Homework 3

Finite Differences and Absolute Stability

due February 9, 2015

Task 1: Finite Difference Scheme

Find the highest order approximation possible of the first derivative based on the grid values u_{i-2}, u_{i-1}, u_i and u_{i+1}. Assume equidistant grid spacing Δx.

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_i} \approx f(u_{i-2}, u_{i-1}, u_i, u_{i+1})$$

a) Give the approximation for the derivative.

b) What is the leading error term? What is the order of this scheme?

c) Implement this scheme for the approximation of the derivative in a similar way as you did in Task 2a) of Homework 2 and numerically assess the order of its accuracy. Note that you do not need to compute truncation and round-off errors separately, just the global order of accuracy is required.

Task 2: Stability Criterion

The range of absolute stability of the Runge-Kutta 4^{th}-order method is studied. This time-stepping method for an initial value problem of the form $\frac{\mathrm{d}u}{\mathrm{d}t} = f(u,t),\ u(t_0) = u_0$ is:

$$\begin{aligned} u^{n+1} &= u^n + \frac{\Delta t}{6}(f^n + 2k_1 + 2k_2 + k_3) \\ t^n &= n\Delta t \end{aligned}$$

where

$$\begin{aligned} f^n &= f(u^n, t^n) \\ k_1 &= f(u_1, t^{n+\frac{1}{2}}), \quad u_1 = u^n + \frac{\Delta t}{2} f^n, \quad t^{n+\frac{1}{2}} = t^n + \frac{\Delta t}{2} \\ k_2 &= f(u_2, t^{n+\frac{1}{2}}), \quad u_2 = u^n + \frac{\Delta t}{2} k_1 \\ k_3 &= f(u_3, t^{n+1}), \quad u_3 = u^n + \Delta t k_2 \end{aligned}$$

Consider a simple linear test equation (Dahlquist equation):

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u.$$

Show that u^{n+1} can be written as a function of u^n and $z = \Delta t \lambda$ as follows

$$u^{n+1} = u^n \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}\right).$$

The absolute stability criterion is given by

$$|G(z)| = \left|\frac{u^{n+1}}{u^n}\right| \leq 1. \tag{1}$$

Draw the region that correspond to equation (1) on the complex $z-$plane.
(Hint: The curve $|G(z)| = 1$ cuts the imaginary axis at ± 2.83)

Task 3: The Modified Wavenumber

On an equispaced grid, the finite-difference derivative of a Fourier mode e^{ikx} can be found by multiplying the function value on each node with the so-called modified wavenumber $\tilde{k}(k)$.
To better understand this concept consider a periodic function

$$f(x) : \mathbb{R} \to \mathbb{C}, \qquad f(x + 2\pi) = f(x), \qquad \forall x.$$

Let $\underline{f}$ be the discrete representation of $f(x)$ on an equidistant grid where $x_j = j\Delta x$, $\Delta x = 2\pi/N$, $j = 0, 1, \ldots, N-1$ with $N = 20$,

$$\underline{f} = [f_0, f_1, \ldots, f_{N-1}]^T \quad \text{where } f_j = f(x_j).$$

a) Write a MATLAB script that computes the matrix $\underline{\underline{D}}$ corresponding to right-sided finite differences of first order. The matrix $\underline{\underline{D}}$ is defined as:

$$\underline{f}'_{num} = \underline{\underline{D}}\ \underline{f}$$

where the vector $\underline{f}'_{num}$ is

$$\underline{f}'_{num} = [\delta f_0, \delta f_1, \ldots, \delta f_{N-1}]^T,$$

and the operator δf_j is

$$\delta f_j = \frac{f_j - f_{j-1}}{\Delta x}.$$

Remember that $f(x)$ is periodic when computing the derivative at the point $x = x_{N-1}$, *i.e.* $f_N = f_0$.

b) Consider $f(x) = e^{ikx}$ and derive the expression for the modified wavenumber $\tilde{k}$ for the right-sided finite-difference scheme. Non-dimensionalise the wavenumber with the grid spacing, *i.e.* derive $\tilde{k}\Delta x(k\Delta x)$.

c) From now on assume that $k = 5$ (*i.e.* consider a specific wave). Compute the derivative in a discrete (δf_j) and analytical ($f'_{x=x_j}$) manner at every grid point. Use the previously defined $\underline{\underline{D}}$ for the discrete derivative. Plot the real part for both the numerical and analytical derivative as a function of x.

d) Compute the vector μ with the elements

$$\mu_j = \frac{\delta f_j}{f_j}$$

and compare it with the complex number $i\tilde{k}$, where $\tilde{k}$ is the modified wavenumber for the right-sided finite differences as derived in b). Use $k = 5$ and $\Delta x = 2\pi/N$ ($N = 20$).
Does this result confirm that the finite-difference derivative of a Fourier mode e^{ikx} can be found by multiplying the function by the modified wave number? *i.e.* does $i\tilde{k}\underline{f} = \underline{\underline{D}}\ \underline{f}$ hold?