

KTH Teknikvetenskap

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## Contents

4 Curves in the projective plane ..... 1
4.1 Lines ..... 1
4.1.3 The dual projective plane $\left(\mathbb{P}^{2}\right)^{*}$ ..... 2
4.1.5 Automorphisms of $\mathbb{P}^{2}$ ..... 2
4.2 Conic sections ..... 3
4.2.1 Conics as the intersection of a plane and a cone ..... 3
4.2.2 Parametrization of irreducible conics ..... 3
4.2.6 The parameter space of conics ..... 5
4.2.8 Classification of conics ..... 5
4.2.13 The real case vs the complex case ..... 6
4.2.14 Pascal's Theorem ..... 6

## Chapter 4

## Curves in the projective plane

We will in this chapter study different aspects of plane curves by which we mean curves in the projective plane defined by polynomial equations. Here we will start with the more classical setting and consider a plane curve as the set of solutions of one homogenous equation in three variables.
We will start by choosing a field, $k$, which in most cases can be thought of as either $\mathbb{R}$ or $\mathbb{C}$, but sometimes, it is interesting also to look at $\mathbb{Q}$ or finite fields.
The first definition we might try is the following.
Definition 4.0.1. A plane curve $C$ is the set of solutions in $\mathbb{P}_{k}^{2}$ of a non-zero homogeneous equation

$$
f(x, y, z)=0 .
$$

Example 4.0.2. The equation $x^{2}+y^{2}+z^{2}=0$ defines a degree two curve over $\mathbb{C}$ but over $\mathbb{R}$ it gives the empty set.
The equation $x^{2}=0$ has a solution set consising of the line $(0: s: t)$ while the degree of the equation is two.

The example above shows us that there are curves that the definition does not give us any one-one correspondance between curves and equations.

### 4.1 Lines

We will start by the easiest curves in the plane, namely lines. These are defined by linear equations

$$
\begin{equation*}
a x+b y+c z=0 \tag{4.1}
\end{equation*}
$$

where $(a, b, c) \neq(0,0,0)$. Observe that any non-zero scalar multiple of $(a, b, c)$ has the same set of solutions, which show us that we can parametrize all the lines in $\mathbb{P}_{k}^{2}$ by another projective plane with coordinates $[a: b: c]$.
Theorem 4.1.1. Any two distinct lines in $\mathbb{P}^{2}$ intersect at a single point.
Proof. The condition that the lines are distinct is the same thing as the equations defining them being linearly independent, which gives a unique solution to the system of equations.

Theorem 4.1.2. Any line in $\mathbb{P}^{2}$ is isomorphic to $\mathbb{P}^{1}$.
Proof. By a change of coordinates the equation of a line can be written as $x=0$ and the solutions are given by $[0: s: t]$ where $(s, t) \neq(0,0)$, which as a set equals $\mathbb{P}^{1}$.
In fact, using this parametrization, we can define a map $\mathbb{P}^{1} \longrightarrow \mathbb{P}^{2}$, which has the given line as the image.
We will come back to what we mean by isomorphism later on in order to make this more precise.

### 4.1.3 The dual projective plane $\left(\mathbb{P}^{2}\right)^{*}$

As mentioned above, the coefficients $a, b, c$ of Equation 4.1, give us natural coordinates on the space of lines in $\mathbb{P}^{2}$ and we will call this the dual projective plane, denoted by $\left(\mathbb{P}^{2}\right)^{*}$.
Theorem 4.1.4. The set of lines through a given point in $\mathbb{P}^{2}$ is parametrized by a line in $\left(\mathbb{P}^{2}\right)^{*}$.

Proof. Equation 4.1 is symmetric in the two sets of variables, $\{x, y, z\}$ and $\{a, b, c\}$. Thus, fixing $[x: y: z]$ gives a line in $\left(\mathbb{P}^{2}\right)^{*}$.

### 4.1.5 Automorphisms of $\mathbb{P}^{2}$

A linear change of coordinates on $\mathbb{P}_{k}^{2}$ is given by a non-singular $3 \times 3$-matrix with entries in $k$ :

$$
\left[\begin{array}{l}
x^{\prime} \\
y^{\prime} \\
z^{\prime}
\end{array}\right]=\left[\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right]\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right] .
$$

Because of the identification $[x: y: z]=[\lambda x: \lambda y: \lambda z]$, the scalar matrices correspond to the identity. The resulting group of automorphisms is called $\operatorname{PGL}(3, k)$.

### 4.2 Conic sections

We will now focus on quadratic plane curves, or conics. These are defined by a homogeneous quadratic equation

$$
a x^{2}+b x y+c y^{2}+d x z+e y z+f z^{2}=0 .
$$

### 4.2.1 Conics as the intersection of a plane and a cone

The name conic is short for conic section and comes from the fact that each such curve can be realized as the intersection of a plane and a circular cone

$$
x^{2}+y^{2}=z^{2}
$$

in $\mathbb{P}^{3}$.


Figure 4.1: The circular cone

### 4.2.2 Parametrization of irreducible conics

The conic section is irreducible if the polynomial defining it is not a product of two non-trivial polynomials.

Theorem 4.2.3. If $C$ is a plane irreducible conic with at least two rational points, then $C$ is isomorphic to $\mathbb{P}_{k}^{1}$.

Proof. Let $P$ be a rational point of $C$ and let $L$ denote the line in $\left(\mathbb{P}^{2}\right)^{*}$ parametrizing lines through $P$. In the coordinates of each line, the polynomial equation reduces to a homogeneous quadratic polynomial in two variables with at least one rational root. Without loss of generality, we may assume that $P$ is $[0: 0: 1]$ and the equation of $C$ has the form

$$
a x^{2}+b x y+c y^{2}+d x z+e y z=0 .
$$



Figure 4.2: The hyperbola, parabola and ellips as a plane sections of a cone

The lines through $P$ are parametrized by a $\mathbb{P}^{1}$ with coordinates $[s: t]$ and we ge the residual intersection between the curve and the line $s x+t y=0$ as

$$
R=\left[e s t-d t^{2}: d s t-e s^{2}: c s^{2}-b s t+a t^{2}\right] .
$$

Since $C$ has another rational point, $Q$, we cannot have $d=e=0$ since $C$ is irreducible. Hence the residual point $R$ is not equal to $P$ except for one $[s, t]$. Moreover, by the next exercise, we get that the three coordinates are never zero at the same time. Hence we have a non-trivial map from $\mathbb{P}^{1}$ to $\mathbb{P}^{2}$ whose image is in $C$. If the image was a line, $C$ would be reducible and we conclude that $C$ is the image of the map.

Excercise 4.2.4. Let $C$ be a conic passing throught the point [0:0:1], i.e., having equation of the form

$$
a x^{2}+b x y+c y^{2}+d x z+e y z=0
$$

Show that $C$ has is reducible if and only if $c d^{2}-b d e+a e^{2}=0$.
Example 4.2.5. The example $x^{2}+y^{2}=0$ with $k$ a field with no square root of -1 shows that we cannot drop the condition that $C$ has at least two rational points.

### 4.2.6 The parameter space of conics

Exacty as for the lines, we have that the equation

$$
a x^{2}+b x y+c y^{2}+d x z+e y z+f z^{2}=0
$$

defines the same curve when multiplied with a non-zero constant. Hence all the conics can be parametrized by a $\mathbb{P}^{5}$ with coordinates $[a: b: c: d: e: f]$. In this parameter space we can look at loci where the conics have various properties. For example, we can look at the locus of degenerate conics that are double lines. These are parametrized by a $\mathbb{P}^{2}$ and the locus of such curves is the image of the Veronese embedding of $\mathbb{P}^{2}$ in $\mathbb{P}^{5}$ defined by

$$
[s: t: u] \mapsto\left[s^{2}: 2 s t: t^{2}: 2 s u: 2 t u: u^{2}\right] .
$$

If we want to look at all the curves that are degenerate as a union of two lines, we look at the image of a map

$$
\Phi: \mathbb{P}^{2} \times \mathbb{P}^{2} \longrightarrow \mathbb{P}^{5}
$$

given by
$\left(\left[s_{1}: t_{1}: u_{1}\right],\left[s_{2}: t_{2}: u_{2}\right]\right) \mapsto\left(s_{1} s_{2}: s_{1} t_{2}+t_{1} s_{2}: t_{1} t_{2}: s_{1} u_{2}+u_{1} s_{2}: t_{1} u_{2}+u_{1} t_{2}: u_{1} u_{2}\right]$.
The image of $\Phi$ is a hypersurface in $\mathbb{P}^{5}$ which means that is is defined by one single equation in the coordinates $[a: b: c: d: e: f: g]$.
Excercise 4.2.7. Find the equation of the hypersurface definied by the image of the map $\Phi: \mathbb{P}^{2} \times \mathbb{P}^{2} \longrightarrow \mathbb{P}^{5}$ defined above.

### 4.2.8 Classification of conics

When we want to classify the possible conics up to projective equivalence, we need to see how the group of linear automorphisms acts. One way to go back to our knowledge of quadratic forms. If 2 is invertible in $k$, i.e., if $k$ does not have characteristic 2 , we may write the equation

$$
a x^{2}+b x y+c y^{2}+d x z+e y z+f z^{2}=0
$$

as $Q(x, y, z)=0$, where $Q$ is the quadratic form associated to the matrix

$$
A=\frac{1}{2}\left[\begin{array}{ccc}
2 a & b & d \\
b & 2 c & e \\
d & e & 2 f
\end{array}\right]
$$

Now, a matrix $P$ from $\operatorname{PGL}(3, k)$ acts on $A$ by

$$
Q \mapsto P^{T} A P
$$

Theorem 4.2.9. Up to projective equivalence, the equation of a conic can be written in one of the three forms

$$
x^{2}=0, \quad x^{2}+\lambda y^{2}=0 \quad \text { and } \quad x^{2}+\lambda y^{2}+\mu z^{2}=0 .
$$

Proof. The first thing that we observe is invariant is the rank of the matrix. If the rank is one, we can choose two of the columns of $P$ to be in the kernel of $A$ and hence after a change coordinates, the equation is $\lambda x^{2}=0$, but this is equivalent to $x^{2}=0$.
If the rank is two, we choose one of the columns to be a generator of the kernel and we get that we can assume that $d=e=f=0$. By completing the square, we can change it into $\kappa x^{2}+\mu y^{2}$, which is equivalent to $x^{2}+\lambda y^{2}$, where $\lambda=\mu / \kappa$.
If the rank is three, proceed by completing the squares in order to write the form as $x^{2}+\lambda y^{2}+\mu z^{2}$.

Remark 4.2.10. In order to further characterize the conics, we need to know about the multiplicative group of our field. In particular, we need to know the quotient of $k^{*}$ by the subgroup of squares.

Theorem 4.2.11. Let $k=\mathbb{C}$. Then there are only three conics up to projective equivalence:

$$
x^{2}=0, \quad x^{2}+y^{2}=0 \quad \text { and } \quad x^{2}+y^{2}+z^{2}=0 .
$$

Proof. Since every complex number is a square, we can change coordinates so that $\lambda=\mu=1$ in Theorem 4.2.9.

Theorem 4.2.12. Let $k=\mathbb{R}$. Then there are four conics up to projective equivalence:

$$
x^{2}=0, \quad x^{2}+y^{2}=0 \quad x^{2}-y^{2}=0, \quad x^{2}+y^{2}-z^{2}=0 .
$$

Proof. Here, only the positive real numbers are squares and we have to distinguish between the various signs of $\lambda$ and $\mu$. If $\lambda=\mu=1$ we get the empty curve, so there is only one non-degenerate curve $x^{2}+y^{2}=z^{2}$.

### 4.2.13 The real case vs the complex case

### 4.2.14 Pascal's Theorem

We will look at a classical theorem by Pascal about conics.


Figure 4.3: Pacsal's Theorem
Theorem 4.2.15 (Pascal's Theorem). Let $C$ be a plane conic and $H$ be a hexagon with its vertices on $C$. The three pairs of opposite sides of the hexagon meet in three collinear points.

There are several ways to understand this theorem and we will now look at one way.

Proof. Start by dividing the lines into two groups of three lines so that no two lines in the same group intersect on the conic $C$.


Figure 4.4: The two groups of lines
Each group of three lines defines a cubic plane curve, given by the product of the three linear equations defining the lines. Since each line in one group
meets each of the lines from the other group, we have nine points of intersections of lines from the two groups. Six of these are on the conic and it remains for us to prove that the remaining three are collinear.
Choose two of the points and take the line $L$ through them. Together with the conic, the line defines a cubic curve, i.e., there is a cubic polynomial vanishing on the line and the conic. In particular, this cubic curve passes through eight of our nine points. We already have two cubic curves passing through all nine points. If the last cubic didn't pass through all nine points, we would have three linearly independent cubic polynomials passing through our eight points.
Denote the three cubic polynomials by $f_{1}, f_{2}$ and $f_{3}$. They can generate seven or eight linearly independent polynomials of degree four. If they generate eight, we get that it will generate a space of codimension 7 in all higher degree, by multiplication by a linear form not passing through any of the points. If they generate only seven linearly independent forms of degree four, we must have two linearly independent syzygies, i.e., relations of the form

$$
\left\{\begin{array}{l}
\ell_{1} f_{1}+\ell_{2} f_{2}+\ell_{3} f_{3}=0 \\
\ell_{4} f_{1}+\ell_{5} f_{2}+\ell_{6} f_{3}=0
\end{array}\right.
$$

Since there is a unique solution to this system up to multiplication by a polynomial, we get that

$$
\left(f_{1}, f_{2}, f_{3}\right)=\ell\left(\ell_{2} \ell_{6}-\ell_{3} \ell_{5}, \ell_{3} \ell_{4}-\ell_{1} \ell_{6}, \ell_{1} \ell_{5}-\ell_{2} \ell_{4}\right)
$$

showing that the three cubics share a common linear factor. However, this cannot be the case, since the two original cubics did not have a common factor.
We conclude that the cubic passing throug eight of the nine point also pass throug the ninth, which shows that the three that were not on the conic have to be collinear.

The property that any cubic passing through eight of the nine points also has to pass through the ninth point is known as the Cayley-Bacharach property and similar consequences occur in much more general situations.

