Wave equations and properties of waves in ideal media

T. Johnson

Outline

- Derivation of the wave equation and definition of wave quantities:
  – dispersion equation / dispersion relation / refractive index
  – wave polarization
- Some math for wave equations (mainly linear algebra)
  – relation between damping and antihermitian part of the dielectric tensor
- Waves in ideal anisotropic media
  – birefringent crystals
- Group velocity
- Plasma oscillations
- Elementary plasma waves
  – Langmuir waves
  – ion-acoustic waves
  – high frequency transverse wave
  – Alfven waves
- Wave resonances & cut-offs

The wave equation in vacuum

- Wave equations can be derived for $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{A}$.
- **Waves in vacuum**, i.e. no free charge or currents; then ϕ=*const*!
  Using Fourier transformed quantities in the Coloumb gauge:

$$\mathbf{E}(\omega,\mathbf{k}) = i\omega\mathbf{A}(\omega,\mathbf{k}) \quad , \quad \mathbf{B}(\omega,\mathbf{k}) = i\mathbf{k}\times\mathbf{A}(\omega,\mathbf{k}) \quad , \quad i\mathbf{k}\bullet\mathbf{A}(\omega,\mathbf{k}) = 0$$

- Ampere's law:

$$i\mathbf{k}\times\mathbf{B} + i\omega\mathbf{E}/c^2 = \mu_0\mathbf{J} \quad \Rightarrow \quad \mathbf{k}\times(\mathbf{k}\times\mathbf{A}) + \omega^2/c^2\mathbf{A} = -\mu_0\mathbf{J}$$

  where $\mathbf{k}\times(\mathbf{k}\times\mathbf{A}) = \mathbf{k}(\mathbf{k}\bullet\mathbf{A}) - |\mathbf{k}|^2\mathbf{A} = -|\mathbf{k}|^2\mathbf{A}$

- Homogeneous wave equation:

$$\boxed{\left(|\mathbf{k}|^2 - \omega^2/c^2\right)\mathbf{A} = 0}$$

- Solutions exists for: $\left(|\mathbf{k}|^2 - \omega^2/c^2\right) = 0$, the *dispersion equation*!

Dispersion relations

- A wave satisfying a dispersion equation is called a *Wave Mode*.

- Solutions to the dispersion equation can be written as a relation between ω and $\mathbf{k}$ called a *dispersion relation*, e.g.

$$\left(|\mathbf{k}|^2 - \omega^2/c^2\right) = 0 \quad \Rightarrow \quad \omega = \omega_M(\mathbf{k})$$

  – Note: here ω is the frequency and $\omega_M(\mathbf{k})$ is a function of $\mathbf{k}$
  – the sub-index M is for wave mode.
  – the function ω_M represents the dielectric response and therefore is a property of the media

- **Example**: In vacuum the dispersion relation reads:

$$\omega = \pm|\mathbf{k}|c \quad \Rightarrow \quad \omega_{M\pm}(\mathbf{k}) = \pm|\mathbf{k}|c$$

i.e. light waves

Refractive index

- Dispersion relations can be written using the refractive index n

$$n \equiv \frac{|\mathbf{k}|c}{\omega} = \frac{c}{\omega/|\mathbf{k}|} \sim \frac{\text{"speed of light"}}{\text{"phase velocity"}}$$

- A dispersion relation for a wave mode can be rewritten…
  - by replacing $\omega^2 = (|\mathbf{k}|c/n)^2$

$$n \equiv n_M(\mathbf{k})$$

  - or by replacing $\mathbf{k} = k\mathbf{e}_k \; , \; k = \omega n/c$

$$n \equiv n_M(\omega, \mathbf{e}_k)$$

- The dispersion relation for waves in vacuum then reads

$$n = \pm 1$$

i.e. the *phase velocity* of vacuum waves is the *speed of light*

Plane waves

- In this lecture we only consider infinite domains
  - and exclusively homogeneous and linear media
- Then the wave equation has plane wave solutions

$$A_i(\mathbf{x},t) = \hat{A}_i \exp(i\mathbf{k} \bullet \mathbf{x} - i\omega t)$$

- Take the plane wave for all perturbed quantities in Maxwell's equation and the equation of motion; then

$$\nabla \rightarrow i\mathbf{k} \quad , \quad \frac{\partial}{\partial \mathrm{t}} \rightarrow -i\omega$$

  - just as when we do Fourier transforms!
  - for linear differential equations: Fourier transforms and plane wave anzats give the same equation
  - e.g. the same wave equations, dispersion relation…!!

The dispersion relation describes the plane waves eigenmodes, i.e. what wave exists in abscense of external currents or charges

Degrees of freedom of the plane wave

- The solution to the wave equation is a sum of plane waves.

$$A_i(\mathbf{x},t) = \hat{A}_i \exp(i\mathbf{k} \bullet \mathbf{x} - i\omega t)$$

- The dispersion relation determines the refractive index
  – i.e. a relation between ω and k for a given direction of propagation

$$A_i(\mathbf{x},t) = \hat{A}_i \exp(i\mathbf{k} \bullet \mathbf{x} - i\omega_M(\mathbf{k})t)$$

- Is there anything else we can say about the solution?

- **Example**: If the wave is launched by an antenna, which degrees of freedom are determined by
  – the antenna?
  – the media?

The wave equation in dispersive media

- Ex: Temporal Gauge, ϕ=0, the fields are described by $\mathbf{A}$ alone

$$\mathbf{E}(\omega,\mathbf{k}) = i\omega\mathbf{A}(\omega,\mathbf{k}) \quad , \quad \mathbf{B}(\omega,\mathbf{k}) = i\mathbf{k}\times\mathbf{A}(\omega,\mathbf{k})$$

- Ampere's law:

$$i\mathbf{k}\times\mathbf{B} + i\omega\mathbf{E}/c^2 = \mu_0\mathbf{J} \quad \Rightarrow \quad \mathbf{k}\times(\mathbf{k}\times\mathbf{A}) + (\omega/c)^2\mathbf{A} = -\mu_0\mathbf{J}$$

- Split $\mathbf{J}=\mathbf{J}_{\mathrm{ext}}+\mathbf{J}_{\mathrm{ind}}$, where $\mathbf{J}_{\mathrm{ext}}$ external drive and $\mathbf{J}_{\mathrm{ind}}$ is induced parts

$$J_{\mathrm{ind},\,i} = \alpha_{ij}A_j$$

where α_{ij} is *polarisation response tensor*

- Inhomogeneous wave equation:

Wave operator

$$\boxed{\Lambda_{ij}A_j = -\frac{\mu_0 c^2}{\omega^2}J_{\mathrm{exp},\,i}} \qquad \text{where} \quad \Lambda_{ij} = \frac{c^2}{\omega^2}\underbrace{\left(k_ik_j - |\mathbf{k}|^2\delta_{ij}\right)}_{\mathbf{k}\times\mathbf{k}\times\ldots} + K_{ij}$$

Dielectric tensor: $K_{ij} = \delta_{ij} + \dfrac{1}{\omega^2\varepsilon_0}\alpha_{ij}$

Dispersion relations in dispersive media

- Homogeneous wave equation:

$$\Lambda_{ij}(\omega,\mathbf{k})A_j(\omega,\mathbf{k}) = 0$$

(*the book includes only the Hermitian part Λ^H, but this is a technicality At the end of this calculations we get the same dispersion relation*)

- Solutions exist if and only if:

$$\Lambda(\omega,\mathbf{k}) \equiv \det\left[\Lambda_{ij}(\omega,\mathbf{k})\right] = 0$$

this is the *dispersion equation*.

- From this equation the *dispersion relation* can be derived

$$\omega = \omega_M(\mathbf{k})$$

where

$$\Lambda\left(\omega_M(\mathbf{k}),\mathbf{k}\right) = 0$$

Non-linear and linear eigenvalue problems ★

- This wave equation is a *non-linear eigenvalue problem*, to see this…
- Remember *linear eigenvalue problems*:
  for a matrix $\mathbf{A}$ find the eigenvalues λ and the eigenvectors $\mathbf{x}$ such that:

$$\mathbf{Ax} - \lambda\mathbf{x} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

  or alternatively

$$\left(A_{ij} - \lambda\delta_{ij}\right)x_j = \Lambda_{ij}(\lambda)x_j = 0$$

  Thus for the linear eigenvalue problem Λ_{ij} is linear in λ .
- Our wave equation has the same form, except $\Lambda_{ij}(\omega)$ is non-linear in ω .
- Thus, we are looking for the eigenvalues ω_M and the eigenvectors $\mathbf{A}$ to the equation

$$\Lambda_{ij}(\omega_M, \mathbf{k})A_j = 0$$

- **Exercise**: show that when $K_{ij}=K_{ij}(\mathbf{k})$, the wave equation is a linear eigenvalue problem in ω^2.
  However, inertia in Eq. of motion (when deriving media responce) gives $K_{ij}=K_{ij}(\omega,\mathbf{k})$.

Polarization vector

- So the wave equation is an eigenvalue problem
  - The eigenvalue is the frequency
  - The normalised eigenvector is called the *polarisation vector,* $\mathbf{e}_M(\mathbf{k})$

$$\mathbf{e}_M(\mathbf{k}) = \frac{\mathbf{A}\big(\omega_M(\mathbf{k}),\mathbf{k}\big)}{\big|\mathbf{A}\big(\omega_M(\mathbf{k}),\mathbf{k}\big)\big|}$$

the direction of the $\mathbf{A}$-field!

  - *Note*: the $\mathbf{A}$-field is parallel to the $\mathbf{E}$-field
- The polarisation vector is complex – what does this mean?
  - e.g. take $\mathbf{e}_M = (\,2\,,\,\mathrm{i}\,,0\,)\,/\,5^{1/2}$, then the vector potential is

$$\mathbf{A}(t,\mathbf{x}) \propto \mathrm{Re}\Big\{\,[\,2\,,\,i\,,\,0\,]\,\exp(i\mathbf{k}\cdot\mathbf{x} + i\omega t)\Big\} =$$

$$= \Big[\,2\cos(\mathbf{k}\cdot\mathbf{x} + \omega t)\,,\;\cos(\mathbf{k}\cdot\mathbf{x} + \omega t + 90^o)\;,\,0\,\Big]$$

  - The difference in “phase” of e_{M1} and e_{M2} (in complex plane; one being real and the other imaginary) makes A_1 and A_2 oscillate *90º* out of phase – elliptic polarisation!

Longitudinal & Transverse waves

Definition:

Longitudinal & Transverse waves have $\mathbf{e}_M$ parallel & perpendicular to $\mathbf{k}$

- **Examples**:
  - *Light waves* have $\mathbf{E} \parallel \mathbf{A}$ perpendicular to $\mathbf{k}$, i.e. a *transverse* wave
  - *Sounds waves* (wave equation for the fluid velocity $\mathbf{v}$) have $\mathbf{v} \parallel \mathbf{k}$, i.e. a *longitudinal* wave.

Outline

- Derivation of the wave equation and definition of wave quantities:
  - dispersion equation / dispersion relation / refractive index
  - wave polarization
- Some math for wave equations (mainly linear algebra)
  - relation between damping and antihermitian part of the dielectric tensor
- Waves in ideal anisotropic media
  - birefringent crystals
- Group velocity
- Plasma oscillations
- Elementary plasma waves
  - Langmuir waves
  - ion-acoustic waves
  - high frequency transverse wave
  - Alfven waves
- Wave resonances & cut-offs

Linear algebra: cofactors ★

- An (i,j):th *cofactor*, λ_{ij} of a matrix $\mathbf{\Lambda}$ is the determinant of the "reduced" matrix, obtained by removing row i and column j , times $(-1)^{i+j}$
- In tensor notation (*you don't have to understand why!*):

$$\lambda_{ai} = \frac{1}{2}\varepsilon_{abc}\varepsilon_{ijl}\Lambda_{bj}\Lambda_{cl} \quad \text{e.g.} \quad \lambda_{21} = (-1)^{i+j}\det\begin{vmatrix} * & \Lambda_{12} & \Lambda_{13} \\ * & * & * \\ * & \Lambda_{32} & \Lambda_{33} \end{vmatrix} = (-1)^{i+j}\begin{vmatrix} \Lambda_{12} & \Lambda_{13} \\ \Lambda_{32} & \Lambda_{33} \end{vmatrix}$$

reduced matrix

- Alternative definition for cofactors:

$$\Lambda_{ik}\lambda_{kj} = \Lambda\delta_{ij}$$

  – Thus, for $\Lambda=0$ each column $(\lambda_{1j}, \lambda_{2j}, \lambda_{3j})^T$ is an eigenvector!
- It can be shown that

$$\lambda_{ai} = \lambda_{kk}e_{Mi}e^{*}_{Mj}$$

where λ_{kk} is the *trace* of λ and e_{Mi} are the normalised eigenvectors

Linear algebra: determinants

- The determinant can be written as (Melrose page 139)

$$\det[\Lambda] = \frac{1}{6}\varepsilon_{abc}\varepsilon_{ijl}\Lambda_{ai}\Lambda_{bj}\Lambda_{cl}$$

- Derivatives (note that the three derivates are identical)

$$\frac{\partial}{\partial x}\det[\Lambda(x)] = \underbrace{\frac{1}{2}\varepsilon_{abc}\varepsilon_{ijl}\Lambda_{ai}\Lambda_{bj}}_{\text{Cofactors } \lambda_{bj}!}\frac{\partial \Lambda_{cl}}{\partial x} = \lambda_{bj}\frac{\partial \Lambda_{bj}}{\partial x}$$

- Special case; take derivative w.r.t. the one tensor component

$$\frac{\partial}{\partial \Lambda_{ij}}\det\left[\Lambda(\Lambda_{11},\Lambda_{12},\Lambda_{21},\Lambda_{22}\ldots)\right] = \lambda_{nm}\underbrace{\frac{\partial \Lambda_{nm}}{\partial \Lambda_{ij}}}_{\delta_{ni}\delta_{jm}} = \lambda_{ij}$$

Linear algebra: Taylor expansion

- The determinant of this matrix is a function of the matrix components

$$\det\left[\Lambda\right] = f(\Lambda_{11},\Lambda_{12},\ldots)$$

- Perturbing the matrix components $\Lambda_{ij} \to \Lambda_{ij} + \delta\Lambda_{ij}$ we can then Taylor expand

$$\det\left[\Lambda + \delta\Lambda\right] = f(\Lambda_{ij} + \delta\Lambda_{ij}) =$$

$$= f(\Lambda_{ij}) + \frac{\partial}{\partial\Lambda_{ij}} f(\Lambda_{ij})\delta\Lambda_{ij} + O(\delta\Lambda^2) =$$

$$= \det\left[\Lambda\right] + \frac{\partial}{\partial\Lambda_{ij}} \det\left[\Lambda\right]\delta\Lambda_{ij} + O(\delta\Lambda^2) =$$

$$= \det\left[\Lambda\right] + \lambda_{ij}\delta\Lambda_{ij} + O(\delta\Lambda^2)$$

Damping of waves

- Next we'll show that for low amplitude waves
  - the Hermitian part of Λ_{ij} provides the dispersion relation
  - the anti-Hermitian part of the dielectric tensor $K^A{}_{ij}$ describes wave damping, i.e. the decay of the wave
- Consider a plane wave with complex frequency $\omega + i\omega_I$

$$A_i(\mathbf{x},t) = \hat{A}_i \exp(\omega_I t)\exp(i\mathbf{k} \bullet \mathbf{x} - i\omega t)$$

  - The wave amplitude decays at a rate $-\omega_I$
  - Note: the wave energy ($\sim|\mathbf{E}|^2$) decays at a rate $\gamma = -2\omega_I$
- The dispersion relation

$$\det\left[\Lambda_{ij}(\omega + i\omega_I, \mathbf{k})\right] = \det\left[\Lambda^H_{ij}(\omega + i\omega_I, \mathbf{k}) + \Lambda^A_{ij}(\omega + i\omega_I, \mathbf{k})\right] = 0$$

Exercise: show that $\mathbf{\Lambda}^A = \mathbf{K}^A$

- To simplify this expression we will assume *weak damping* …

Weak damping of waves

- Assume the damping to be weak by:

$$K_{ij}^{A} \rightarrow 0 \quad \text{and} \quad \omega_I \rightarrow 0$$

- Also assume $\omega_I \sim K^A$
  - Interpretation of the relation $\omega_I \sim K^A$: reduce K^A by factor, then ω_I reduces by the same factor, thus the they go to zero *together*
- Expand in small ω_I :

$$\Lambda_{ij}(\omega + i\omega_I, \mathbf{k}) \approx \Lambda_{ij}(\omega, \mathbf{k}) + i\omega_I \frac{\partial}{\partial\omega}\Lambda_{ij}(\omega, \mathbf{k}) + O(\omega_I^{\,2})$$

$$\approx \Lambda_{ij}^{H}(\omega, \mathbf{k}) + \underbrace{K_{ij}^{A}(\omega, \mathbf{k}) + i\omega_I \frac{\partial}{\partial\omega}\Lambda_{ij}^{H}(\omega, \mathbf{k})}_{1^{\text{st}} \text{ order in } \omega_I} + i\omega_I \frac{\partial}{\partial\omega} K_{ij}^{A}(\omega, \mathbf{k}) + O(\omega_I^{\,2})$$

Both small, i.e. $\sim\omega_I^2$

- Dispersion equation then reads

$$\det\left[\Lambda_{ij}^{H} + \delta\Lambda_{ij}\right] = 0 \quad , \quad \delta\Lambda_{ij} = K_{ij}^{A}(\omega, \mathbf{k}) + i\omega_I \frac{\partial\Lambda_{ij}^{H}(\omega, \mathbf{k})}{\partial\omega} + O(\omega_I^{\,2})$$

  - Expand the determinant in small $\delta\Lambda_{ij}$

Weak damping of waves

- The dispersion equation (repeated from previous page):

$$\det\left[\Lambda_{ij}^{H}+\delta\Lambda_{ij}\right]=0 \quad , \quad \delta\Lambda_{ij}=K_{ij}^{A}(\omega,\mathbf{k})+i\omega_{I}\frac{\partial\Lambda_{ij}^{H}(\omega,\mathbf{k})}{\partial\omega}+O(\omega_{I}{}^{2})$$

- Taylor expand the determinant

$$\det\left[\Lambda_{ij}^{H}+\delta\Lambda_{ij}\right]=\det\left[\Lambda_{ij}^{H}\right]+\delta\Lambda_{ij}\lambda_{ij}+O\left(\delta\Lambda_{ij}{}^{2}\right)$$

  - where λ_{ij} are the cofactors of

- NOTE: (see "Linear Algebra" pages): $\lambda_{ij}\dfrac{\partial}{\partial\omega}\Lambda_{ij}^{H}(\omega,\mathbf{k})=\dfrac{\partial}{\partial\omega}\det\left[\Lambda_{ij}^{H}\right]$

- The dispersion equation can then be written as

$$\det\left[\Lambda_{ij}^{H}(\omega,\mathbf{k})\right]+\lambda_{ij}K_{ij}^{A}(\omega,\mathbf{k})+i\omega_{I}\frac{\partial}{\partial\omega}\det\left[\Lambda_{ij}^{H}(\omega,\mathbf{k})\right]+O(\omega_{I}{}^{2})=0$$

Weak damping of waves

- Note that the dispersion equation with weak damping has both *real* and *imaginary* parts
  - The matrix of cofactors is Hermitian, thus $\lambda_{ij} K_{ij}^{A}$ is imaginary
  - Also: $\det(\Lambda_{ij}^{H})$ is real

$$0 = \mathrm{Re}\left\{\det\left[\Lambda\left(\omega,\mathbf{k}\right)\right]\right\} \approx \det\left[\Lambda_{ij}^{H}\left(\omega,\mathbf{k}\right)\right] + O(\omega_I^{\,2})$$

$$0 = \mathrm{Im}\left\{\det\left[\Lambda\left(\omega,\mathbf{k}\right)\right]\right\} \approx -i\lambda_{ij}K_{ij}^{A}\left(\omega,\mathbf{k}\right) + \omega_I \frac{\partial}{\partial\omega}\det\left[\Lambda_{ij}^{H}\left(\omega,\mathbf{k}\right)\right] + O(\omega_I^{\,2})$$

- The first equation gives dispersion relation for real frequency

$$\omega = \omega_M(\mathbf{k}) \quad \text{such that:} \quad \det\left[\Lambda_{ij}^{H}\left(\omega_M(\mathbf{k}),\mathbf{k}\right)\right] + O(\omega_I^{\,2}) = 0$$

and the second equations gives the damping rate

$$\boxed{\omega_I = \frac{i\lambda_{ij}K_{ij}^{A}\left(\omega_M(\mathbf{k}),\mathbf{k}\right)}{\frac{\partial}{\partial\omega}\det\left[\Lambda_{nm}^{H}\left(\omega,\mathbf{k}\right)\right]_{\omega=\omega_M(\mathbf{k})}} + O(\omega_I^{\,2})}$$

Energy dissipation rate, γ_M

- Alternatively we can form the *energy dissipation rate*, i.e. rate at which the wave energy is damped $\gamma_M = -2\omega_I$
  - express the cofactor in terms of polarisation vectors $\lambda_{ij} = \lambda_{kk} e_{Mi} e^*_{Mj}$

$$\gamma_M = -2i\omega_M(\mathbf{k}) R_M(\mathbf{k}) \left\{ \underbrace{e^*_{Mi}(\mathbf{k})}_{\text{Vector}} \underbrace{K^A_{ij}}_{\text{Matrix}}(\omega_M(\mathbf{k}),\mathbf{k}) \underbrace{e_{Mj}(\mathbf{k})}_{\text{Vector}} \right\}$$

Note: this is related to the hermitian part of the conductivity, $\sigma^H_{ij} \propto iK^A_{ij}$

$$\gamma_M \propto e^*_{Mi} \left[iK^A_{ij} \right] e_{Mj} \propto e^*_{Mi} \sigma^H_{ij} e_{Mj}$$

  - here R_M is the *ratio of electric to total energy*

$$R_M(\mathbf{k}) = \left\{ \frac{\lambda_{ss}(\omega,\mathbf{k})}{\omega \dfrac{\partial}{\partial\omega} \det\left[\Lambda^H_{nm}(\omega,\mathbf{k}) \right]} \right\}_{\omega=\omega_M(\mathbf{k})}$$

and plays an important role in Chapter 15

Determinant and the cofactors in the general case ★

- Explicit forms for dispersion equation and cofactors
- Write $\mathbf{\Lambda}$ in terms of the refractive index n and the unit vector along $\mathbf{k}$, i.e. $\boldsymbol{\kappa} = \mathbf{k}/|\mathbf{k}|$

$$\Lambda_{ij} = \frac{c^2}{\omega^2}\left(k_i k_j - |\mathbf{k}|^2 \delta_{ij}\right) + K_{ij} \;\rightarrow\; \Lambda_{ij} = n^2\left(\kappa_i \kappa_j - \delta_{ij}\right) + K_{ij}$$

- Brute force evaluation give

$$\det[\Lambda] = n^4 \kappa_i \kappa_j K_{ij} - n^2\left(\kappa_i \kappa_j K_{ij} K_{ss} - \kappa_i \kappa_j K_{is} K_{sj}\right) + \det[K]$$

and the cofactors (related to the eigenvector) are

$$\lambda_{ij} \approx n^4 \kappa_i \kappa_j - n^2\left(\kappa_i \kappa_j K_{ss} - \delta_{ij} \kappa_r \kappa_s K_{rs} - \kappa_i \kappa_s K_{sj} - \kappa_s \kappa_j K_{is}\right) +$$
$$+ \tfrac{1}{2}\delta_{ij}\left(K_{ss}{}^2 - K_{rs} K_{sr}\right) + K_{is} K_{sj} + K_{ss} K_{ij}$$

Outline

- Derivation of the wave equation and definition of wave quantities:
  - dispersion equation / dispersion relation / refractive index
  - wave polarization
- Some math for wave equations (mainly linear algebra)
  - relation between damping and antihermitian part of the dielectric tensor
- Waves in ideal anisotropic media
  - birefringent crystals
- Group velocity
- Plasma oscillations
- Elementary plasma waves
  - Langmuir waves
  - ion-acoustic waves
  - high frequency transverse wave
  - Alfven waves
- Wave resonances & cut-offs

Ex. 1: Isotropic, not spatially dispersive, media

- Isotropic, not spatially dispersive, media: $K_{ij}(\omega) = K(\omega)\delta_{ij}$

unit vector, $\boldsymbol{\kappa}=\mathbf{k}/k$

- Place z-axis along $\mathbf{k}$: $\Lambda_{ij} = n^2\left(\kappa_i\kappa_j - \delta_{ij}\right) + K_{ij} = \begin{pmatrix} K-n^2 & 0 & 0 \\ 0 & K-n^2 & 0 \\ 0 & 0 & K \end{pmatrix}$
  (n = refractive index)

- Dispersion equation: $\left(K-n^2\right)^2 K = 0$

K is the square root of the **refractive index**

- Dispersion relations: $\begin{cases} n^2 = K(\omega) \rightarrow n_M(\omega)^2 \equiv K(\omega) \\ K(\omega) = 0 \end{cases}$
  - Note: $K(\omega)=0$ means oscillations, NOT waves! (See section on Group velocity)
- The waves $n^2=K(\omega)$ are transverse waves
  - Plug dispersion relation into Λ_{ij} to see that the eigenvectors are perpendicular to $\mathbf{k}$!
- Polarisation vectors of transverse waves are *degenerate* (not unique eigenvector per mode); discussed in detail in Chapter 14.

Ex 2: Isotropic media with spatial dispersion

- Isotropic media with spatial dispersion (e.g. align z-axis: $\mathbf{e}_z=\boldsymbol{\kappa}$)

$$K_{ij}(\omega,\mathbf{k}) = K^L(\omega,k)\kappa_i\kappa_j + K^T(\omega,k)\left(\delta_{ij} - \kappa_i\kappa_j\right) = \begin{bmatrix} K^T & 0 & 0 \\ 0 & K^T & 0 \\ 0 & 0 & K^L \end{bmatrix}$$

- Dispersion equation

$$K^L(\omega,k)\left[K^T(\omega,k) - n^2\right]^2 = 0$$

- The longitudinal dispersion relation $K^L(\omega,k) = 0$
  - Dispersion give us a longitudinal wave! (eigenvector parallel to **k**)

- Transverse dispersion relation $K^T(\omega,k) - n^2 = 0$
  - Again the transverse waves are degenerate.

Ex 3: Birefringent media

- Uniaxial and biaxial crystals are *birefringent*
  - A light ray entering the crystal splits into two rays; the two rays follow different paths through the crystal.
  - Why?
- Consider a uniaxial crystal;
  - align z-axis with the distinctive axis of the crystal

$$K(\omega) = \begin{pmatrix} K_\perp(\omega) & 0 & 0 \\ 0 & K_\perp(\omega) & 0 \\ 0 & 0 & K_\parallel(\omega) \end{pmatrix}$$

  - Align coordinates $\mathbf{k}$ in x-z plane; let θ be the angle between z-axis and $\mathbf{k}$.

$$\kappa = (\sin\theta \ , 0 \ , \ \cos\theta)$$

Birefringent media (cont.)

- Dispersion equation in uniaxial media

$$\left(K_\perp - n^2\right)\left[K_\perp K_\parallel - n^2\left(K_\perp \sin^2\theta + K_\parallel \cos^2\theta\right)\right]^2 = 0$$

- Two modes, different refractive index (*naming conventions differ!*)
  - The (ordinary) O-mode: $n_O{}^2 = K_\perp$
  - The (extraordinary) X-mode: $n_X{}^2 = \dfrac{K_\perp K_\parallel}{K_\perp \sin^2\theta + K_\parallel \cos^2\theta}$
- **O-mode**: is transverse: $\mathbf{e}_O(\mathbf{k}) = \left(0\ ,\ 1\ ,\ 0\right)$
  - E-field along the crystal plane
- **X-mode**: is ***not*** transverse and ***not*** longitudinal:

$$\mathbf{e}_X(\mathbf{k}) \propto \left(K_\parallel \cos\theta\ ,\ 0\ ,\ K_\perp \sin\theta\right)$$

  - E-field has components both along and perpendicular to crystal plane

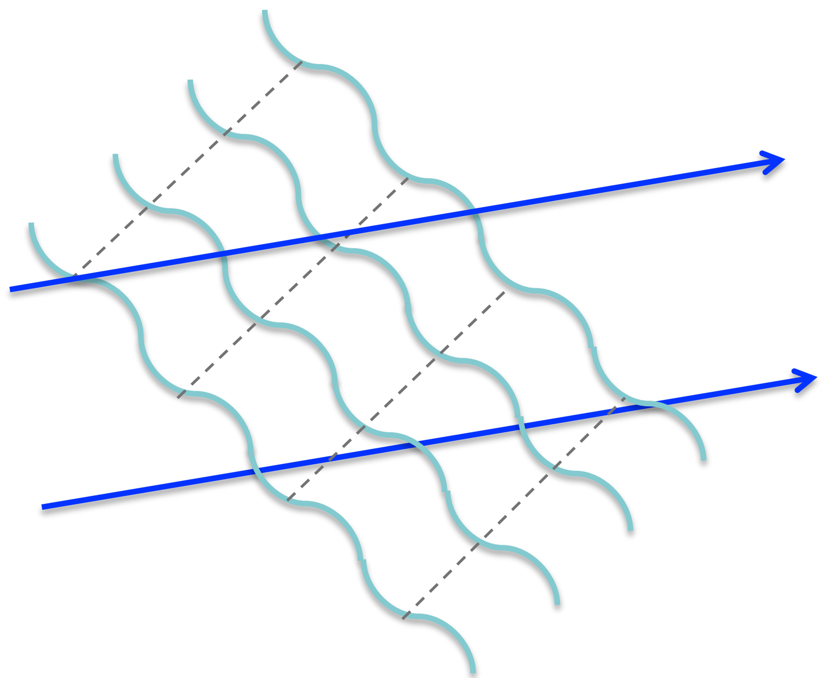
Wave splitting

- Let a light ray fall on a birefringent crystal with electric field components in all directions (x,y,z).
  - The y-component will enter the crystal as an O-mode! (polarisation vector is in y-direction)
  - The x,z-components as X-modes (polarisation vector is in xz-plane)

- The O-mode and X-mode have different refractive index (they travel with different speed), i.e. the wave will refract differently!

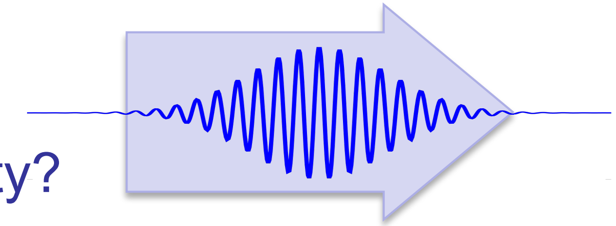
Outline

- Derivation of the wave equation and definition of wave quantities:
  - dispersion equation / dispersion relation / refractive index
  - wave polarization
- Some math for wave equations (mainly linear algebra)
  - relation between damping and antihermitian part of the dielectric tensor
- Waves in ideal anisotropic media
  - birefringent crystals
- Group velocity
- Plasma oscillations
- Elementary plasma waves
  - Langmuir waves
  - ion-acoustic waves
  - high frequency transverse wave
  - Alfven waves
- Wave resonances & cut-offs



The group velocity

- The propagation of waves is a transfer of energy
  - e.g. the light from the sun transfer energy to earth
    - It's warm in the sun since the sunlight bring energy!
- Consider a wave package from an antenna
  - Does this package travel with the phase velocity?



In dispersive media the answer is **no**!!

The velocity at which the shape of the wave's amplitudes (modulation/envelope) moves is called the **group velocity**

- The group velocity is *often* the velocity of information or energy
  - Warning! There are exceptions; experiments have shown that group velocity can go above speed of light, but then the information does not travel as fast

The velocity of a wave package, 1(2)

- The concept of group velocity can be illustrated by the motion of a wave package
  - This motion can easily be identified for a 1D wave package
  - travelling in a wave mode with dispersion relation: $\omega = \omega_M(k)$
  - assuming the wave is almost monocromatic

$$\omega_M(k) \approx \omega_{M0} + \omega'_{M0}(k - k_0) \quad , \quad \omega'_{M0} \equiv \frac{d\omega_{M0}}{dk}$$

- Let the wave have a Fourier transform

$$E(\omega,k) = A(k)\delta\big(\omega - \omega_M(k)\big) + \overbrace{A^*(-k)\delta\big(\omega + \omega_M(k)\big)}^{\text{complex conjugate of the first term: below denoted c.c.}}$$

- To study how the wave package travel in space-time, take the inverse Fourier transform

$$E(t,r) = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk A(k)\delta\big(\omega - \omega_M(k)\big)\exp\{ikx - i\omega t\} + \text{c.c.}$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} dk A(k)\exp\{ikx - i\omega_M(k)t\} + \text{c.c.}$$

The velocity of a wave package, 2(2)

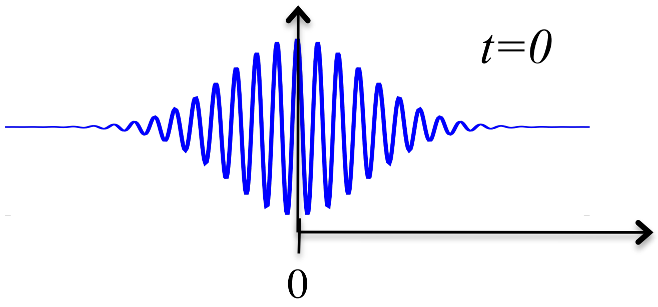
- Now apply the assumption of having “almost chromatic waves”

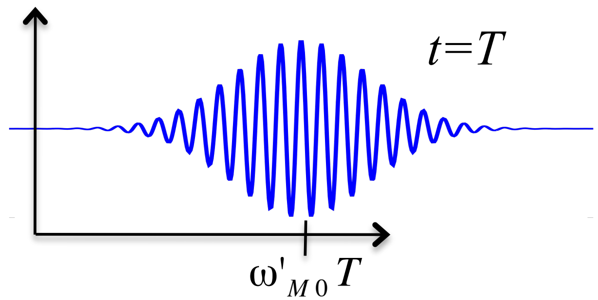
$$\omega_M(k) \approx \omega_{M0} + \omega'_{M0}(k - k_0) \Rightarrow$$

$$E(t,x) \approx \frac{1}{4\pi}\int_{-\infty}^{\infty} dk A(k) \exp\{ikx - i\omega_M(k)t\} + \text{c.c.} =$$

$$= \frac{e^{-i(\omega_{M0} + \omega'_{M0}k_0)t}}{4\pi}\int_{-\infty}^{\infty} dk A(k) \exp\{ik(x - \omega'_{M0}t)\} + \text{c.c.} = e^{-i(\omega_{M0} + \omega'_{M0}k_0)t} fcn(x - \omega'_{M0}t) + \text{c.c.}$$

  – i.e. if a wave package is centered around $x=0$ at time $t=0$, then at time $t=T$ wave package has the identical shape but now centred around $x = \omega'_{M0}T$





- Wave package moves with a speed called the *group velocity* : $$v_g = \omega'_{M0} \equiv \frac{d\omega_{M0}}{dk}$$

Hamilton's equations of motion

- The concept of group velocity can also be studied in terms of rays
- How do you follow the path of a ray in a dispersive media?
  - Hamilton studied this problem in the mid 1800's and developed a ***particle theory for waves***; i.e. like photons! (long before Einstein)
  - Hamilton's theory is now known as *Hamiltonian mechanics*
  - Hamilton's equations of motion are for a particle:

$$\dot{q}_i(t) = \frac{\partial H(p,q,t)}{\partial p_i}$$

$$\dot{p}_i(t) = -\frac{\partial H(p,q,t)}{\partial q_i}$$

  - where
    - $q_i = (x,y,z)$ are the position coordinates
    - $p_i = (mv_x, mv_y, mv_z)$ are the canonical momentum coordinates
    - The Hamiltonian H is the sum of the kinetic and potential energy
- But what are q_i, p_i and H for waves?

Hamilton's equations for rays

- What are q_i, p_i and H for waves?
  - The position coordinates $q_i = (x, y, z)$
  - In quantum mechanics the wave momentum is $\hbar k$; in Hamilton's theory the momentum is $p_i = (k_x, k_y, k_z)$
  - The Hamiltonian energy H is $\omega_M(k)$ (energy of wave in quanta $\hbar\omega$), i.e. the solution to the dispersion relation for the mode *M*!
- Consequently, the group velocity of a wave mode *M* is:

$$\mathbf{v}_{gM} \equiv \dot{\mathbf{q}} = \frac{\partial \omega_M(\mathbf{k})}{\partial \mathbf{k}}$$

- The second of Hamilton equations tells us how $\mathbf{k}$ changes when passing through a weakly inhomogeneous media, i.e. one in which the dispersion relation changes *slowly* as the wave propagates through the media, $\omega_M(\mathbf{k},\mathbf{q})$

$$\dot{\mathbf{k}} = -\frac{\partial \omega_M(\mathbf{k},\mathbf{q})}{\partial \mathbf{q}}$$

Warning! Hamiltons equations only work for *almost homogeneous media*. If the media changes rapidly the ray description may not work!

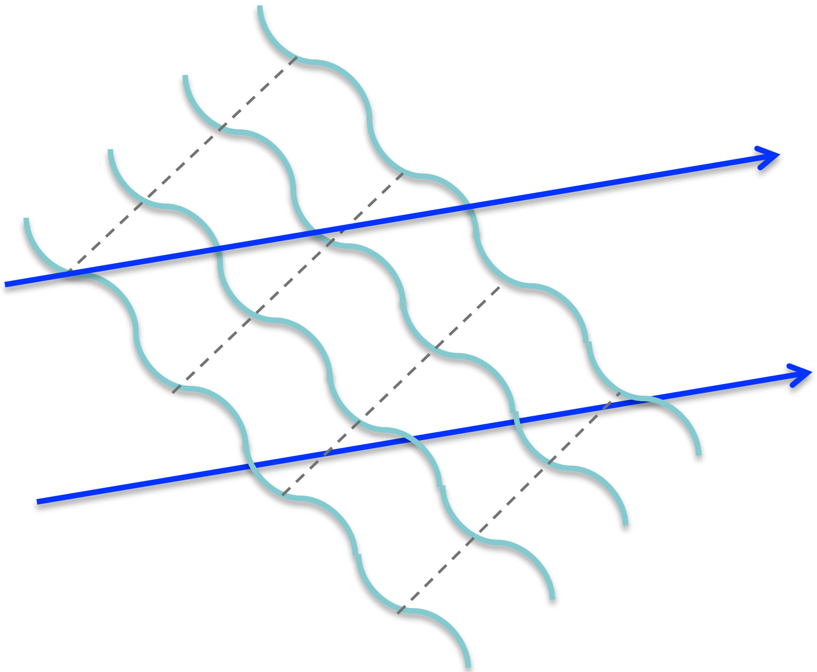
Examples of group velocities

- Let us start with the ordinary light wave $\omega_L(\mathbf{k}) = ck = c\sqrt{k_i k_i}$
  - The group velocity: $v_{gM,i} \equiv \frac{\partial}{\partial k_i}\omega_L(\mathbf{k}) = \frac{\partial}{\partial k_i} ck = c\kappa_i$
  - The phase velocity: $v_{phM,i} \equiv \frac{\omega_L(\mathbf{k})}{k}\kappa_i = c\kappa_i$
- High frequency waves: $\omega_M(\mathbf{k})^2 = c^2k^2 + \omega_{pe}^2$

  (response of electron gas; discussed shortly)
  - The group velocity: $v_{gM,i} = \frac{\partial}{\partial k_i}\sqrt{\omega_{pe}^2 + c^2k^2} = \frac{c}{\sqrt{\omega_{pe}^2/(kc)^2 + 1}}\kappa_i$
  - The phase velocity: $v_{phT,i} = \left(\omega_{pe}^2/(kc)^2 + 1\right)^{1/2} c\kappa_i$
  - Note:
    - phase velocity may be *faster* than speed of light
    - group velocity is *slower* than speed of light
      - Note: information travel with v_g ; cannot travel faster than speed of light

Outline

- Derivation of the wave equation and definition of wave quantities:
  - dispersion equation / dispersion relation / refractive index
  - wave polarization
- Some math for wave equations (mainly linear algebra)
  - relation between damping and antihermitian part of the dielectric tensor
- Waves in ideal anisotropic media
  - birefringent crystals
- Group velocity
- **Plasma oscillations**
- Elementary plasma waves
  - Langmuir waves
  - ion-acoustic waves
  - high frequency transverse wave
  - Alfven waves
- Wave resonances & cut-offs



Plasma oscillations

- Plasma oscillations: "*the linear reaction of cold and unmagnetised electrons to electrostatic perturbations*"
  - Cold electrons are electrons where the temperature is negligible.
- Model equations:
  - Electrostatic perturbations follow Poisson's equation

$$\Delta\phi = \rho/\varepsilon_0$$

  where $\rho = q_i n_i + q_e n_e$ is the charge density.
  - Electron response

$$m_e \frac{\partial v_e}{\partial t} = q_e \nabla\phi$$

  - Ion response; ions are heavy and do not have time to move: $\mathbf{v}_i = 0$
  - Charge continuity

$$\frac{\partial\rho}{\partial t} + \nabla\cdot \mathbf{J} = 0 \quad \text{, where} \quad \mathbf{J} = q_i n_i \mathbf{v}_i + q_e n_e \mathbf{v}_e$$

Plasma oscillations

- Consider small oscillations near a static equilibrium:

$$\left.\begin{aligned} \mathbf{v}_e(t) &= 0 + \mathbf{v}_{e1}(t) \\ \phi(t) &= 0 + \phi_1(t) \\ n_e(t) &= n_0 + n_{e1}(t) \\ n_i(t) &= n_0 q_e / q_i + 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \mathbf{J} &= q_e\left(n_{e0}\mathbf{v}_{e1} + n_{e1}\mathbf{v}_{e1}\right) \approx q_e n_{e0}\mathbf{v}_{e1} \\ \rho &= q_e n_{e1} \end{aligned}\right.$$

Non-linear (small term)

  - where all the small quantities have sub-index 1.
- Next Fourier transform in time and space

$$\left.\begin{aligned} -k^2\phi_1 &= q_e n_{e1}/\varepsilon_0 \\ -i\omega m_e \mathbf{v}_{e1} &= i q_e \mathbf{k}\phi_1 \\ -i\omega(q_e n_{e1}) + i\mathbf{k}\cdot(q_e n_{e0}\mathbf{v}_{e1}) &= 0 \end{aligned}\right\} \quad \left[\omega^2 - \underbrace{n_{e0}q_e^{\,2}/(\varepsilon_0 m_e)}_{\equiv\, \omega_{pe}^{\,2}}\right] n_{e1} = 0$$

ω_{pe} is the plasma frequency (see previous lecture)

Plasma oscillations

- Equation for the density oscillation is a dispersion equation

$$\left[\omega^2 - \omega_{pe}{}^2\right] n_{e1} = 0$$

  - Eigen-oscillations appear when

$$\omega^2 = \omega_{pe}{}^2$$

  - These are *plasma oscillations*!

- Note: $$v_{gM,i} \equiv \pm \frac{\partial}{\partial k_i} \omega_{pe} = 0$$

  Thus, plasma oscillation is *not a wave* since no information is propagated by the oscillation!

- However, if we let the electrons have a finite temperature the plasma oscillations are turned into *Langmuir waves*!

Plasma oscillations in the dielectric tensor

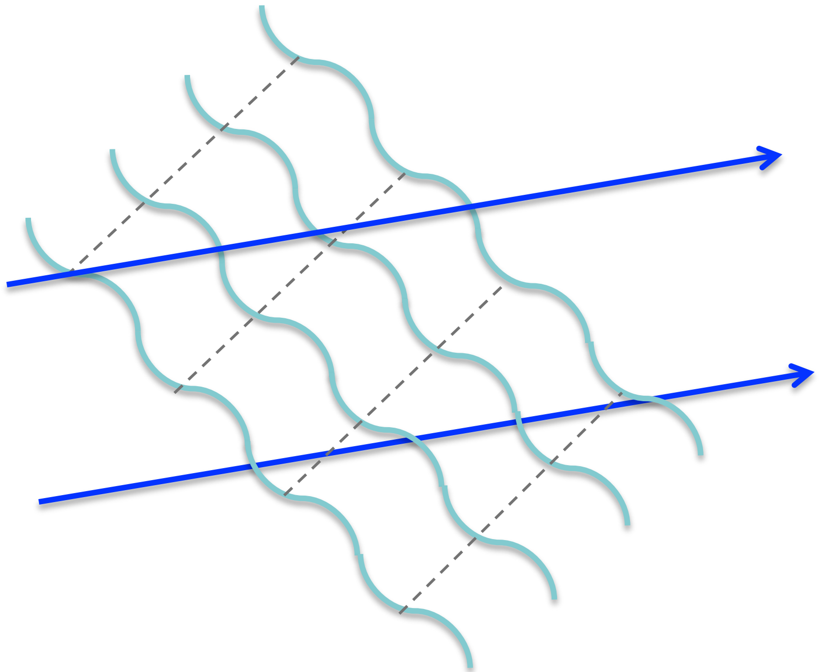
- Let us derive plasma oscillations for from the dielectric tensor.
- Use cold magnetised plasma tensor (see previous lecture)
  - **Assume**: $\mathbf{k}$ parallel to $\mathbf{B}_0$ (the z-direction)

$$K = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \xrightarrow{\text{Wave eq}} \det\begin{pmatrix} S-n^2 & -iD & 0 \\ iD & S-n^2 & 0 \\ 0 & 0 & P \end{pmatrix} = 0$$

- For longitudinal waves: $P = 0$.
  - since $P = 1 - \omega^2/\omega_p^2$, the solution is a plasma oscillation, $\omega=\omega_p$!
- Plasma oscillation can be found
  - in *non-magnetised plasmas* (previous page),
  - when E-field along the *direction of the magnetic field* (see above) and
  - in *almost any media* at very *high frequencies*
    (at high frequency electrons response like free particle)

Outline

- Derivation of the wave equation and definition of wave quantities:
  - dispersion equation / dispersion relation / refractive index
  - wave polarization
- Some math for wave equations (mainly linear algebra)
  - relation between damping and antihermitian part of the dielectric tensor
- Waves in ideal anisotropic media
  - birefringent crystals
- Group velocity
- Plasma oscillations
- Elementary plasma waves
  - Langmuir waves
  - ion-acoustic waves
  - high frequency transverse wave
  - Alfven waves
- Wave resonances & cut-offs



Langmuir waves

- In warm plasmas, plasma oscillations turn into *Langmuir waves*.
- Consider a *non-magnetised warm* plasma with **k** in the *z*-direction

$$K = \begin{pmatrix} K_T & 0 & 0 \\ 0 & K_T & 0 \\ 0 & 0 & K_L \end{pmatrix} \Longrightarrow \det\begin{pmatrix} K_T - n^2 & 0 & 0 \\ 0 & K_T - n^2 & 0 \\ 0 & 0 & K_L \end{pmatrix} = 0$$

$$K_L = 1 + \sum_i \frac{1}{k^2\lambda_{Di}^2}\left[1 - \phi(y_i) + i\sqrt{\pi}\, y_i e^{-y_i^2}\right], \quad K_T = \ldots$$

$$\left\{ \begin{aligned} v_{thi} &\equiv \sqrt{T_i / m_i} \\ \lambda_{Di} &\equiv v_{thi} / w_{pi} \\ y_i &\equiv \omega / 2^{1/2} k v_{thi} \end{aligned} \right.$$

- Langmuir wave, the longitudinal solution: $\Re\{K_L\} \approx 0$
- Neglect ions response and expand in small thermal electron velocity (almost cold electrons); use expansion in Eq. (10.30), gives approximate dispersion relation for *Langmuir waves*

$$\omega^2 = \omega_L^2(k) \approx \omega_{pe}^2 + \boxed{3k^2 v_{the}^2}$$

Letting v_{the}=0 give plasma oscillations!

Polarization and damping of Langmuir waves

- Polarization vector e_i can be obtained from wave equation when inserting the dispersion relation $K_L \approx K_L^H = 0$

$$\begin{pmatrix} K_T - n^2 & 0 & 0 \\ 0 & K_T - n^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad e_i = \delta_{i3}$$

- Thus, the wave damping can be written as

$$\gamma_L = -2i\omega_L(\mathbf{k}) R_L(\mathbf{k}) \left\{ e_{Li}^*(\mathbf{k}) K_{ij}^A(\omega_L(\mathbf{k}),\mathbf{k}) e_{Lj}(\mathbf{k}) \right\} =$$

$$= -2i\omega_L(\mathbf{k}) R_L(\mathbf{k}) K_{33}^A(\omega_L(\mathbf{k}),\mathbf{k}) =$$

$$= -2i\omega_L(\mathbf{k}) R_L(\mathbf{k}) \Im\left\{ K_L(\omega_L(\mathbf{k}),\mathbf{k}) \right\}$$

where $\dfrac{1}{R_L(k)} = \omega \left. \dfrac{\partial \mathrm{Re}\left[K_L(\omega,k)\right]}{\partial \omega} \right|_{\omega = \omega_L(k)}$

Absorption of Langmuir waves

- Inserting the dispersion relation and the expression for K_L gives the energy dissipation rate

$$\gamma_L \approx \left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_{pe}^{\;4}}{v_{the}^{\;3} k^3} N_{res} \quad , \text{ where } N_{res} = \exp\left[-v^2 / 2v_{the}^{\;2}\right]\Big|_{v=\omega_L(k)/k}$$

  - Damping (dissipation) is due to Landau damping, i.e. for electrons with velocities v such that $\omega_L(k) - kv = 0$
  - Here N_{res} is proportional to the number of Landau resonant electrons
- Damping is small for small & large thermal velocities

$$kv_{the} / \omega_L(k) \to 0 \quad \Longrightarrow \quad \gamma_L \sim \lim_{v_{the} \to 0} v_{the}^{\;-3} \exp\left[-v^2 / 2v_{the}^{\;2}\right] \to 0$$

$$kv_{the} / \omega_L(k) \to \infty \quad \Longrightarrow \quad \gamma_L \sim \lim_{v_{the} \to \infty} v_{the}^{\;-3} \exp\left[-0\right] \to 0$$

  - Maximum in damping is when $v_{the} \approx \omega_L(k) / k$

Ion acoustic waves

- In addition to the Langmuir waves there is another important longitudinal plasma wave (i.e. $K_L=0$) called the ion acoustic wave.
- This mode require motion of both ions and electrons. Assume:
  - Fast electrons: $v_{the} >> \omega/k$, expansions (10.29)
  - Slow ions: $v_{thi} << \omega/k$, expansions (10.30)

$$\Re\{K_L\} = 1 + \underbrace{\frac{1}{k^2\lambda_{De}^2}}_{\text{electron}} - \underbrace{\frac{\omega_{pi}^{\ 2}}{\omega^2}}_{\text{ion}} \Rightarrow \begin{cases} \omega = \omega_{IA}(k) \approx \dfrac{kv_s}{\sqrt{1+k^2\lambda_{De}^2}} \\ \gamma_L \approx \left(\dfrac{\pi}{2}\right)^{1/2} \omega_{IA}(k) \left(\dfrac{v_s}{v_{the}} + \left(\dfrac{\omega_s(k)}{kv_{the}}\right)^3 N_{res} \right) \end{cases}$$

  - Here v_s is the sounds speed: $v_s = \omega_{pi}\lambda_{De}^2 = \sqrt{T/m_e}$
  - Again, N_{res} is proportional to the number of Landau resonant electrons
- Ion acoustic waves reduces to normal sounds waves for small $k\lambda_{De}$

$$\omega = \omega_{Sound}(k) \approx kv_s$$

Transverse waves - Modified light waves

- High frequency wave transverse, $\omega >> \omega_{pe}$, behave almost like light waves.
- Expanding cold response in small ω_{pe}/ω gives: $K_T = 1 - \omega^2/\omega_{pe}{}^2$
- Transverse dispersion relation:

$$K_T - n^2 = 0 \quad \Longrightarrow \quad \omega^2 = \omega_T(k)^2 \approx \omega_{pe}{}^2 + c^2 k^2$$

- These waves are very weakly damped;
  - Phase velocity

    $$v_{ph}{}^2 = c^2 + \omega_{pe}{}^2 / k^2 > c^2$$

    thus ***no*** resonant particles and thus ***no Landau damping***!
  - damping can be obtained from collisions;
    for “collision frequency” = ν_e the energy decay rate is

    $$\gamma_T(k) \approx \nu_e \frac{\omega_{pe}{}^2}{\omega^2}$$

Alfven waves (1)

- Next: Low frequency waves in a cold magnetised plasma including both ions and electrons
- These waves were first studied by Hannes Alfvén, here at KTH in 1940. The wave he discovered is now called the *Alfvén wave*.
- To study these waves we choose:
  $\mathbf{B} \parallel \mathbf{e}_z$ and $\mathbf{k} = (k_x , 0 , k_{||})$
- The dielectric tensor for these waves were derived in the previous lecture assuming $\omega << \omega_{ci}, \omega_{pi}$

$$K = \begin{pmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & P \end{pmatrix} \quad \left\{ \begin{array}{l} S \approx c^2 \dfrac{\mu_0 \sum_j m_j n_j}{B^2} = \dfrac{c^2}{V_A^{\,2}} \\ P \approx \dfrac{1}{\omega^2} \sum_j \dfrac{n_j q_j^{\,2}}{m_j \varepsilon_0} = \dfrac{\omega_p^{\,2}}{\omega^2} \end{array} \right.$$

V_A = "*Alfvén speed*"

Alfven waves (2)

- Wave equation
  - for $n_j = ck_j/\omega$

$$\begin{pmatrix} S - n_{\|}^2 & 0 & -n_{\|}n_x \\ 0 & S - n^2 & 0 \\ -n_{\|}n_x & 0 & P - n_x^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_{\|} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- If you put in numbers, then P is huge!
  - Thus, third equations gives $E_{||} \approx 0$ ($E_{||}$ is the E-field along $\mathbf{B}$)
- Why is $E_{||} \approx 0$ for low frequency waves have?
  - electrons can react very *quickly* to any $E_{||}$ perturbation (along $\mathbf{B}$) and *slowly* to $\mathbf{E}$-perturbations perpendicular to $\mathbf{B}$
  - Thus, they allow E-fields to be perpendicular, but not parallel to $\mathbf{B}$!
- We are then left with a 2D system:

$$\begin{pmatrix} S - n_{\|}^2 & 0 & -- \\ 0 & S - n^2 & -- \\ -- & -- & -- \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Alfven waves (3)

- There are two eigenmodes:

$$\begin{pmatrix} S - n_{\|}^{2} & 0 & -- \\ 0 & S - n^{2} & -- \\ -- & -- & -- \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \det\left[\Lambda_{ij}\right] = \left(S - n_{\|}^{2}\right)\left(S - n^{2}\right) = 0$$

  - The shear Alfvén wave (shear wave): $S = n_{\|}^{2}$, or $\omega_A(\mathbf{k}) = k_{\|} V_A$
    - Important in almost all areas of plasma physics e.g. fusion plasma stability, space/astrophysical plasmas, molten metals and other laboratory plasmas
    - Polarisation: *see exercise!*
  - The compressional Alfvén wave: $S = n^{2}$, or $\omega_F(\mathbf{k}) = k V_A$ (fast magnetosonic wave)
    - E.g. used in radio frequency heating of fusion plasmas (my research field)
    - Polarisation: *see exercise!*

Ideal MHD model for Alfven waves

The most simple model that gives the Alfven waves is the linearized *ideal MHD* model for a

- quasi-neutral, low pressure plasma – described by fluid velocity $\mathbf{v}$
- in a static magnetic field $\mathbf{B}_0$
- at low frequency and long wave length

$$nm\frac{d\mathbf{v}}{dt} = \mathbf{j} \times \mathbf{B}_0$$

Momentum balance
(sum of electron and ion momentum balance; $n_e q_e \mathbf{v}_e + n_i q_i \mathbf{v}_i = \mathbf{J}$)

$$\mathbf{E} + \mathbf{v} \times \mathbf{B}_0 = \mathbf{0}$$

Ohms law
(electron momentum balance when $m_e \rightarrow 0$)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Faraday's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

Ampere's law

Wave equation for shear Alfven waves ★

Derivation of wave equation for the shear wave

1. Substitute $\mathbf{E}$ from Ohms law into Faraday's law

$$\nabla \times (\mathbf{v} \times \mathbf{B}_0) = -\frac{\partial \mathbf{B}}{\partial t}$$

2. Take the time derivative of the equation above and use the momentum balance to eliminate the velocity

$$\nabla \times \left(\left(\frac{\mathbf{j} \times \mathbf{B}_0}{mn} \right) \times \mathbf{B}_0 \right) = -\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

3. Assume the induced current to be perpendicular to $\mathbf{B}_0$

$$\frac{|B_0|^2}{mn} \nabla \times \mathbf{j} = -\frac{\partial^2 \mathbf{B}}{\partial t^2} \qquad \text{Note}: \quad \frac{|B_0|^2}{mn} = \mu_0 V_A{}^2$$

4. Finally use Ampere's law to eliminate $\mathbf{j}$

$$\boxed{\nabla \times (\nabla \times \mathbf{B}) + V_A{}^{-2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0}$$

Wave equation with phase & group velocity V_A

Physics of the shear Alfven waves

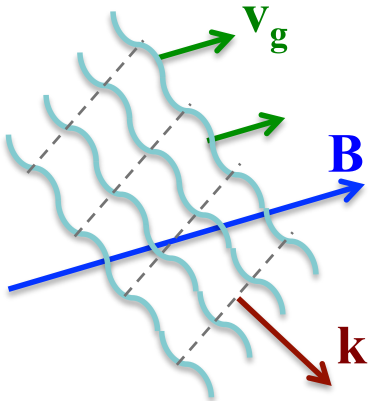
- In MHD the plasma is “frozen into the magnetic field” (see course in Plasma Physics)
  - When plasma move, it “pulls” the field line along with it (eq. 1 prev. page)
  - The plasma give the field lines inertia, thus field lines bend back – like guitar strings!
  - Energy transfer during wave motion:
    - **B**-field is *bent* by plasma motion; work needed to bend field line
      - kinetic energy transferred into field line bending
    - Field lines want to unbend and push the plasma back:
      - energy transfer from *field line bending* to kinetic energy
    - … wave motion!

- **B**-field lines can act like strings:
  - The Alfven wave propagates along field lines like waves on a string!
  - Note: the group velocity always points in the direction of the magnetic field, thus it propagate *along* the fields lines!

Group velocities of the shear wave

- Dispersion relation for the shear Alfven wave: $\omega_A(\mathbf{k}) = V_A k_{\|} = V_A \mathbf{k} \bullet \mathbf{B} / |\mathbf{B}|$
  – phase velocity: $\mathbf{v}_{phA} \equiv \pm \frac{V_A k_{\|}}{k} \frac{\mathbf{k}}{k}$
  – group velocity: $\mathbf{v}_{gA} \equiv \frac{\partial}{\partial \mathbf{k}} \left(\pm V_A k_{\|} \right) = \pm V_A \frac{\mathbf{B}}{|\mathbf{B}|}$
    - *wave front* moves with $\mathbf{v}_{phA}$, along $\mathbf{k} = (\, k_x \,,\, 0 \,,\, k_{||} \,)$
    - wave-energy moves with $\mathbf{v}_{gA}$, along $\mathbf{B} = (\, 0 \,,\, 0 \,,\, B_0 \,)$!
  – Thus, a shear Alfven waves is "trapped to follow magnetic field lines"
    - like waves propagating along a string
  – Note also:

$$\left| \mathbf{v}_{gA} \right| = V_A \geq \left| \mathbf{v}_{phA} \right|$$



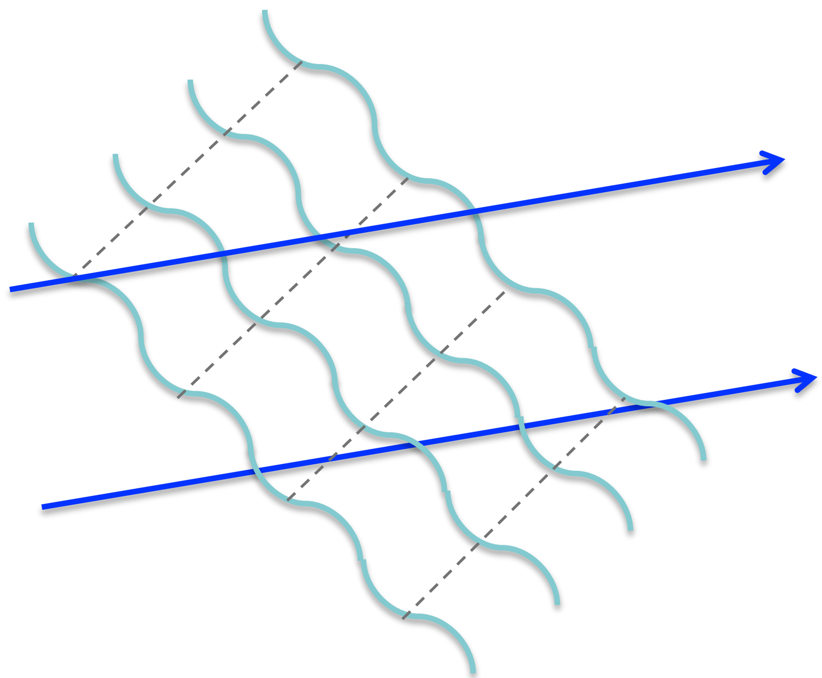
- Fast magnetosonic wave $\omega_F(\mathbf{k}) = V_A k$ is not dispersive!

$$\mathbf{v}_{gF,i} = \mathbf{v}_{phF,i} = V_A \frac{\mathbf{k}}{k}$$

- Thus, an external source may excite two Alfven wave modes propagating in different directions, with different speed

Outline

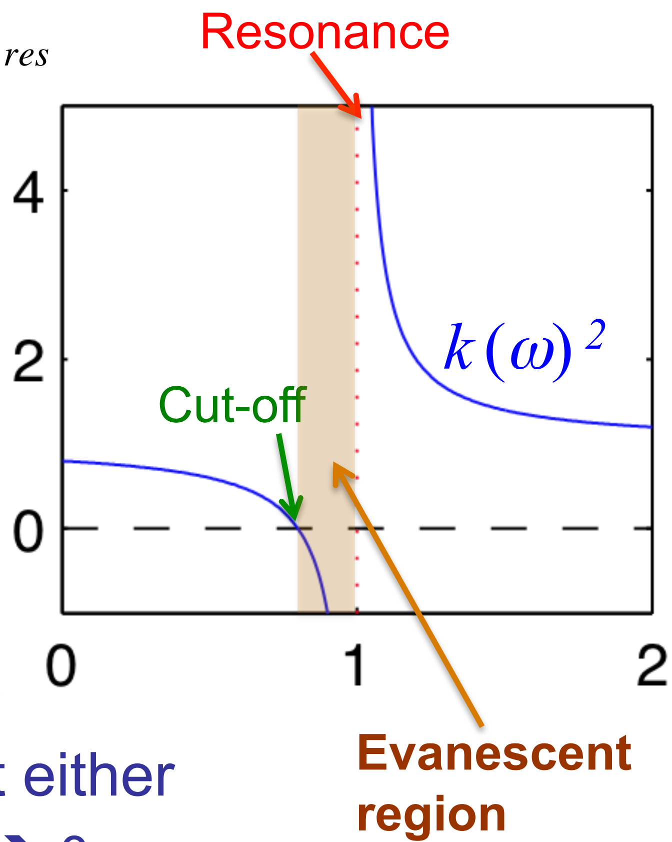
- Derivation of the wave equation and definition of wave quantities:
  - dispersion equation / dispersion relation / refractive index
  - wave polarization
- Some math for wave equations (mainly linear algebra)
  - relation between damping and antihermitian part of the dielectric tensor
- Waves in ideal anisotropic media
  - birefringent crystals
- Group velocity
- Plasma oscillations
- Elementary plasma waves
  - Langmuir waves
  - ion-acoustic waves
  - high frequency transverse wave
  - Alfven waves
- Wave resonances & cut-offs



Resonances, cut offs & evanescent waves

- Dispersion relation often has singularities of the form

$$k(\omega)^2 \sim 1 + \frac{\omega_1}{\omega - \omega_{res}} = \frac{\omega - \omega_{cut}}{\omega - \omega_{res}} \quad , \quad \omega_{cut} = \omega_1 - \omega_{res}$$



- 3 regions with different types of waves
  - $\omega < \omega_{cut}$ & $\omega > \omega_{res}$, then $k^2 > 0$

$$E \sim E_1 \exp(i|k|x) + E_2 \exp(-i|k|x)$$

  - $\omega_{cut} < \omega < \omega_{res}$, then $k^2 < 0$

$$E \sim E_1 \exp(|k|x) + E_2 \exp(-|k|x)$$

    called **evanescent waves** (growing/decaying)

- The transitions to evanescent waves occur at either
  - ***resonances*** ; k ➔ ∞ , i.e. the wave length λ ➔ 0
    - here at $\omega = \omega_{res}$
  - ***cut-offs*** ; k ➔ 0, i.e. the wave length λ ➔ ∞
    - here at $\omega = \omega_{cut}$

CMA diagram for cold plasma with ions and electrons

- This plasma model can have either 0, 1 or 2 wave modes

- The modes are illustrated in the ***CMA diagram***

- 3 symbols representing different types of anisotropy:
  - ellipse: 
  - “eight”: 8
  - “infinity”: ∞

  (*don’t need to know the details*)

- When moving in the diagram mode disappear/appear at:
  - ***resonances*** ; k ➔ ∞
  - ***cut-offs*** ; k ➔ 0

figure missing

Conclusions

- The electro-magnetic wave equation can be written as a matrix eq.

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{A})+(\omega/c)^2\mathbf{A}+\mu_0\mathbf{J}_{ind}(\mathbf{A})=-\mu_0\mathbf{J}_{ext}$$

$$\Lambda_{ij}A_j=-\frac{\mu_0c^2}{\omega^2}J_{\exp,\,i}\quad\text{where}\quad\Lambda_{ij}=\frac{c^2}{\omega^2}\left(k_ik_j-|\mathbf{k}|^2\delta_{ij}\right)+K_{ij}$$

- Solutions to the wave equation for small damping

$$\omega=\omega_M(\mathbf{k})\quad\text{such that :}\quad\det\left[\Lambda_{ij}^H\left(\omega_M(\mathbf{k}),\mathbf{k}\right)\right]+O(\omega_I^{\;2})=0$$

$$\omega_I=\frac{i\lambda_{ij}K_{ij}^A\left(\omega_M(\mathbf{k}),\mathbf{k}\right)}{\frac{\partial}{\partial\omega}\det\left[\Lambda_{nm}^H(\omega,\mathbf{k})\right]_{\omega=\omega_M(\mathbf{k})}}+O(\omega_I^{\;2})$$

- **Polarization vectors**; the normalised eigenvectors of Λ.

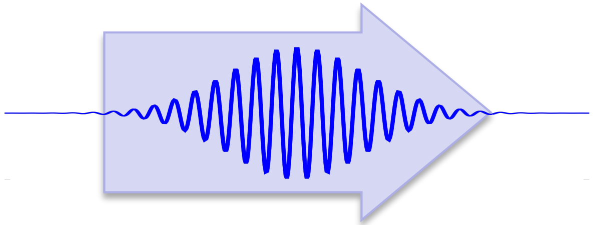
Conclusions

- Birefringency:

$$K(\omega) = \begin{pmatrix} K_\perp(\omega) & 0 & 0 \\ 0 & K_\perp(\omega) & 0 \\ 0 & 0 & K_\parallel(\omega) \end{pmatrix} \Rightarrow \begin{cases} n_O{}^2 = K_\perp \\ n_X{}^2 = \dfrac{K_\perp K_\parallel}{K_\perp \sin^2\theta + K_\parallel \cos^2\theta} \end{cases}$$

- Group velocity: velocity of information/energy

$$v_g = \frac{d\omega_{M0}}{dk}$$



- **Examples:**

  (illustrating dispersion, difference between group/phase velocity)

  – Plasma oscillations, Langmuir waves, ion-acoustic waves, high frequency transverse wave, Alfven waves

- Wave resonances ($k^2 \to \infty$) & cut-offs ($k^2 \to 0$)