Study questions
2015-02-08

1. Describe the Euler and the Lagrange coordinate systems and derive the expression for the rate of change of a given quantity F in the Euler coordinate system.

2. State the definition of the material derivative, $\frac{\mathrm{D}}{\mathrm{D}t}$, and explain the meaning of its different components.

3. State the compressible continuity, momentum and energy equations in non-dimensional form and give the definition of the flow parameters.

4. What is the definition of a Newtonian fluid? Give the expression for viscous stress tensor for such a fluid (define the coefficients involved). What is the Stokes hypothesis in this context?

5. Consider the incompressible Navier–Stokes equation in 2D:

   a) Derive the boundary-layer equation in x-direction.
   b) Which condition for the pressure field is found from the boundary-layer approximations applied to the y-momentum equation?

6. Describe when a system of partial differential equations together with initial and boundary conditions is well posed. Give one example of an equation with initial and boundary conditions that is well posed, and one example that is not well posed.

7. Numerical accuracy

   a) In homework 2, the "machine epsilon" ε has been discussed. Give its approximative value for default MATLAB variables. What is this accuracy level usually referred to, and how many bytes of storage is required for a single floating-point variable of that type? Give also a definition of ε, relating it to the error of a floating-point number.
   b) Assume a (fairly inaccurate) computer with $\varepsilon = 10^{-3}$. Using the formula for error propagation, compute the relative error of adding the two floating-point numbers $x = 1.00$ and $y = 1.01$. How large is the absolute error (*i.e.* $x + y = 2.01 \pm ??$)?
   c) Using the same computer as in b), compute now the numerical difference $x - y$ and give again the relative and absolute error of the result.

8. Show that the heat equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

   is parabolic.

9. Describe in words how the solution of elliptic, parabolic and hyperbolic equations behave, *i.e.* what physical processes are described by such equations. Discuss which of the types allow wave-like solutions.

10. Derive the difference formula and the corresponding leading error term for the second derivative using a central three-point scheme. What is the order of that scheme?

11. Consider the integration of an ordinary differential equation $u' = \lambda u$.

    a) Write down the explicit and implicit first-order Euler discretisation. Discuss briefly the main difference between the two methods, and comment on advantages and disadvantages.

    b) Derive the region of absolute stability for the **explicit** Euler scheme. Compute the amplification factor G. Sketch the solution in the complex plane $z = \lambda \Delta t$. Is the scheme absolutely stable?

    c) Set $\lambda = -1$. Which integration scheme(s) would you use and what is the maximum possible time step? Motivate your answer.

    d) Compared to the implicit scheme, does the explicit scheme give less accurate results, *i.e.* what is the order of the two schemes?

12. Derive the region of absolute stability for the **implicit** Euler scheme using the test equation $u' = \lambda u$. Compute the amplification factor G. Sketch the solution in the complex plane $z = \lambda \Delta t$. Is the scheme absolutely stable?

13. Derive the region of absolute stability for the **implicit** Crank-Nicolson scheme using the test equation $u' = \lambda u$. Compute the amplification factor G. Sketch the solution in the complex plane $z = \lambda \Delta t$. Is the scheme absolutely stable?

14. Imagine you have a very stiff problem, *i.e.* in the equation $u' = \lambda u$ the λ goes to $-\infty$. Derive for the explicit Euler, the implicit Euler and the Crank-Nicolson scheme the amplification factor G for stiff problems. Which of these schemes provide a good approximation to the real (analytical) amplification for one time step?

15. You want to solve the ordinary differential equation $u' = \lambda u$ with $\lambda = 3\sqrt{-1}$. Which integration scheme(s) would you use? Motivate your answer.

16. Consider the advection-diffusion equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \ .$$

    Discretise this equation in space using central schemes and in time using the explicit first-order Euler scheme.

    a) Write down the discretised equation. Use as abbreviations $\sigma = a\Delta t/\Delta x$ and $\beta = \nu \Delta t / \Delta x^2$.

b) Perform a von-Neumann stability analysis using the Fourier modes $\hat{u}_\xi e^{\mathrm{i}\xi x}$. In particular, compute the amplification factor $\hat{G}(\xi \Delta x)$. A condition for stability depending on β alone can then be derived by setting $\xi \Delta x = \pi$. Another condition relating σ and β can be found close to $\xi \Delta x = 0$ using the expansions $\sin \xi \Delta x \approx \xi \Delta x$ and $\cos \xi \Delta x - 1 \approx -\frac{1}{2}(\xi \Delta x)^2$. Neglect terms of order $(\xi \Delta x)^4$ and above. What are then the conditions for stability in terms of σ and β?

c) Can you derive an explicit condition for the maximum time step? Does it depend on the spatial grid spacing Δx?

17. Consider the diffusion equation,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \ .$$

Discretise this equation in space using central schemes and in time using the explicit first-order Euler scheme.

a) Write down the fully discretised equation. Use as abbreviation $\beta = \nu \Delta t / \Delta x^2$.

b) Perform a von-Neumann stability analysis using the Fourier modes $\hat{u}_\xi e^{\mathrm{i}\xi x}$. In particular, compute the amplification factor $\hat{G}(\xi \Delta x)$. Derive a condition for stability as a function of β and the discretisation parameters Δt and Δx.

c) What is the main difference of this stability condition compared to the CFL condition of the advection equation? What possibilities do you see to improve the stability of the viscous term?

d) Instead of an explicit Euler scheme, derive the stability limit for the implicit Euler scheme of first order. To this end, perform a von-Neumann analysis and compute the amplification factor $\hat{G}(\xi \Delta x)$. What is the condition on stability for this scheme?

d) Consider now a two-dimensional grid with directions x and y. Rederive the stability condition for the explicit scheme, assuming $\Delta x = \Delta y$. How does the result differ from part b)?

18. On the example of the flow around an airplane wing, discuss – as a function of the angle of attack – the regions in which viscosity is important and regions which can be treated inviscidly. Which equations would you use in the different regions of the flow?

19. Write down the compressible Euler equations in conservative form. Briefly discuss the physical meaning of the individual terms, and the physical concept that leads to the formulation of the equations. Write down the system in such a way that it is completely closed, *i.e.* the same number of equations as the number of unknowns. Of what type are the unsteady Euler equations (no derivation needed)? What are the conservative variables? What are the primitive variables?

20. Define and use the Rankine-Hugoniot jump condition to compute the shock speed for the following problem

$$\begin{aligned} u_t + uu_x &= 0 \quad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= \begin{cases} 1 & x \leq 0 \\ 0 & \text{otherwise} \ . \end{cases} \end{aligned}$$

How would the solution look like if the initial condition is reversed, *i.e.*

$$u(x,0) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \ . \end{cases}$$

21. Define the entropy condition for a scalar conservation law.

$$u_t + f(u)_x = 0 \quad -\infty < x < \infty, \quad t > 0$$

with a convex flux function $f(u)$. The shock is moving with speed s and the state to the left is given by u_L and the state to the right by u_R.

In what case do we need an entropy condition (expansion fan, shock)?

22. Investigate the one-sided difference scheme

$$u_j^{n+1} = u_j^n - a\frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)$$

for the advection equation

$$u_t + au_x = 0 \ .$$

Consider the cases $a > 0$ and $a < 0$.

a) Prove that the scheme is consistent and find the order of accuracy. Assume $\Delta t/\Delta x$ constant.

b) Determine the stability requirement for $a > 0$ and show that it is unstable for $a < 0$.

23. Apply the Lax–Friedrichs scheme to the advection equation

$$u_t + au_x = 0$$

that is,

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)$$

a) Write down the modified differential equation.

b) What type of equations is this?

c) What kind of behaviour can we expect from the solution?

24. Sketch the effect of diffusive and dispersive errors on the advection of a top-hat (a signal with discontinuity) signal. What terms are known to cause such errors? If you consider the advection of a pure sine wave, what are the effects of diffusive and dispersive errors?

25. For the general wave $u(x,t) = \hat{u}\exp(kx - \omega t)$, derive the dispersion relation between k and ω for

    a) the advection equation $u_t + cu_x = 0$,
    b) the heat equation $u_t = \nu u_{xx}$,
    c) the dispersive equation $u_t = \alpha u_{xxx}$,

    and compute for all cases the group and phase velocity. Discuss your results in terms of the expected behaviour of the solutions to these equations.

26. The three-point centred scheme applied to

$$u_t + au_x = 0, \quad a > 0$$

    yields the approximation

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)$$

    Show that this approximation is not stable (using an appropriate method to show stability) even though the CFL condition is fulfilled (discuss why the CFL condition is indeed fulfilled for instance with a sketch).

27. What does the Lax(–Richtmyer) equivalence theorem state?

28. What is the condition on the $n \times n$ real matrix $A(\mathbf{u})$ for the system

$$\mathbf{u}_t + A\mathbf{u}_x = 0$$

    to be hyperbolic?

29. The barotropic gas dynamic equations

$$\begin{aligned} \rho_t + \rho u_x &= 0 \\ u_t + uu_x + \frac{1}{\rho}p_x &= 0 \end{aligned} \tag{1}$$

    where

$$p = p(\rho) = C\rho^\gamma$$

    and C a constant, can be linearised by considering small perturbations (ρ', u') around a motionless gas.

    a) Let $\rho = \rho_0 + \rho'$ and $u = u_0 + u'$ where $u_0 = 0$. Linearise the system (1) and show that this yields the following linear system (the primes have been dropped)

$$\begin{aligned} \rho_t + \rho_0 u_x &= 0 \\ u_t + \frac{a^2}{\rho_0}\rho_x &= 0 \end{aligned} \tag{2}$$

       where a is the speed of sound. a and ρ_0 are constants.

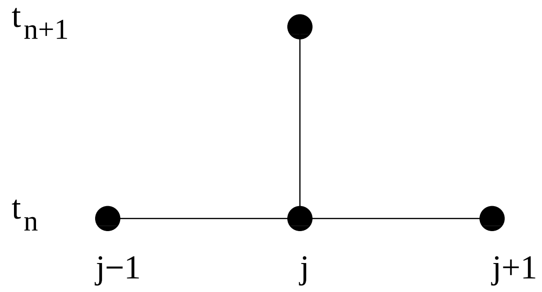


Figure 1: Computational stencil.

b) Is the system given by (2) a hyperbolic system? Motivate your answer.

c) Determine the characteristic variables in terms of ρ and u.

d) Determine the partial differential equations that are fulfilled by the characteristic variables, *i.e.* the characteristic formulation.

e) Let $-\infty < x < \infty$ (no boundaries) and the initial conditions at $t = 0$ are

$$\rho(0, x) = \sin(x) \quad u(0, x) = 0 \ .$$

Determine the analytical solution of equation (2) for $t > 0$. Hint: Start from the characteristic formulation.

30. The linearised form of the barotropic gas dynamics equations (1) is given by

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \underbrace{\begin{pmatrix} 0 & \rho_0 \\ c^2/\rho_0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0, \tag{3}$$

where c is the speed of sound. c and ρ_0 are constants.

a) Draw the domain of dependence of the solution to the system (3) in a point P in the x-t plane.

b) The system is solved numerically on a grid given by $x_j = j\Delta x, j = 0, 1, 2...$ and $t_n = n\Delta t, n = 0, 1, 2, ...$ using an explicit three-point scheme, see Figure 1.

Draw the domain of dependence of the numerical solution at P (in the same figure as a)) of the three-point scheme in the case when

i) the CFL condition is fulfilled

ii) the CFL condition is NOT fulfilled.

Assume that P is a grid point.

31. Consider the Euler equations in 1D

$$\begin{aligned} \rho_t + \rho u_x + u\rho_x &= 0 \\ u_t + uu_x + \frac{1}{\rho} p_x &= 0 \\ p_t + \rho c^2 u_x + up_x &= 0 \ . \end{aligned}$$

How many boundary conditions must be added at the
inflow boundary when the flow is

a) Supersonic
b) Subsonic

outflow boundary when the flow is

c) Supersonic
d) Subsonic

Motivate your answer!

32. Consider the shock tube problem described by the isentropic Euler equations in one space dimension:

$$\begin{pmatrix} \rho \\ \rho u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \end{pmatrix}_x = 0 \ . \tag{4}$$

At $t = 0$ a membrane is separating a region with a high-pressure gas from a region with gas at a lower pressure.

a) Describe how the solution is evolving as a function of time once the membrane is removed.

b) What type of discontinuity is excluded when solving equation (4) instead of the full Euler equations, and why?

33. Give at least one reason for using artificial viscosity when solving a conservation law using the MacCormack scheme. Why does one not use any artificial viscosity in an upwind discretisation?

34. During the lecture, we have encountered various names for the variables appearing in conservation laws. Give an explanation what primitive/quasi-linear, conservative and characteristic variables/formulations are. You can show your finding on the example of a 2×2 system.

35. Projection on a divergence-free space.

   a) Show that a vector field w_i can be decomposed into

   $$w_i = u_i + \frac{\partial p}{\partial x_i}$$

   where u is divergence free and parallel to the boundary.

   b) Apply this to the Navier–Stokes equations, show that the pressure term disappears and recover an equation for the pressure from the gradient part.

36. From the differential form of the Navier–Stokes equations obtain the Navier–Stokes equations in integral form used in finite-volume discretisations.

37. Finite volume (FV) discretisation.
    (a) Derive the finite volume (FV) discretisation of the continuity equation ($\partial u_i/\partial x_i = 0$) on an arbitrary grid,
    (b) derive the FV discretisation for the Laplace equation on an arbitrary grid,
    (c) show that both are equivalent to a central difference approximation for Cartesian grids.
38. State the difficulties associated with the finite-volume discretisations of the Navier–Stokes equations on a collocated grid and show the form of the spurious solution which exist.
39. Staggered grid.
    (a) Define an appropriate staggered grid that can be used for the discretisation of the Navier–Stokes equations,
    (b) Write down the FV discretisation of the Navier–Stokes equations on a staggered Cartesian grid,
    (c) Discuss how to treat no-slip and inflow/outflow boundary conditions.
40. Describe the artificial-compressibility method for solving the Navier–Stokes equations for a steady incompressible flow.
41. Time dependent flows.
    (a) Define a simple projection method for the time dependent incompressible Navier–Stokes equations
    $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{N}(u) & \mathbf{G} \\ \mathbf{D} & \mathbf{0} \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$
    (b) Show in detail the equation for the pressure to be solved in each time step and discuss the boundary conditions for the pressure.
42. Time-step restriction for Navier–Stokes solutions.
    (a) Motivate the use of an appropriate form of the advection-diffusion equation as a model equation for stability analysis,
    (b) Identify the various terms that appear in the advection-diffusion equation, and given them a physical meaning,
    (c) Derive the time-step limitations for the two main parts of that equation separately. Which of the limits is most dangerous for low Reynolds number (*i.e.* high viscosity), and which one for high Reynolds number?
    (d) How would one proceed when calculating the combined stability limit using both contributions at the same time?
    (e) State the 2D equivalent of that restriction.

43. Consider the project for this course: We were developing a MATLAB code for solving the incompressible Navier–Stokes equations using second-order finite differences on a staggered grid using first order time integration. List at least 5 points of possible improvements for this code in the order of importance (according to your opinion), and motivate why a better method for these aspects is appropriate.

44. Consider one-dimensional derivative matrices $\underline{\underline{D}}_1$ and $\underline{\underline{D}}_2$ for the first and second derivative of a vector $\underline{u}$ based on central differences. Write down the complete matrices for the case of $\underline{u}$ and $\underline{f}$ having a length of 5. Include homogeneous Dirichlet conditions on the inlet boundary and homogeneous Neumann conditions on the outlet boundary (if needed). In short, derive the derivative matrices such that the following systems can be solved

$$\frac{\mathrm{d}u}{\mathrm{d}x} = f \quad \text{and} \quad \frac{\mathrm{d}^2u}{\mathrm{d}x^2} = f$$

for u given f and using the mentioned boundary conditions.

45. Iterative techniques for linear systems.

    (a) Consider the algebraic equation system $LU = b$. To define a general iterative method write $L = P + A$. Which properties should matrices A and P have to insure convergence of the iterative method?

    (b) Which choices of P and A correspond to Jacobi respective Gauss–Seidel methods?

    (c) Define Gauss–Seidel iterations for the Laplace equation, give the convergence rate and derive an approximation for the number of iterations required for error reduction of $\mathcal{O}(h^2)$.

    (d) Describe the idea behind multigrid methods.

    (e) Define the 2-level multigrid method for the Laplace equation.

46. Coordinate transformation.

    (a) Define the coordinate transformation from a Cartesian one (x, y, z) to a general one (ξ, η, ζ). State the Jacobian matrix of transformation and describe a practical way of computing it.

    (b) Derive the transformation of the 2D Navier–Stokes equations

$$\text{from} \quad \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = 0 \quad \text{to} \quad \frac{\partial \mathbf{U}'}{\partial t} + \frac{\partial \mathbf{F}'}{\partial \xi} + \frac{\partial \mathbf{G}}{\partial \eta} = 0,$$

    and give the vectors $\mathbf{U}', \mathbf{F}'$ and $\mathbf{G}'$ in terms of $\mathbf{U}, \mathbf{F}$ and $\mathbf{G}$.

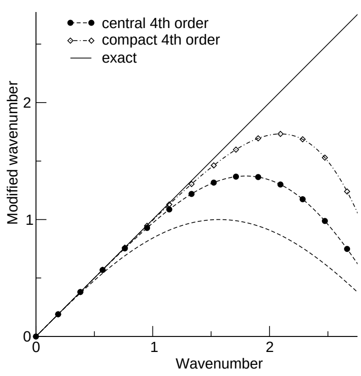
47. Compact finite-difference scheme. Consider the general approximation of type

$$\beta(f'_{i+2} + f'_{i-2}) + \alpha(f'_{i+1} + f'_{i-1}) + f'_i =$$
$$\frac{c}{6h}(f_{i+3} - f_{i-3}) + \frac{b}{4h}(f_{i+2} - f_{i-2}) + \frac{a}{2h}(f_{i+1} - f_{i-1}),$$

(a) and derive the equations which should be satisfied to get different order of accuracy for discretisation of first derivative f_i'.

(b) By Fourier analysis of the differencing error of the scheme above derive an expression for the modified wavenumber.

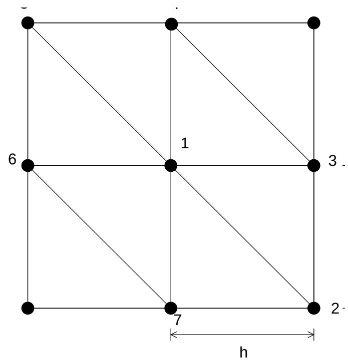
(c) What do the curves in the figure tell us?



48. Unstructured node-centred finite volume.

    (a) Define the dual grid.

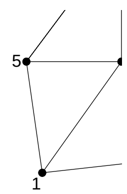
    (b) Present a finite-volume approximation of $u_t = u_{xx} + u_{yy}$. Examine the consistency of the scheme and give the order of the accuracy (use the grid given here).



    (c) Show that the u_x at node c can be approximated by the following finite-volume approximation and proof that its accuracy is $\mathcal{O}(h)$ (first order),

$$(u_x)_c \approx \frac{1}{V_c} \sum_i \frac{u_c + u_i}{2} \delta y_i.$$

    (V_c is the volume of the dual grid)



49. Upwind discretisation

    (a) Consider equation $u_t + au_x = 0$, where a is the convective velocity. Give a first-order accurate upwind discretisation of his equation which is stable independent of the sign of a.

    (b) Define a flux splitting scheme for discretisation of one-dimensional

Euler equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} = 0, \quad \mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ E_t \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} \rho u \\ \rho u^2 \\ (E_t + p)u \end{pmatrix}.$$