mathbf{F}\{H(t)\} = \lim_{\eta\to 0}\frac{i}{\omega+i\eta}$

    This generalised function is often written as: $\frac{1}{\omega+i0} := \lim_{\eta\to 0}\frac{1}{\omega+i\eta}$

Plemelj formula

- Relation between *H(t)* and *sgn(t)*:

$$2H(t) = 1 + \mathrm{sgn}(t)$$

with the Fourier transform:

$$\frac{1}{\omega + i0} = \wp\frac{1}{\omega} - i\pi\delta(\omega)$$

This is known as the *Plemelj formula*

  - Note: How we treat w=0 matters! …but why?
  - We will use the Plemelj formula when describing resonant wave damping (see later lectures)

Driven oscillator with dissipation

- Example of the Plemelj formula: a driven oscillator with eigenfrequency Ω:

$$\frac{\partial^2 f(t)}{\partial t^2} + \Omega^2 f(t) = E(t)$$

with dissipation coefficient ν:

$$\frac{\partial^2 f(t)}{\partial t^2} + 2\nu \frac{\partial f(t)}{\partial t} + \Omega^2 f(t) = E(t)$$

- Fourier transform: $\left(-\omega^2 - i2\nu\omega + \Omega^2\right) f(\omega) = E(\omega)$

- Solution:

$$f(\omega) = \frac{E(\omega)}{-\omega^2 - i2\nu\omega + \Omega^2} = \frac{E(\omega)}{2\hat{\Omega}}\left[\frac{1}{\omega - \hat{\Omega} + i\nu} - \frac{1}{\omega + \hat{\Omega} + i\nu}\right]$$

where $\hat{\Omega} = \sqrt{\Omega^2 - \nu^2}$

- Take limit when damping ν goes to zero:

$$f(\omega) = \frac{E(\omega)}{2\Omega}\left[\frac{1}{\omega - \Omega + i0} - \frac{1}{\omega + \Omega + i0}\right]$$

use Plemelj formula

Later we'll look at the inverse transform

$$f(\omega) = \frac{E(\omega)}{2\Omega}\left[\wp\left(\frac{1}{\omega - \Omega}\right) - \wp\left(\frac{1}{\omega + \Omega}\right) - i\pi\delta(\omega - \Omega) + i\pi\delta(\omega + \Omega)\right]$$

Physics interpretation of Plemej formula

- For oscillating systems:
  eigenfrequency Ω will appear as *resonant denominator*

$$f(\omega) \sim \frac{1}{\omega \pm \Omega} \quad \Leftrightarrow \quad f(t) \sim e^{\pm i\Omega t}$$

Including infinitely small dissipation and applying Plemelj formula

$$\frac{1}{\omega - \Omega + i0} = \wp \frac{1}{\omega - \Omega} - i\pi\delta(\omega - \Omega)$$

- Later lectures on the dielectric response of plasma:
  When the dissipation goes to zero for a kinetic plasma there is still a wave damping called Landau damping, a “collisionless” damping, which comes from the δ-function

$$\text{"}damping\text{"} \sim i\pi\delta(\omega - \Omega)$$

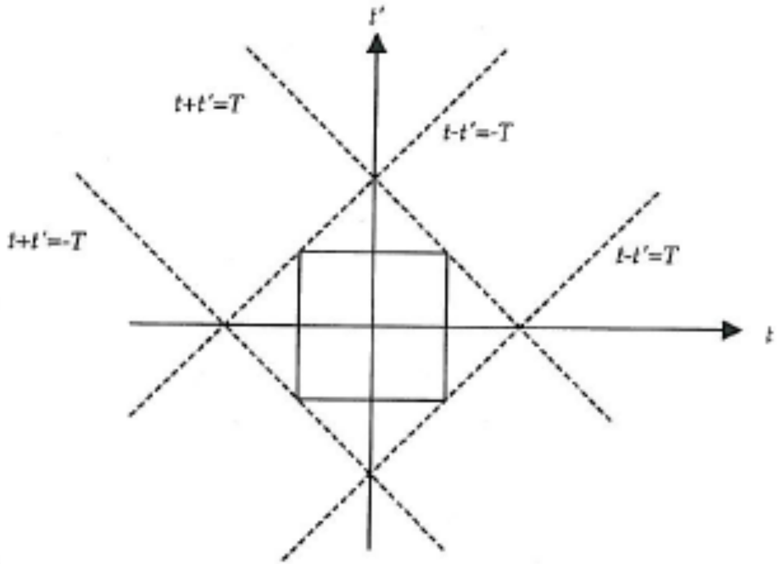
Square of δ-function

- To evaluate square of δ-function

$$\left[2\pi\delta(\omega)\right]^2 = \int_{T/2}^{T/2} dt e^{i\omega t} \int_{T/2}^{T/2} dt' e^{i\omega t'}$$

$$= \int_{T}^{T} d(t-t') \int_{T}^{T} d(t+t') e^{i\omega(t+t')}$$

$$= T 2\pi\delta(\omega)$$



  - Thus also the integral of the δ^2 goes to infinity as $T \rightarrow \infty$!
  - Luckily, in practice you usually find δ^2 in the form δ^2 / T , which is integrable!

In the first exercise class on Tuesday we will do example.

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Laplace transforms and complex frequencies (Chapter 8)

- Fourier transform is restricted to handling real frequencies, i.e. not optimal for damped or growing modes
  - For this purpose we need the Laplace transforms, which allow us to study complex frequencies.
- To understand better the relation between Fourier and Laplace transforms we will first study the residual theorem and see it applied to the Fourier transform of causal functions.

The Theorem of Residues

- Expand $f(z)$ around singularity, $z=z_i$:

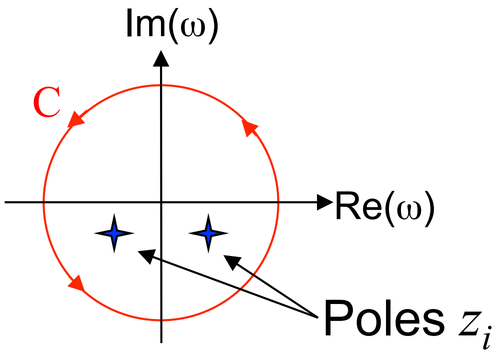
$$f(z) \approx \frac{R_i}{(z-z_i)} + c_0 + c_1(z-z_i) + \dots$$

  - the point $z=z_i$ is called a *pole*
  - the numerator R_i is the *residue*

- The integral along closed contour in the complex plane can be solved using *the theorem of residues*

$$\int_C f(z)dz = 2\pi i \sum_i R_i$$

$$R_i = \lim_{z \to z_i} (z-z_i) f(z)$$



  - where the sum is over all poles z_i inside the contour

Example: Theorem of Residues

- **Example**: $f(z)=1/z$ and C encircling a poles at $z=0$

$$\int_C f(z)dz = \int_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{re^{i\theta}} ire^{i\theta} d\theta = \int_0^{2\pi} i d\theta = 2\pi i$$

$$where \quad z = re^{i\theta} \quad dz = ire^{i\theta} d\theta$$

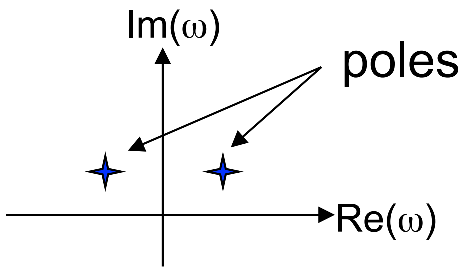
Causal functions

- ***Causal functions***:
  functions f_c that "start" at $t=0$, such that $f_c(t)=0$ for $t<0$.

- **Example**: causal damped oscillation $f_c(t) = e^{-\gamma t} \cos(\Omega t)$, for $t>0$

$$\mathbf{F}\{f_c(t)\} = \int_0^\infty e^{-iwt} e^{-\gamma t} \cos(\Omega t) dt = \frac{i}{2}\left[\frac{1}{\omega - \Omega - i\gamma/2} + \frac{1}{\omega + \Omega - i\gamma/2}\right]$$

  - The two denominators are poles in the complex ω plane
  - Both poles are in the upper half of the complex plane $\mathrm{Im}(\omega)<0$



- Causal function are suitable for Laplace transformations
  - to better understand the relation between Laplace and Fourier transforms; study the *inverse* Fourier transform of the causal damped oscillator

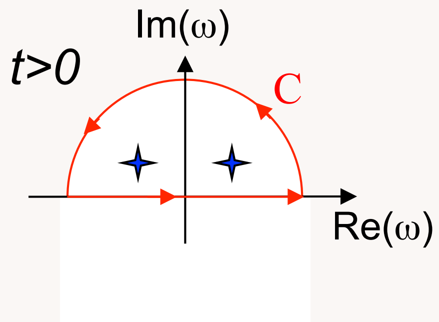
Causal functions and contour integration

- Use Residual analysis for inverse Fourier transform of the causal damped oscillation

$$\mathbf{F}^{-1}\{f_c(t)\} = \frac{1}{2\pi}\int d\omega\, e^{i\omega t}\left\{\frac{i}{2}\left[\frac{1}{\omega-\Omega-i\gamma/2}+\frac{1}{\omega+\Omega-i\gamma/2}\right]\right\}$$

For $t > 0$:

- For $\mathrm{Im}(\omega) \to \infty$, then $e^{i\omega t} \to 0$ and $\lim_{|\omega|\to\infty} \tilde{f}_c(\omega) \sim 1/\omega \to 0$
- Thus, close contour with half circle $\mathrm{Im}(\omega)>0$
- Inverse Fourier transform is sum of *residues* from poles

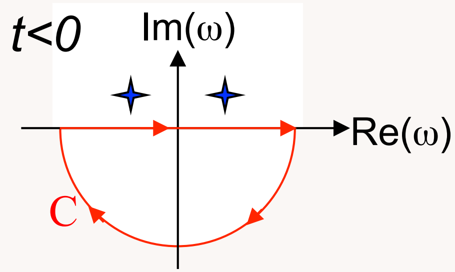


$$f_c(t) = \frac{1}{2\pi}\int_C e^{i\omega t}\frac{i}{2}\left[\frac{1}{\omega-\Omega+i\gamma/2}+\frac{1}{\omega+\Omega+i\gamma/2}\right]d\omega$$

$$= -\sum_i iR_i = -i\frac{i}{2}\left[e^{(i\Omega-\gamma/2)t}+e^{(-i\Omega-\gamma/2)t}\right]$$

For $t < 0$:

- $e^{i\omega t} \to 0$, for $\mathrm{Im}(\omega) \to -\infty$;
  close contour with half circle $\mathrm{Im}(\omega)<0$
- No poles inside contour: $f(t)=0$ for $t<0$



Laplace transform

- Laplace transform of function *f(t)* is

$$F(s) = L\left\{f(t)\right\} = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\infty} e^{-st} f(t) dt$$

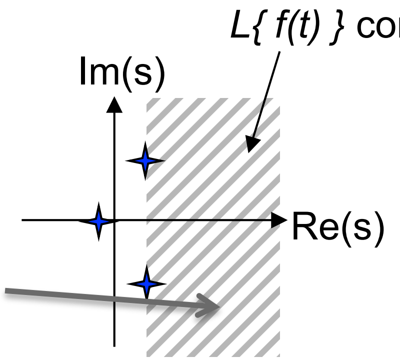
  - Like a Fourier transform for a causal function, but $i\omega \to s$.

- Region of convergence:
  - Note: For $\mathrm{Re}(s)<0$ the integral may not converge since the factor e^{-st} diverges
  - Consider function $f(t) = e^{\nu t} \Rightarrow F(s) = \int_0^{\infty} e^{(\nu - s)t} dt$

    $F(s)$ is integrable only if $\mathrm{Re}(s) > \mathrm{Re}(\nu)$

    > Thus, the Laplace transform is only valid for
    > $\mathrm{Re}(s) > \mathrm{Re}(\nu)$

    Note: $f(t) = e^{\nu t}$ means pole at $s=\nu$, i.e.
    poles must be to the right of the *region of convergence*

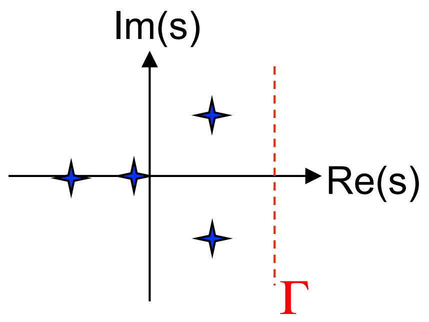


  - Laplace transform allows studies of unstable modes; $e^{\gamma t}$!

Laplace transform

- Laplace transform

$$F(s) = L\{f(t)\} = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\infty} e^{-st} f(t) dt$$



- For causal function the inverse transform is:

$$f(t) = L^{-1}\{F(s)\} = \int_{\Gamma - i\infty}^{\Gamma + i\infty} e^{st} F(s) ds$$

  - Here the parameter Γ should be in the ***region of convergence***, i.e. chosen such that all poles lie to the ***right*** of the integral contour Re(s)=Γ.

  - **Causality**: since all poles lie right of integral contour, $L^{-1}\{ f(s) \}(t)=0$, for $t<0$.
    - Proof: see inverse Fourier transform fo causal damped harmonic oscillator (Hint: close contour with semicircle Re(s)>0)

  - Thus, **only for causal function** is there an inverse $f(t) = L^{-1}\{L\{f(t)\}\}$

- Again: Laplace transform allows studies of unstable modes; $e^{\gamma t}$!

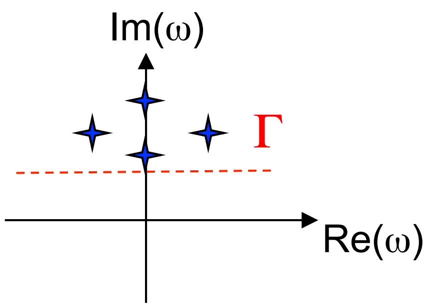
Complex frequencies

- Formulas for Laplace and Fourier transform very similar
  - Laplace transform for *complex* growth rate s / Fourier for *real* frequencies ω
  - For causal function, Laplace transform is more powerful
  - For causal function, Fourier transforms and Laplace transforms are similar!

- Let $s=i\omega$; provides alternative formulation of the Laplace transform for causal $f(t)$

$$\hat{F}(\omega) = L\{f(t)\} = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{+\infty} e^{-i\omega t} f(t)dt = \int_{-\infty}^{+\infty} e^{-i\omega t} f(t)dt$$

- Here ω is a *complex frequency*
- The inverse transform for causal functions is



$$f(t) = L^{-1}\{\hat{F}(\omega)\} = \int_{-i\Gamma-\infty}^{-i\Gamma+\infty} e^{i\omega t} \hat{F}(\omega)dt$$

  - for decaying modes all poles are above the real axis and $\Gamma=0$.
- **Thus, the Laplace and Fourier transforms are the same for amplitude integrable causal function, but only the Laplace transform is defined for complex frequencies.**

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Greens functions (Chapter 5)

- **Greens functions**: technique to solve *inhomogeneous* equations
- Linear differential equation for f given source S:

$$L(z)f(z) = S(z)$$

  - Where the differential operator L is of the form:

$$L = A_n \frac{d^n}{dz^n} + A_{n-1}\frac{d^{n-1}}{dz^{n-1}} + \ldots A_0$$

- Define Greens function G to solve:

$$L(z)G(z,z') = \delta(z-z')$$

  - the response from a point source – e.g. the fields from a particle!
- **Ansatz**: given the Greens function, then there is a solution:

$$f(z) = \int G(z,z')S(z')dz'$$

- **Proof**:

$$L(z)f(z) = \int L(z)G(z,z')S(z')dz' = \int \delta(z-z')S(z')dz' = S(z)$$

How to calculate Greens functions?

- For differential equations without explicit dependence on z, then

$$L(z) = L(z - z')$$

  - we may rewrite G as:

$$G(z,z') \rightarrow G(z - z')$$

- Fourier transform from z-z' to k :

$$L(z - z')G(z - z') = \delta(z - z')$$

$$\Downarrow$$

$$L(ik)G(k) = 1$$

$$\Downarrow$$

$$G(k) = \frac{1}{L(ik)}$$

Example:

$$\left(\frac{\partial^2}{\partial t^2} + \Omega^2\right)G(t - t') = \delta(t - t')$$

$$\Downarrow$$

$$\left(-\omega^2 + \Omega^2\right)G(\omega) = 2\pi$$

$$\Downarrow$$

$$G(\omega) = -\frac{2\pi}{\omega^2 - \Omega^2}$$

- Inverse Fourier transform

$$G(z - z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(k)e^{-ik(z-z')}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{L(ik)}e^{-ik(z-z')}dk$$

Solve integral!

Greens function for the Poisson's Eq. for static fields

- Poisson's equation

$$-\varepsilon_0 \nabla^2 \phi(\mathbf{x}) = \rho(\mathbf{x})$$

- Green's function $-\varepsilon_0 \nabla^2 G(\mathbf{x}-\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$

$$-\varepsilon_0 \left|\mathbf{k}^2\right| G(\mathbf{k}) = 1$$

$$G(\mathbf{x}-\mathbf{x}') = \frac{1}{(2\pi)^3 \varepsilon_0} \int \frac{\exp\left[i\mathbf{k} \bullet (\mathbf{x}-\mathbf{x}')\right]}{\left|\mathbf{k}^2\right|} d^3\mathbf{k}$$

$$G(\mathbf{x}-\mathbf{x}') = \frac{1}{4\pi\varepsilon_0 \left|\mathbf{x}-\mathbf{x}'\right|}$$

- Thus, we obtain the familiar solution; a sum over all sources

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}$$

Greens Function for d'Alembert's Eq. (time dependent field)

- D'Alembert's Eq. has a Green function $G(t,x)$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(t-t',\mathbf{x}-\mathbf{x}') = \mu_0 \delta(t-t')\delta^3(\mathbf{x}-\mathbf{x}')$$

- Fourier transform $(t-t',\mathbf{x}-\mathbf{x}') \rightarrow (\omega,\mathbf{k})$ gives

$$\left(\frac{\omega^2}{c^2} - |\mathbf{k}|^2\right) G(\omega,\mathbf{k}) = \mu_0$$

$$G(\omega,\mathbf{k}) = \frac{-\mu_0}{\omega^2/c^2 - |\mathbf{k}|^2}$$

$$G(t-t',\mathbf{x}-\mathbf{x}') = \frac{\mu_0}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta(t-t'-|\mathbf{x}-\mathbf{x}'|/c)$$

  - Information is propagating radially away from the source at the speed of light

Greens Function for the Temporal Gauge

- Temporal gauge gives different form of wave equation

$$\frac{\omega^2}{c^2}\mathbf{A}(\omega,\mathbf{k}) + \mathbf{k}\times\mathbf{k}\times\mathbf{A}(\omega,\mathbf{k}) = -\mu_0\mathbf{J}(\omega,\mathbf{k})$$

$$\left[\left(\frac{\omega^2}{c^2} - |\mathbf{k}|^2\right)\delta_{ij} + k_i k_j\right] A_j(\omega,\mathbf{k}) = -\mu_0 J_i(\omega,\mathbf{k})$$

  - Different response in *longitudinal* : $\mathbf{k}\cdot\mathbf{A}(\omega,\mathbf{k}) = -\frac{\mu_0 c^2}{\omega^2}\mathbf{k}\cdot\mathbf{J}(\omega,\mathbf{k})$
  - and *transverse* directions: $\mathbf{k}\times\mathbf{A}(\omega,\mathbf{k}) = -\frac{\mu_0}{\omega^2/c^2 - |\mathbf{k}|^2}\mathbf{k}\times\mathbf{J}(\omega,\mathbf{k})$

- To separate the longitudinal and transverse parts the Greens function become a 2-tensor G_{ij}

$$\left[\left(\frac{\omega^2}{c^2} - |\mathbf{k}|^2\right)\delta_{ij} + k_i k_j\right] G_{jl}(\omega,\mathbf{k}) = -\mu_0\delta_{il}$$

- Solution has poles $\omega = \pm |\mathbf{k}|\, c$:

$$G_{ij}(\omega,\mathbf{k}) = -\frac{-\mu_0}{\omega^2/c^2 - |\mathbf{k}|^2}\left(\delta_{ij} + \frac{\omega^2}{c^2}k_i k_j\right)$$

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Self-study: Linear algebra

- The inner product

$$\mathbf{a} \bullet \mathbf{b} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_3 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{j=1}^{3} a_j b_j \equiv a_j b_j$$

  – The repeated indexed are called "dummy" indexes

Einsteins summation convention:
"always sum over repeated indexes"

- The outer product

$$\mathbf{b} \otimes \mathbf{a} = \begin{bmatrix} b_1 \\ b_3 \\ b_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_2 b_1 & a_3 b_1 \\ a_1 b_2 & a_2 b_2 & a_3 b_2 \\ a_1 b_3 & a_2 b_3 & a_3 b_3 \end{bmatrix}$$

  – Express **a** and **b** in a basis [$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$]

$$\mathbf{b} \otimes \mathbf{a} = \sum_{i=1}^{3} a_i \mathbf{e}_i \otimes \sum_{j=1}^{3} b_j \mathbf{e}_j = b_j a_i \left[\mathbf{e}_i \otimes \mathbf{e}_j \right]$$

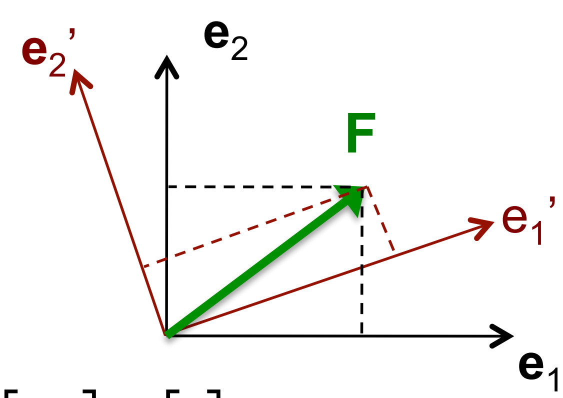
Note: 9 terms

  – e.g.

$$\left[\mathbf{e}_2 \otimes \mathbf{e}_3 \right] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Self-study: Vectors

- Vectors are defined by a **length** and a **direction**.
  – Note that the **direction is independent** of the coordinate system, thus the **components depend** on the coordinate system



$$\mathbf{F} = F_i \mathbf{e}_i = F_i' \mathbf{e}_i'$$

  – thus in the (x, y) systems the components may be: $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

  – then for 30 degrees between the coordinate systems (u, v) components are: $\begin{bmatrix} F_1' \\ F_2' \end{bmatrix} = \begin{bmatrix} \sqrt{1/2} \\ \sqrt{3/2} \end{bmatrix}$

- The relation between vectors are given by transformation matrixes
  – if transformation is a rotation then transformation matrixes

$$F_i' = R_{ij} F_j \quad ; \quad [\mathrm{R}_{\mathrm{ij}}] \equiv \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} \cos(v) & -\sin(v) \\ \sin(v) & \cos(v) \end{bmatrix}$$

Self-study: Tensors

- Tensors are also independent of coordinate system
- Examples:
  - A **scalar** is a tensor of *order zero*.
  - A **vector** is a tensor of *order one*.
- Tensors of *order two* in 3d space has 3 directions and 3 magnitudes
  - For a given coordinate system a tensor T of order two (or a 2-tensor) can be represented by a matrix

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \equiv T_{ij}\mathbf{e}_i\mathbf{e}_j \quad ; \quad \left[T_{ij}\right] \equiv \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

- Transformation of 2-tensors
  - Transformation the basis: $\mathbf{F} = F_i\left[R_{ik}\right]\left[R_{km}\right]^{-1}\mathbf{e}_m = F_k'\left[R_{km}\right]^{-1}\mathbf{e}_m \equiv F_k'\mathbf{e}_k'$

$$\mathbf{e}_i' \equiv \left[R_{\ln}\right]^{-1}\mathbf{e}_j \quad ; \quad \mathbf{T} = T_{ij}\mathbf{e}_i\mathbf{e}_j$$

$$= T_{ij}\left[R_{ik}\right]\left[R_{km}\right]^{-1}\mathbf{e}_m\left[R_{jl}\right]\left[R_{\ln}\right]^{-1}\mathbf{e}_n \quad \Longrightarrow \quad T_{ij}' = R_{ik}T_{kl}R_{jl}$$

$$= \left[R_{ik}\right]T_{ij}\left[R_{jl}\right]\mathbf{e}_k'\mathbf{e}_l' \equiv T_{kl}'\mathbf{e}_k'\mathbf{e}_l'$$