Wave Response of Ideal Media

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Questions

- What is meant by the electromagnetic respons of a media?
- What is meant by temporal dispersion, spatial dispersion and anisotropy?
- Under what conditions do these three effects appear?

Discuss in groups of 2-3 people.

Overview

- Introduction to the concept of a response
- First example: Response of electron gas
  - Changing the speed of light
  - Dispersion
- Polarization of atoms and molecules (brief)
  - Response of crystals
- Medium of oscillators
  - Detailed study of the resonance region
    - Hermitian / antihermitian parts of the dielectric tensor
    - Application of the Plemej formula
- Dielectric response for plasmas
  - Magnetoionic theory (anisotropic/gyrotropic)
  - Cold plasmas (Alfven velocity)
  - Warm plasmas (Landau damping)

What do we mean by dielectric response?

- When an electromagnetic wave passes through a media, e.g. air, water, copper, a crystal or a plasma, then:
  - The electromagnetic fields exert a force on the particles of the media
  - The force may then "pull" the particles to induce
    - charge separation ρ ⇒ drive $\mathbf{E}$-field in Poisson's equation

$$\nabla \bullet \mathbf{E}_{media} = \rho_{media} / \varepsilon_0$$

      - $\mathbf{E}$-field is coupled to the $\mathbf{B}$-field through Maxwells equations
    - currents $\mathbf{J}$ ⇒ drive $\mathbf{E}$- & $\mathbf{B}$-fields through Ampere's law

$$\nabla \times \mathbf{B}_{media} - \frac{1}{c^2} \frac{\partial \mathbf{E}_{media}}{\partial t} = \mu_0 \mathbf{J}_{media}$$

  - The fields induced by the media are called the *dielectric response*
  - The total fields are:

$$\mathbf{E} = \mathbf{E}_{external} + \mathbf{E}_{media}$$

$$\mathbf{B} = \mathbf{B}_{external} + \mathbf{B}_{media}$$

See previous lecture for representation in terms of:
- Polarization P
- Magnetization M

Equations for calculating the dielectric response

E- & B-field exerts a force on the particles in media

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \textit{solve for } \mathbf{v}\,!$$

The induced motion of charge particles form a current and a charge density

$$\mathbf{J}_{media} = \sum_{species} qn\mathbf{v} \qquad \frac{\partial}{\partial t}\rho_{media} + \nabla \circ \mathbf{J}_{media} = 0$$

(n=particle density)

The **response** can be quantified by the conductivity σ

$$J_i(\mathbf{k},\omega) = \sigma_{ij}(\mathbf{k},\omega) E_j(\mathbf{k},\omega)$$

Current and charge drive the **electromagnetic response**

$$\nabla \bullet \mathbf{E}_{media} = \rho_{media} / \varepsilon_0$$

$$\nabla \times \mathbf{B}_{media} - \frac{1}{c^2}\frac{\partial \mathbf{E}_{media}}{\partial t} = \mu_0 \mathbf{J}_{media}$$

Response of *electron gas* to oscillating E-field

Example: Consider electron response to electric field oscillations (e.g. high frequency, long wave length waves in a plasma)

- Align x-axis with the electric field: $\mathbf{E}(t) = \mathbf{e}_x E_x(t)$
- Electron equation of motion:

$$m\ddot{x}(t) = qE_x(t) \quad \Rightarrow \quad x(\omega) = -\frac{q}{m\omega^2} E_x(\omega)$$

- The current driven in the medium (let n be the electron density)

$$J_x(t) \equiv qn\dot{x}(t) \quad \Rightarrow \quad J_x(\omega) = i\frac{q^2 n}{m\omega} E_x(\omega)$$

- Thus we have derived the conductivity of this media

$$\sigma(\omega) = i\frac{nq^2}{m\omega}$$

- Here: $\sigma \sim 1/\omega$, means that the media is *dispersive*!

Response of *electron gas* to oscillating E-field (2)

- This media is ***isotropic*** (the same response in all directions)
  - **Proof 1**: rotate E-field to align with y-axis or z-axis and repeat calculation
  - **Proof 2**: use argument that the medium have no "intrinsic direction" (there is no static the magnetic field, no structure like in a crystal, or similar), thus the media have to be isotropic
  - Being an isotropic media the components of the conductivity tensor are:

$$\sigma_{ij}(\omega) = \sigma(\omega)\delta_{ij} \quad \Rightarrow \quad \sigma_{ij}(\omega) = i\frac{q^2 n}{m\omega}\delta_{ij} \equiv i\varepsilon_0 \frac{\omega_p^2}{\omega}\delta_{ij}$$

where ω_p is known as the *plasma frequency*: $\omega_p^2 \equiv \dfrac{nq^2}{\varepsilon_0 m}$

- Relations to:
  - susceptibility: $$\chi_{ij}(\omega) \equiv \frac{i}{\varepsilon_0\omega}\sigma_{ij}(\omega) = \frac{i\sigma(\omega)}{\varepsilon_0\omega}\delta_{ij} = -\frac{\omega_p^2}{\omega^2}\delta_{ij}$$
  - polarisation response: $$\alpha_{ij}(\omega) \equiv i\omega\sigma_{ij}(\omega) = -\varepsilon_0\omega_p^2\delta_{ij}$$
  - dielectric tensor: $$K_{ij}(\omega) \equiv \delta_{ij} + \chi_{ij}(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\delta_{ij}$$

Application of response

- How does the electron response affect the propagation of waves?
  - Consider: high frequency, long wave length waves in a plasma
    - then response tensor from previous page is valid (more details later)
- Split currents into antenna current J_{ant} and the current induced in the media J_{media}. Then Amperes and Faradays equations give:

$$\mathbf{k}\times\mathbf{k}\times\mathbf{E}+\frac{\omega^2}{c^2}\mathbf{E}+i\mu_0\omega\mathbf{J}_{media}=-i\mu_0\omega\mathbf{J}_{ant}$$

Note: total field E driven by both J_{media} and J_{ant}

- Use the conductivity $\sigma=i\varepsilon_0\frac{\omega_p^2}{\omega}$ of the media:

$$\frac{\omega^2}{c^2}\mathbf{E}+i\mu_0\omega\mathbf{J}_{media}=\frac{\omega^2}{c^2}\mathbf{E}+i\mu_0\omega\sigma\mathbf{E}=\omega^2\underbrace{\frac{1}{c^2}\left(1-\frac{\omega_p^2}{\omega^2}\right)}_{1/c_m^2}\mathbf{E}$$

$$\Rightarrow\quad\mathbf{k}\times\mathbf{k}\times\mathbf{E}+\frac{\omega^2}{c_m^2}\mathbf{E}=-i\mu_0\omega\mathbf{J}_{ant}$$

i.e. a wave equation with speed of light: $c_m^2=c^2\left(1-\frac{\omega_p^2}{\omega^2}\right)^{-1}$

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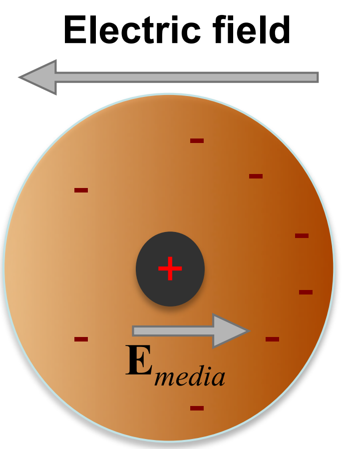
Polarization of atoms and molecules

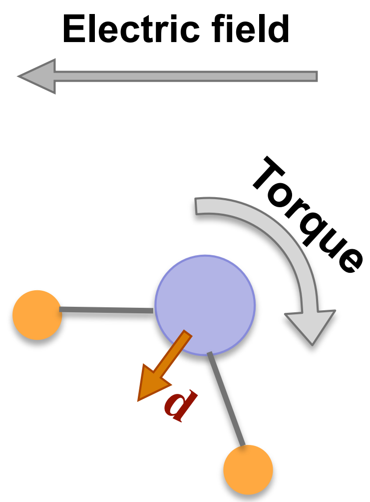
- The polarization of an atom (requires quantum mechanics)
  - The electric field pushes the electrons, inducing a charge separation;
  - Quantum mechanically: perturbs the eigenfunctions (orbitals): $\psi^{(0)} \rightarrow \psi^{(0)}+\psi^{(1)}$

$$\psi_q^{(1)} = \sum a_{qq'}\psi_{q'}^{(0)} \rightarrow J_{media}^{(1)} \;\&\; \rho_{media}^{(1)}$$

  - The field induced by the media is opposite to the total field



- The polarization of a water molecule
  - Water molecules, dipole moment **d**
  - The electric field induce a *torque* that turns it to reduce the total field
  - *Note*: the electron eigen-states of the molecules are also perturbed, like in the atom



Uniaxial crystals

- In solids the response, or electron mobility, is determined by the
  - **Metals**: the valence electron give rapid response
  - **Insulators**: electrons orbitals are bound to a single atom or molecule
- Uniaxial crystals: have an optical axis; e.g. the normal $\hat{n}$ to a sheeth structure
- Stronger bonds within then between the sheeths
  - Graphite: valence electrons are shared only within a sheeth
  - electron ***mobility*** (response) is different within and perpendicular to the sheeths
  - The crystal is ***anisotropic***
- Let the normal to the crystal be in the z-direction (as in figure)

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_\perp & 0 & 0 \\ 0 & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{bmatrix}$$

Graphite

$$\begin{cases} \sigma_\parallel = 2.5 - 5.0 \times 10^{-6} \\ \sigma_\perp = 3 \times 10^{-3} \end{cases}$$

- *Example*: slight birefrigence in optical fibres can cause *modal dispersion*

Biaxial crystals

- Uniaxial crystals has symmetric plane, in which the electron mobility is constant
- Biaxial crystals have no symmetry plane
  - Instead they have different conductivity in all three directions

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_\alpha & 0 & 0 \\ 0 & \sigma_\beta & 0 \\ 0 & 0 & \sigma_\gamma \end{bmatrix}$$

  - When expressed in terms of the dielectric tensor one may introduce three refractive indexes of the media

$$\left[K_{ij}\right] = \left[\delta_{ij} + \frac{i}{\omega\varepsilon_0}\sigma_{ij}\right] = \begin{bmatrix} (n_\alpha)^2 & 0 & 0 \\ 0 & (n_\beta)^2 & 0 \\ 0 & 0 & (n_\gamma)^2 \end{bmatrix}$$

Epsom Salt ($MgSO_4$):
n_j = [1.433, 1.455, 1.461]

These medias are rarely strongly unisotropic

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Reminder: Equations for calculating the dielectric response

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$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \textit{solve for } \mathbf{v}\,!$$

The induced motion of charge particles form a current and a charge density

$$\mathbf{J}_{media} = \sum_{species} qn\mathbf{v} \qquad \frac{\partial}{\partial t}\rho_{media} + \nabla \circ \mathbf{J}_{media} = 0$$

(n=particle density)

The respons can be quantified in e.g. the conductivity σ

$$J_i(\mathbf{k},\omega) = \sigma_{ij}(\mathbf{k},\omega) E_j(\mathbf{k},\omega)$$

Current and charge drive the **dielectric response**

$$\nabla \bullet \mathbf{E}_{media} = \rho_{media} / \varepsilon_0$$

$$\nabla \times \mathbf{B}_{media} - \frac{1}{c^2}\frac{\partial \mathbf{E}_{media}}{\partial t} = \mu_0 \mathbf{J}_{media}$$

Medium of oscillators

- Consider a medium consisting of charged particles with
  - charge q , mass m , density n
- Let the particles position x follow the equation of a forced oscillator
  - i.e. the media has an eigenfrequency Ω and a damping rate Γ
    - damping could be due to collisions (resistivity) and the eigenfrequency could be due to magnetization an acustic eigenfrequency of a crystal

$$\ddot{x}(t) + \Gamma\dot{x}(t) + \Omega^2 x(t) = \frac{q}{m} E_x(t) \quad \Rightarrow \quad x(\omega) = \frac{q/m}{\Omega^2 - \omega^2 - i\Gamma\omega} E_x(\omega)$$

- The current is then

$$J(\omega) = qn\left[-i\omega x(\omega)\right] = -\frac{i\omega n q^2/m}{\Omega^2 - \omega^2 - i\Gamma\omega} E_x(\omega) \equiv \sigma E_x(\omega)$$

- Thus the dielectric tensor reads

$$K_{ij} = \delta_{ij} + \frac{i}{\varepsilon_0 \omega}\sigma_{ij} = \left(1 + \frac{\omega_p^2}{\Omega^2 - \omega^2 - i\Gamma\omega}\right)\delta_{ij} \quad , \quad \text{where} \quad \omega_p^2 \equiv \frac{nq^2}{\varepsilon_0 m}$$

  - again ω_p is the plasma frequency

Medium of oscillators (2)

- Isotropic dielectric tensors K_{ij} can be replaced by a scalar K, consider e.g. the inner product $K_{ij}E_j = K\delta_{ij}E_j = KE_i$
- For the medium of harmonic oscillators

$$K = 1 + \frac{\omega_p^2}{\Omega^2 - \omega^2 - i\Gamma\omega}$$

- In the *high frequency* limit where $\omega >> \Omega$ and $\omega >> \Gamma$, then

$$K = 1 - \frac{\omega_p^2}{\omega^2} + \dots$$

  - this is the response of the electron gas!
- At *low frequency* $\omega << \Omega$ and $\omega \sim \Gamma$, then

$$K = 1 + \frac{\omega_p^2}{\Omega^2}$$

  - here the medium is no longer dispersive (independent of ω)

Medium of oscillators (3)

- The medium is the most dispersive when the frequency is near the characteristic frequency of the medium $\omega \sim \Omega$,
  - first rewrite the denominator

$$D \equiv \Omega^2 - \omega^2 - i\Gamma\omega =$$

$$= \Omega^2 - (\omega + i\Gamma/2)^2 - \Gamma^2/4$$

$$= (\Omega - \omega - i\Gamma/2)(\Omega + \omega + i\Gamma/2) - \Gamma^2/4$$

  - assume here the damping rate to be small $\omega >> \Gamma$ such that the last last term is negligible
  - Next use the relation: $\frac{1}{(a-b)(a+b)} = \frac{1}{2b}\left(\frac{1}{a-b} - \frac{1}{a+b}\right)$
  - The dielectric constant is then

$$K \approx 1 - \frac{\omega_p^2}{(\omega + i\Gamma/2 - \Omega)(\omega + i\Gamma/2 + \Omega)}$$

$$= 1 - \frac{\omega_p^2}{2\Omega}\left[\frac{1}{\omega + i\Gamma/2 - \Omega} - \frac{1}{\omega + i\Gamma/2 + \Omega}\right]$$

Medium of oscillators (4)

- Next we shall use the condition that we are close to resonance; i.e. the frequency is near the characteristic frequency $\omega \sim \Omega$:

$$|\omega - \Omega| << |\omega + \Omega| \quad \Rightarrow \quad \left|\frac{1}{\omega - \Omega + i\Gamma/2}\right| >> \left|\frac{1}{\omega + \Omega + i\Gamma/2}\right|$$

- The dielectric constant then reads

$$K \approx 1 - \frac{\omega_p^2}{2\Omega}\frac{1}{(\omega - \Omega + i\Gamma/2)} = 1 - \frac{\omega_p^2}{2\Omega}\frac{(\omega - \Omega - i\Gamma/2)}{\left[(\omega - \Omega)^2 + \Gamma^2/4\right]}$$

$$\begin{cases} K^H \equiv \Re\{K\} = 1 - \dfrac{\omega_p^2}{2\Omega}\dfrac{\omega - \Omega}{\left[(\omega - \Omega)^2 + \Gamma^2/4\right]} \\ K^A \equiv \Im\{K\} = \dfrac{\omega_p^2}{\Omega}\dfrac{\Gamma}{\left[(\omega - \Omega)^2 + \Gamma^2/4\right]} \end{cases}$$

Hermitian: wave propagation (reactive response)

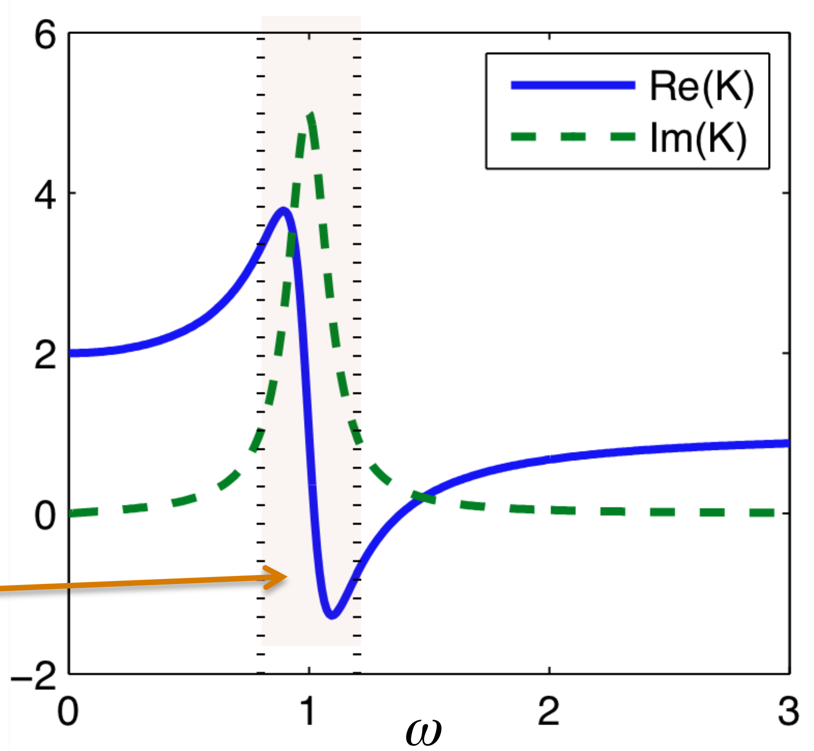
Antihermitian: wave absorption (resistive response)

Medium of oscillators (5)

- Antihermitian part comes from $i\Gamma/2$ in

$$K \approx 1 - \frac{\omega_p^2}{2\Omega}\frac{1}{(\omega - \Omega + i\Gamma/2)}$$

  - which is most important if $|\Gamma/2| \sim |\omega - \Omega|$ (for $|\Gamma/2| << |\omega - \Omega|$ then $K^A << K^H$)
  - Thus, the dissipation occur mainly where $|\Gamma| > |\omega - \Omega|$



- Summary:
  - Low frequency: not dispersive
  - Resonant region: strong damping in ***thin layer*** $|\Gamma| > |\omega - \Omega|$
  - High frequency: response decay with frequency, $\chi \sim K - 1 \sim \omega^{-2}$ like an electron gas.

Medium of oscillators (6)

- What happens in the limit when the damping Γ goes to zero?
- Again assume $\omega \sim \Omega$ then

$$K \approx 1 - \frac{\omega_p^2}{2\Omega} \frac{1}{(\omega - \Omega + i\Gamma/2)}$$

- The limit where Γ goes to zero can be rewritten using the Plemej formula

$$\lim_{\Gamma \to 0} K \approx \lim_{\Gamma \to 0} \left(1 - \frac{\omega_p^2}{2\Omega} \frac{1}{(\omega - \Omega + i\Gamma/2)} \right) = 1 - \frac{\omega_p^2}{2\Omega} \frac{1}{(\omega - \Omega + i0)} =$$

$$= 1 - \frac{\omega_p^2}{2\Omega} \left[\wp \frac{1}{\omega - \Omega} - i\pi\delta(\omega - \Omega) \right]$$

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Dielectric response for plasmas

- A first example of a plasma model is the Magnetoionic theory:
  - ***Assume***: ions are static; unperturbed by the wave field (no response)
  - ***Assume***: electrons are cold; they are initially static, but move in the presence of the wave field
  - ***Assume***: the plasma has a static magnetic field; align the coordinate system: $\mathbf{B}_0 = B_0 \mathbf{e}_z$

- What is the dielectric response of a magnetoionic media?
  - align also y-axis such that: $\mathbf{k} = k_y \mathbf{e}_y + k_z \mathbf{e}_z$
  - the response of the electrons is then given by Newtons equation

$$\dot{\mathbf{v}} = \frac{q}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

  - Next: add a friction with the ions (a force $-m\nu_f \mathbf{v}$) and use $\mathbf{v} = \dot{\mathbf{r}}$

$$\ddot{\mathbf{r}} = \frac{q}{m}\left(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}\right) - \nu_f \dot{\mathbf{r}}$$

Dielectric response for plasmas (2)

- Note that the magnetic field has two components a wave component and a static component

$$\mathbf{B} = \mathbf{B}_{wave} + \mathbf{B}_0$$

  - Thus the Lorentz force is non-linear: $m\dot{\mathbf{r}} \times \left(\mathbf{B}_{wave} + \mathbf{B}_0\right)$

- Assuming that the wave amplitude is small, then we can neglect $\mathbf{B}_{wave}$

$$\ddot{\mathbf{r}} - \dot{\mathbf{r}} \times \mathbf{e}_z \frac{q}{m} B_0 + \nu_f \dot{\mathbf{r}} = \frac{q}{m}\mathbf{E}$$

  - here we can identify the cyclotron frequency $\Omega = qB_0/m$

- Fourier transform: $-\omega^2 \mathbf{r} + i\omega \mathbf{r} \times \mathbf{e}_z \Omega - i\omega\nu_f \mathbf{r} = \frac{q}{m}\mathbf{E}$

$$-\omega^2 r_i + i\omega\varepsilon_{ijk} r_j \delta_{3k}\Omega - i\omega\nu_f r_i = \frac{q}{m} E_i$$

Note: $\mathbf{e}_z = \mathbf{e}_3 = \delta_{3k}\,\mathbf{e}_k$

$$\underbrace{\left[\left(\omega + i\nu_f\right)\delta_{ij} - i\varepsilon_{ij3}\Omega\right]}_{\text{Matrix in the indexes } i,j} r_j = -\frac{q}{m\omega} E_i$$

Matrix in the indexes i, j

Dielectric response for plasmas (3)

- Write equation as a matrix equations:

$$\left[(\omega + i\nu_f)\delta_{ij} - i\varepsilon_{ij3}\Omega\right]\left[r_j\right] = \begin{bmatrix} \omega + i\nu_f & -i\Omega & 0 \\ i\Omega & \omega + i\nu_f & 0 \\ 0 & 0 & \omega + i\nu_f \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \frac{q}{m\omega}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

- Inverting the matrix

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \frac{q}{m\omega}\begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & M_{33} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

$$\begin{cases} M_{11} = M_{22} = \dfrac{\omega + i\nu_f}{(\omega + i\nu_f)^2 - \Omega^2} \\ M_{12} = -M_{21} = \dfrac{i\Omega}{(\omega + i\nu_f)^2 - \Omega^2} \\ M_{33} = \dfrac{1}{\omega + i\nu_f} \end{cases}$$

- The current is then

$$\mathbf{j} = nq(-i\omega\mathbf{r}) = -i\varepsilon_0\omega_p^2\begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & M_{33} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

Dielectric response for plasmas (4)

- The dielectric tensor in the magnetoionic theory then reads:

$$[K_{ij}] = \left[\delta_{ij} + \frac{i}{\varepsilon_0\omega}\sigma_{ij}\right] = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

$$\begin{cases} S = 1 - \dfrac{\omega_p^2}{\omega}\dfrac{\omega + i\nu_f}{(\omega + i\nu_f)^2 - \Omega^2} \\ D = -\dfrac{\omega_p^2}{\omega}\dfrac{\Omega}{(\omega + i\nu_f)^2 - \Omega^2} \\ P = 1 - \dfrac{\omega_p^2}{\omega(\omega + i\nu_f)} \end{cases}$$

or

$$K_{ij} = S\left(\delta_{ij} - b_i b_j\right) + P b_i b_j - iD\varepsilon_{ijk} b_k$$

where b_k are the components of the unit vector parallel to the magnetic field

- This dielectric response tensor is:
  - ***Anisotropic***; response is different for $\mathbf{E}$ in the x, y, or z direciton.
  - ***Gyrotropic***: the off-diagonal terms (involving D) are perpendicular to a characteristic direction of the media

Hermitian part of the dielectric tensor

$$\mathbf{K}^H = \frac{1}{2}\left(\mathbf{K} + \mathbf{K}^{T*}\right) = \frac{1}{2}\left(\begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} + \begin{bmatrix} S^* & (-iD)^* & 0 \\ (iD)^* & S^* & 0 \\ 0 & 0 & P^* \end{bmatrix}^T\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} + \begin{bmatrix} S^* & (iD)^* & 0 \\ (-iD)^* & S^* & 0 \\ 0 & 0 & P^* \end{bmatrix}\right) =$$

Transpose make $-iD^*$ and iD^* change place

$$= \frac{1}{2}\left(\begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} + \begin{bmatrix} S^* & -i(D^*) & 0 \\ i(D^*) & S^* & 0 \\ 0 & 0 & P^* \end{bmatrix}\right) =$$

$$= \frac{1}{2}\begin{bmatrix} S + S^* & -i(D + D^*) & 0 \\ i(D + D^*) & S + S^* & 0 \\ 0 & 0 & P + P^* \end{bmatrix} = \begin{bmatrix} \Re\{S\} & -i\Re\{D\} & 0 \\ i\Re\{D\} & \Re\{S\} & 0 \\ 0 & 0 & \Re\{P\} \end{bmatrix}$$

Antihermitian part of the dielectric tensor

$$\mathbf{K}^A = \frac{1}{2}\left(\mathbf{K} - \mathbf{K}^{T*}\right) =$$

$$= \frac{1}{2}\left(\begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} - \begin{bmatrix} S^* & (-iD)^* & 0 \\ (iD)^* & S^* & 0 \\ 0 & 0 & P^* \end{bmatrix}^T\right) =$$

$$= i\begin{bmatrix} \Im\{S\} & -i\Im\{D\} & 0 \\ i\Im\{D\} & \Im\{S\} & 0 \\ 0 & 0 & \Im\{P\} \end{bmatrix}$$

Cold plasma dielectric response

- A commonly used representation of the plasma is the ***cold plasma***
  - Here ions and electrons are in a stationary equilibrium, and move only in the presence of a wave field
  - Usually the friction between ions and electrons are neglected
  - Each species is then described by the
    - charge q^ν
    - mass m^ν
    - position $\mathbf{r}^\nu$ (or velocity $\mathbf{v}^\nu$)
  - where $\nu = i$ represent the ions and $\nu = e$ represent the electrons
    - NOTE: ν is not a tensor index!
- The linearlised equation of motion for species ν ($\mathbf{B}_0 = B_0 \mathbf{e}_z$) :

$$m^\nu \ddot{\mathbf{r}}^\nu - q^\nu \dot{\mathbf{r}}^\nu \times \mathbf{B}_0 = q^\nu \mathbf{E}$$

$$\ddot{\mathbf{r}}^\nu - \dot{\mathbf{r}}^\nu \times \mathbf{e}_z \Omega^\nu = \frac{q^\nu}{m^\nu} \mathbf{E}$$

  - where $\Omega^\nu = q^\nu B_0 / m^\nu$
  - this equation is solved like in the magnetoionic theory

Cold plasma dielectric response (2)

- The solution of the equation of motion for species ν is

$$\begin{bmatrix} r_1^\nu \\ r_2^\nu \\ r_3^\nu \end{bmatrix} = \frac{q^\nu}{m^\nu \omega} \begin{bmatrix} M_{11}{}^\nu & M_{12}{}^\nu & 0 \\ M_{21}{}^\nu & M_{22}{}^\nu & 0 \\ 0 & 0 & M_{33}{}^\nu \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

$$\begin{cases} M_{11}{}^\nu = M_{22}{}^\nu = \dfrac{\omega}{\omega^2 - \Omega^{\nu\,2}} \\ M_{12}{}^\nu = -M_{21}{}^\nu = \dfrac{i\Omega^\nu}{\omega^2 - \Omega^{\nu\,2}} \\ M_{33}{}^\nu = \dfrac{1}{\omega} \end{cases}$$

- With many species the current is a sum over the all species:

$$\mathbf{j} = \sum_\nu \mathbf{j}^\nu = \sum_\nu n^\nu q^\nu (-i\omega \mathbf{r}^\nu) = \sum_\nu -i\varepsilon_0 \omega_{p\nu}^2 \begin{bmatrix} M_{11}{}^\nu & M_{12}{}^\nu & 0 \\ M_{21}{}^\nu & M_{22}{}^\nu & 0 \\ 0 & 0 & M_{33}{}^\nu \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

  - thus also the conductivity is a sum over species:

$$\sigma = \sum_\nu -i\varepsilon_0 \omega_{p\nu}^2 \begin{bmatrix} M_{11}{}^\nu & M_{12}{}^\nu & 0 \\ M_{21}{}^\nu & M_{22}{}^\nu & 0 \\ 0 & 0 & M_{33}{}^\nu \end{bmatrix}$$

Cold plasma dielectric response (3)

- The dielectric tensor for the cold plasma reads

$$[K_{ij}] = \left[\delta_{ij} + \frac{i}{\varepsilon_0 \omega}\sigma_{ij}\right] = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

$$\begin{cases} S = 1 - \sum_\nu \dfrac{\omega_{p\nu}^2}{\omega^2 - \Omega^{\nu 2}} \\ D = -\sum_\nu \dfrac{\omega_{p\nu}^2}{\omega} \dfrac{\Omega^\nu}{\omega^2 - \Omega^{\nu 2}} \\ P = 1 - \sum_\nu \dfrac{\omega_{p\nu}^2}{\omega^2} \end{cases}$$

- Low frequency limit $\omega \ll \Omega^\nu, \omega_{p\nu}$

$$S = \ldots = 1 + \frac{c^2}{V_A^2} \approx \frac{c^2}{V_A^2}$$

Alfven velocity

$$D = \ldots \approx 0$$

  - i.e. non-dispersive in S !
- Low frequency tensor:
  - compare: uniaxial crystal
  - describes Alfven wave and plasma oscillations (*see next lecture*)

$$[K_{ij}] = \begin{bmatrix} c^2/V_A^2 & 0 & 0 \\ 0 & c^2/V_A^2 & 0 \\ 0 & 0 & 1 - \sum_\nu \omega_{p\nu}^2/\omega^2 \end{bmatrix}$$

Cold plasma dielectric response (4)

- High frequency limit $\omega >> \Omega^\nu$, $\omega_{p\nu}$

$$S = 1 - \sum_\nu \frac{\omega_{p\nu}^2}{\omega^2} = P$$

$$D = -\sum_\nu \frac{\omega_{p\nu}^2 \Omega^\nu}{\omega^3} \sim O(\omega^{-3})$$

$$K_{ij} \approx \left(1 - \sum_\nu \frac{\omega_{p\nu}^2}{\omega^2}\right)\delta_{ij} + O(\omega^{-3})$$

Like an electron gas!

Kinetic descriptions of gases and plasmas

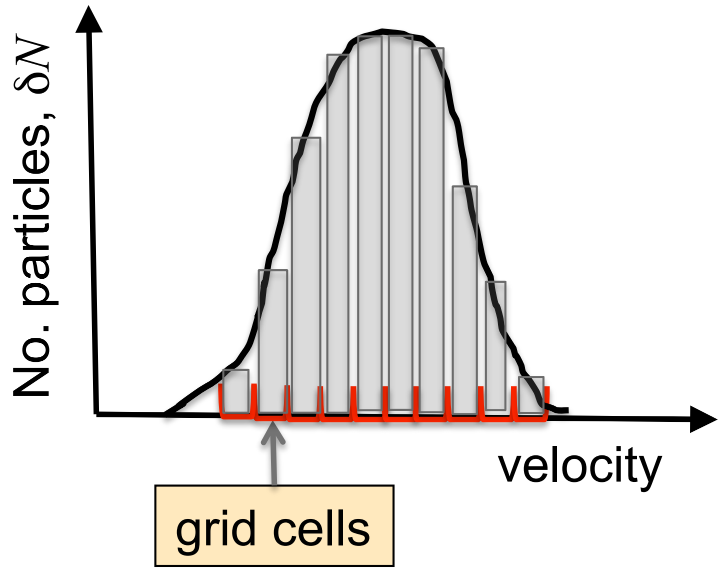
- Gases and plasmas are made up of particles that move "randomly"
  - This randomness makes them pratially impossible to predict exactly
- Instead: study them *statistically* :
  - Select a velocity grid:
    $v^i = i * \delta v$ for $i=0,1,2\ldots$
  - Construct histgram over particle velocity
    - counter number of particle in each grid cell
  - A density of particles in a velcity-space

$$f(v^i) = \frac{\delta N(v^i)}{\delta v}$$

Distribution function="*density in phase-space*"

- i.e. combine real and velocity space
- consider a box: $(x,x+\delta x)$, $(y,y+\delta y)$, $(z,z+\delta z)$

$$f(\mathbf{x},\mathbf{v}) \equiv f(x,y,z,v_x,v_y,v_z) = \frac{\delta N(v_x,v_y,v_z)}{\delta x \delta y \delta z \delta v_x \delta v_y \delta v_z}$$



The Maxwellian distribution function

- The "most" important/common distribution function is called the Maxwellian distribution function. For a gas/plasma with mass per particle m, temperature T and density n

$$f^M(v) = \frac{n}{(\sqrt{2\pi} m V)^3} \exp\left[-\frac{v^2}{2V^2} \right]$$

  - here V is the thermal velocity; $T = m V^2$
- E.g. when a gas or a plasma relaxed over a long time it will approach an equilibrium state. This state can be shown to be a Maxwellian!
  - The Maxwellian maximizes the entropy

Response of a warm plasma

- In Maxwells equations we need to know the charge density and current deinsity.
  - How can we calculate them from the distribution function?
- Note that the number density of particles

$$n(\mathbf{x}) = \sum_{\text{velocities cells}} \frac{\delta N(v_x, v_y, v_z)}{\delta x \delta y \delta z} = \sum_{\text{velocities cells}} f(\mathbf{x}, \mathbf{v}) \delta v_x \delta v_y \delta v_z$$

- How to calculate the density n and the average fluid velocity $\langle \mathbf{v} \rangle$:

$$\begin{cases} n = \int f(v) d^3 v \\ n \langle \mathbf{v} \rangle = \int \mathbf{v} f(v) d^3 v \end{cases}$$

- Thus, for an ensamble of species ν (e.g. ion and electron)

$$\rho = \sum_{\nu} q^{\nu} n^{\nu} = \sum_{\nu} q^{\nu} \int f^{\nu}(v) d^3 v$$

$$\mathbf{J} = \sum_{\nu} q^{\nu} n^{\nu} \langle \mathbf{v} \rangle^{\nu} = \sum_{\nu} q^{\nu} \int \mathbf{v} f^{\nu}(v) d^3 v$$

Response of a warm plasma

- When subject to a wave field, the equation of motion reads

$$m^{\nu}\dot{v}_i(t,\mathbf{r},\mathbf{v}) = q^{\nu}\left[E_i + \varepsilon_{ijk}v_jB_k\right]$$

- The distribution then evolves according to the Vlasov equation
(continuity equation in real and velocity space)

$$\left\{\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i} + \dot{v}_i(t,\mathbf{r},\mathbf{v})\frac{\partial}{\partial v_i}\right\}f^{\nu}(t,\mathbf{r},\mathbf{v}) = 0$$

$$\left\{\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i} + \frac{q^{\nu}}{m^{\nu}}\left[E_i(t,\mathbf{r}) + \varepsilon_{ijk}v_jB_k(t,\mathbf{r})\right]\frac{\partial}{\partial v_i}\right\}f^{\nu}(t,\mathbf{r},\mathbf{v}) = 0$$

- **Note**: the wave field perturbs both E, B and f, thus this equations is non-linear in the perturbation!

Response of a warm plasma (2)

- Separate unperturbed and perturbed quantities

$$\begin{cases} f(t,r,v) = f^{M\nu}(v) + f^{1\nu}(t,r,v) \\ \mathbf{E}(t,r,v) = 0 + \mathbf{E}^1(t,r,v) \\ \mathbf{B}(t,r,v) = 0 + \mathbf{B}^1(t,r,v) \end{cases}$$

- The Vlasov equation:

$$\left\{ \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} + \underbrace{\frac{q^\nu}{m^\nu}\left[E_i^1 + \varepsilon_{ijk} v_j B_k^1 \right] \frac{\partial}{\partial v_i}}_{\text{Non-linear terms}} \right\} f^{1\nu}(t,r,v) = -\frac{q^\nu}{m^\nu}\left[E_i^1 + \varepsilon_{ijk} v_j B_k^1 \right] \frac{\partial}{\partial v_i} f^{M\nu}(v)$$

- Linearlised equations and use Faraday's law $\mathbf{B}^1 = \mathbf{k} \times \mathbf{E}^1 / w$

$$\left\{ \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} \right\} f^{1\nu}(t,r,v) = -\frac{q^\nu}{m^\nu}\left[E_i^1 + \varepsilon_{ijk}\varepsilon_{knm} v_j \frac{k_n}{\omega} E_m^1 \right] \frac{\partial}{\partial v_i} f^{M\nu}(v)$$

- Fourier transform $f^{1\nu}(\omega,k,v) = \underbrace{\frac{-i}{\omega - \mathbf{k} \bullet \mathbf{v}}}_{} \frac{q^\nu}{m^\nu}\left[\left(1 - \frac{\mathbf{k} \bullet \mathbf{v}}{\omega}\right)\delta_{im} + \frac{v_m k_i}{\omega} \right] E_m^1 \frac{\partial}{\partial v_i} f^{M\nu}(v)$

Resonance when particles travel at phase velocity of the wave!

Landau-resonance

- The resonance in the solution to the linearised Vlasov equation is related to a damping

$$f^{1v}(\omega,k,v) = \frac{i}{\omega - \mathbf{k} \bullet \mathbf{v}} \cdots$$

  – This was first realised by Lev Landau in 1946.

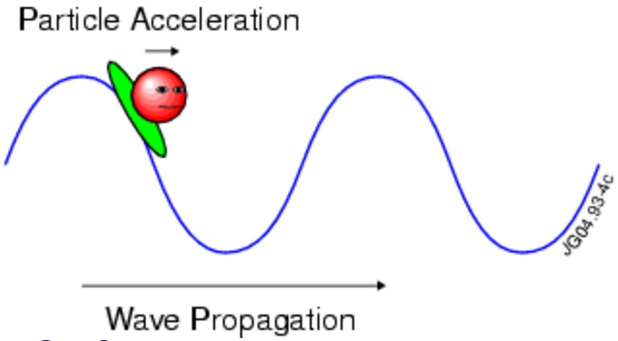
- What is the physics of this resonance?

The physics of the Landau-resonance

- Consider a plane wave $E \sim \exp(i\mathbf{k} \bullet \mathbf{x} - i\omega t)$
- Let a particle travel with the constant velocity $\mathbf{x} = \mathbf{v}t$

$$E \sim \exp(i\mathbf{k} \bullet \mathbf{v}t - i\omega t) = \exp(i[\mathbf{k} \bullet \mathbf{v} - \omega]t) = \exp(-i\omega' t)$$

- Thus, the particles will *see* a field oscillating with the frequency ω'
  - ω' is the Doppler shifted velocity!



- The resonance condition $\omega - \mathbf{k} \bullet \mathbf{v} = 0$,
  - i.e. the Doppler shifted frequency is zero
  - i.e. particle travels with the *same* speed as the wave
  - i.e. the E-field will *accelerate the particle forever* – the wave is damped!
- **Note**: we have linearised the equations, thus we assume that changes in particle velocity are small no matter now long the acceleration time!
  - in reality non-linear effects come in and then the damping remains only if the dissipation (Γ) is more important than non-linearity

Response of a warm plasma (3)

- The current is obtained from the integral over velocity space

$$j_n(\omega,k) = \sum_{\nu} q^{\nu} \int v_n f^{1\nu}(\omega,k,v) d^3v$$

  - using the perturbed distribution from the previous page

$$f^{1\nu}(\omega,k,v) = \frac{-i}{\omega - \mathbf{k}\bullet\mathbf{v}} \frac{q^{\nu}}{m^{\nu}} \left[\left(1 - \frac{\mathbf{k}\bullet\mathbf{v}}{\omega}\right)\delta_{im} + \frac{v_m k_i}{\omega} \right] \frac{\partial f^{M\nu}(v)}{\partial v_i} E_m^1$$

- The current can be written as

Add a weak dissipation to allow for use of Plemej formula

$$j_n(\omega,k) = \underbrace{\left\{ -i\varepsilon_0 \sum_{\nu} \omega_{p\nu}^2 \int \left[\delta_{im} + \frac{v_m k_i}{\omega - \mathbf{k}\bullet\mathbf{v} + i0} \right] \frac{v_n v_i}{n^{\nu} V^{\nu}} f^{M\nu}(v) d^3v \right\}}_{\text{The conductivity tensor!}} E_m^1$$

The conductivity tensor!

Plemei formula in kinetic plasmas

- In cold plasmas the Plemej formula appear for resonances like:

$$\sim \frac{1}{\omega - \Omega + i0}$$

  - where Ω is a natural frequency of the system.
  - i.e. only at exactly the correct resonance is there an antihermitian ternsor component
  - In practice this does not generate damping.

- In warm plasmas the resonance condition is

$$\sim \frac{1}{\omega - \mathbf{k} \bullet \mathbf{v} + i0}$$

  - i.e. all particles travelling with *the right speed* are in resonance.
  - For smooth distributions there are always some particle with the right speed.
  - Thus, there is an antihermitian part of the dielectric tensor, and thus damping, at all frequencies!

Response of a warm plasma (3)

- After *some* algebra it is possible to rewrite the dielectric tensor as a tensor with different longitudinal and transverse responses.

$$K_{ij} = K^L \kappa_i \kappa_i + K^T \left(\delta_{ij} - \kappa_i \kappa_i \right)$$

  - The key parameter in the reponse is the ratio between the phase velocity and the thermal velocity: $y_\nu = \omega / \sqrt{2} k V^\nu$
  - Thus, the thermal velocity is at the Landau resonance if $y_\nu = 1$

- The tensor components reads:

$$K^L = 1 + \sum_\nu \left(\frac{\omega_{p\nu}}{k V^\nu} \right)^2 \left[1 - \phi(y_\nu) + i \sqrt{\pi} y_\nu \exp(-y_\nu{}^2) \right]$$

$$K^T = 1 + \sum_\nu \left(\frac{\omega_{p\nu}}{\omega} \right)^2 \left[\phi(y_\nu) - i \sqrt{\pi} y_\nu \exp(-y_\nu{}^2) \right]$$

$$\phi(z) = 2 z e^{-z^2} \int_0^z e^{-t^2} dt$$

Damping in warm plasma

- Consider longitudinal waves
  - the damping is then proportional to (see later lectures for details)

$$\Im\{K^L\} = \sum_\nu \left(\frac{\omega_{p\nu}}{kV^\nu}\right)^2 \sqrt{\pi}\, y_\nu \exp(-y_\nu{}^2)$$

- This function has a maximum when $y_\nu \sim 0.7$, or $\omega/k \sim V_{th}$, i.e. when the phase velocity of the wave is roughly equal to the thermal velocity
  - this the when the Landau resonance is most effective