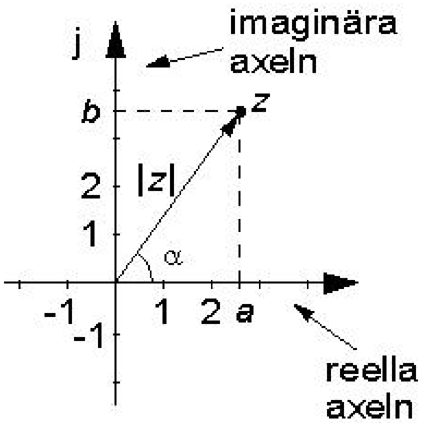
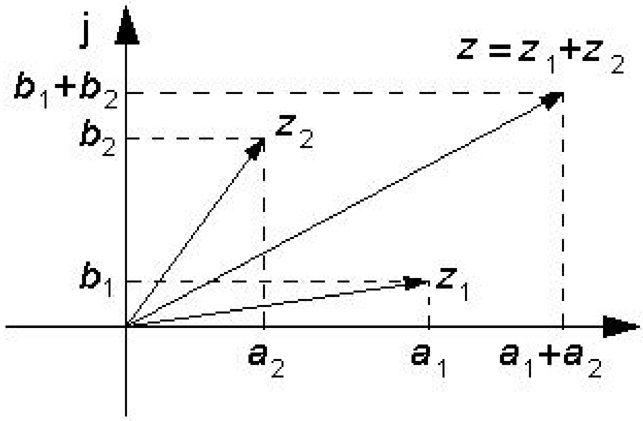
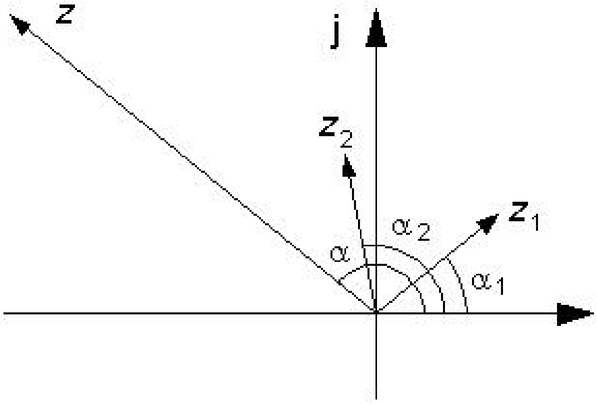
Booklet: Refresher, complex numbers





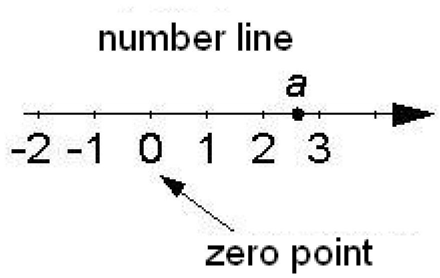


Refresher, complex numbers ...

Basic properties of complex numbers

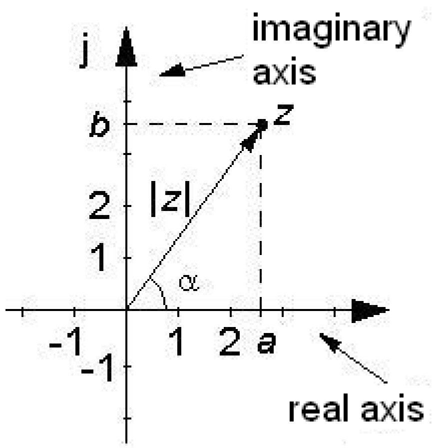
Definitios

A common, **real number** is usually illustrated as a point on the so-called number line. The magnitude is represented by the distance from the point in question to zero..



A complex number z consists of two components. It can be written as *a* + j*b*. Here, *a* and *b* are real numbers. j is the square root of -1 and is called the imaginary unit. *a* is the complex number real part Re (*z*). b is the imaginary part, Im (*z*).

Every complex number can be represented as a point in a two-dimensional coordinate system, the **complex plane**.



Number *z* is represented by a point with coordinates *a* and *b*.

The distance from the point to the origin represents the **amount** or **number value** |*z*|.

$$|z| = \sqrt{a^2 + b^2}$$

or

$$|z| = \sqrt{\left[\mathrm{Re}(z)\right]^2 + \left[\mathrm{Im}(z)\right]^2}$$

The angle α is called the argument of z, arg(z) and as seen in the figure

$$\tan(\alpha) = \frac{b}{a}$$

or

$$\arg(z) = \alpha = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right) + n \cdot 2\pi \qquad a > 0$$

$$\arg(z) = \alpha = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right) + \pi + n \cdot 2\pi \qquad a < 0$$

We can also express z in **polar form**, eg with $|z|$ and α. As seen in the figure

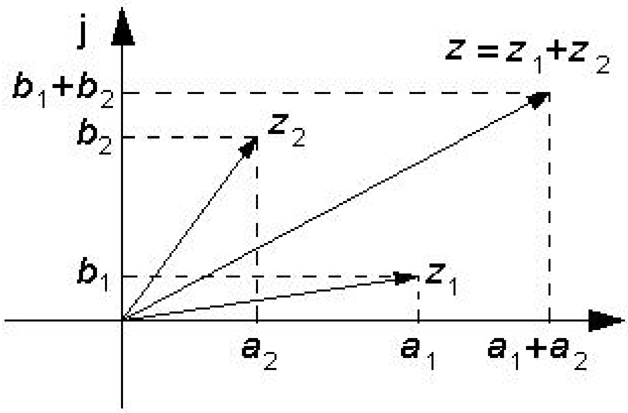
$$a = |z|\cos(\alpha) \quad b = |z|\sin(\alpha)$$
$$z = |z|\left(\cos(\alpha) + \mathrm{j} \cdot \sin(\alpha)\right)$$

One can then imagine that it's the connecting line between the point and the origin that represents the number. We can see this as a pointer (vector) with the length $|z|$ and a direction that is defined by the angle α.

Basic properties

Complex numbers can be treated algebraically, the following rules apply.

Addition



$$z = z_1 + z_2 = \mathrm{Re}(z_1) + \mathrm{Re}(z_2) + \mathrm{j}\big(\mathrm{Im}(z_1) + \mathrm{Im}(z_2)\big)$$

$$z_1 = a_1 + \mathrm{j}b_1 \qquad z_2 = a_2 + \mathrm{j}b_2$$

$$z = z_1 + z_2 = \big(a_1 + a_2\big) + \mathrm{j}\big(b_1 + b_2\big) = \mathrm{Re}(z) + \mathrm{j} \cdot \mathrm{Im}(z)$$

The figure shows what the addition means in the complex plane. The pointer of z equals the geometric sum of the pointers of z_1 and z_2. For $|z|$ and $\arg(z)$ applies the previously mentioned general terms.

Subraction

$$z = z_1 - z_2 = \mathrm{Re}(z_1) - \mathrm{Re}(z_2) + \mathrm{j}\big(\mathrm{Im}(z_1) - \mathrm{Im}(z_2)\big)$$

In the complex plane z equals the geometric difference between the z_1 and z_2.

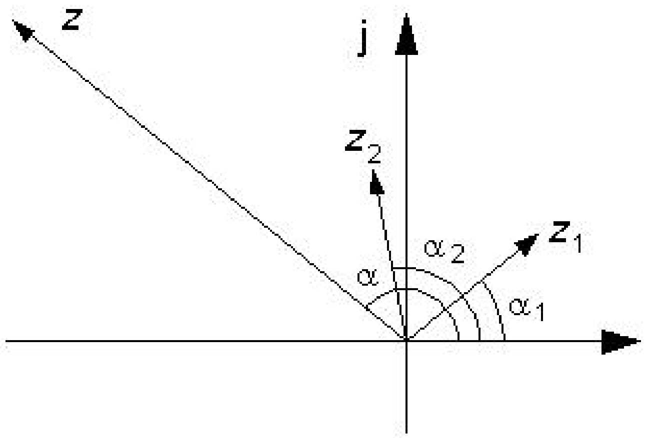
Multiplication

The multiplication rule, we demonstrate most easily with an example.

$$z_1 = a_1 + \mathrm{j}b_1 \qquad z_2 = a_2 + \mathrm{j}b_2 \qquad (\mathrm{j})^2 = -1$$

$$z = z_1 \cdot z_2 = (a_1 + \mathrm{j}b_1)(a_2 + \mathrm{j}b_2) = (a_1a_2 - b_1b_2) + \mathrm{j}(a_1b_2 + a_2b_1)$$

The multiplication can also be implemented with the numbers expressed in polar form.



$$\begin{aligned} z &= z_1 \cdot z_2 \\ &= |z_1|\big(\cos(\alpha_1) + \mathrm{j}\sin(\alpha_1)\big) \cdot |z_2|\big(\cos(\alpha_2) + \mathrm{j}\sin(\alpha_2)\big) \\ &= |z_1| \cdot |z_1|\big(\cos(\alpha_1 + \alpha_2) + \mathrm{j}\sin(\alpha_1 + \alpha_2)\big) \end{aligned}$$

This means that

$$|z| = |z_1 \cdot z_2| = |z_1| \cdot |z_2| \qquad \arg(z) = \arg(z_1) + \arg(z_2)$$

Division

Algebraic the division is implemented like this:

$$z_1 = a_1 + \mathrm{j}b_1 \qquad z_2 = a_2 + \mathrm{j}b_2$$

$$z = \frac{z_1}{z_2} = \frac{a_1 + \mathrm{j}b_1}{a_2 + \mathrm{j}b_2}$$

Now, one often wants to have the results in the form a+jb and if so, one extends the denominator with the conjugates quantity t a_2 - jb_2. Then one gets

$$z = \frac{(a_1 + \mathrm{j}b_1)(a_2 - \mathrm{j}b_2)}{a_2^2 + b_2^2} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \mathrm{j}\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$$

If numbers are expressed in polar form, the division rule that look like this:

$$z = \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\left(\cos(\alpha_1 - \alpha_2) - \mathrm{j}\sin(\alpha_1 - \alpha_2)\right)$$

$$|z| = \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad \arg(z) = \arg(z_1) - \arg(z_2)$$

Some memory rules

1. If $z = z_1 + z_2$, so is generally $|z| \neq |z_1| + |z_2|$
   (only if $\arg(z_1) = \arg(z_2)$ then $|z| = |z_1| + |z_2|$)

2. When calculating the amount of a product or a quotient of two complex numbers z_1 and z_2 it is generally unnecessary to first calculate the complex result and then form the amount. One calculates instead $|z_1|$ and $|z_2|$ eparately, for as we have seen applies

   $$|z_1 \cdot z_2| = |z_1| \cdot |z_2| \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Exemples

Exemple 1

Redo the expression 2+3/j to form *a*+j*b*.

$$2+\frac{3}{j}=2+\frac{3\cdot j}{j\cdot j}=2+\frac{3\cdot j}{-1}=2-3j$$

Exemple 2

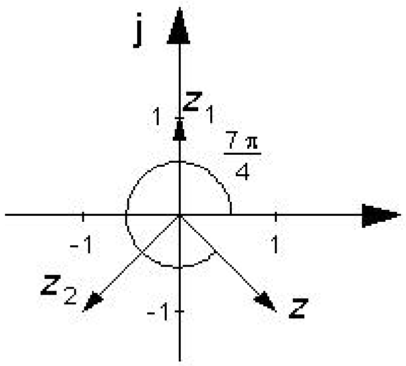
Write expression z = 6 + jA + 1/(jB) in the general form of complex numbers, and write a expression for the amount.

$$z=6+jA-\frac{j}{B}=6+j\left(A-\frac{1}{B}\right)$$

$$|z|=\left|6+j\left(A-\frac{1}{B}\right)\right|=\sqrt{36+\left(A-\frac{1}{B}\right)^2}$$

Exemple 3

Determine |z| and arg(z) when $z = z_1 \cdot z_2$ and z_1 = j and z_2 = -1 -j



Algebraic

$$z = z_1 \cdot z_2 = \mathrm{j} \cdot (-1 - \mathrm{j}) = -\mathrm{j} - \mathrm{j}^2 = 1 - \mathrm{j}$$

$$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\arg(z) = \alpha = \arctan\left(\frac{-1}{1}\right) = \arctan(-1) =$$

$$-\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$$

Polar

$$|z_1| = 1 \qquad |z_2| = \sqrt{2}$$

$$z_1 = 1\left(\cos\frac{\pi}{2} + \mathrm{j}\sin\frac{\pi}{2}\right) \quad z_2 = \sqrt{2}\left(\cos\frac{5\pi}{4} + \mathrm{j}\sin\frac{5\pi}{4}\right)$$

$$z = z_1 \cdot z_2 = 1 \cdot \sqrt{2}\left(\cos\left(\frac{\pi}{2} + \frac{5\pi}{4}\right) + \mathrm{j}\sin\left(\frac{\pi}{2} + \frac{5\pi}{4}\right)\right)$$

$$|z| = \sqrt{2} \qquad \arg(z) = \frac{7\pi}{4}$$

Exemple 4

$z_1 = 3 + \mathrm{j}5$, $z_2 = 5 + \mathrm{j}7$. Calculate

$$|z| = \left|\frac{z_1}{z_2}\right|$$

$$|z| = \left|\frac{z_1}{z_2}\right| = \frac{|3 + \mathrm{j}5|}{|5 + \mathrm{j}7|} = \frac{\sqrt{3^2 + 5^2}}{\sqrt{5^2 + 7^2}} =$$

$$\frac{\sqrt{9 + 25}}{\sqrt{25 + 49}} = \frac{\sqrt{34}}{\sqrt{74}} = \sqrt{\frac{34}{74}} = 0{,}68$$

If instead multiplied with conjugate quantity the calculations had been

$$z = \frac{3 + j5}{5 + j7} = \frac{(3 + j5)(5 - j7)}{(5 + j7)(5 - j7)} = \frac{15 - j21 + j25 - j^2 35}{25 - j^2 49}$$

$$= \frac{50 + j4}{74} \qquad j^2 = -1$$

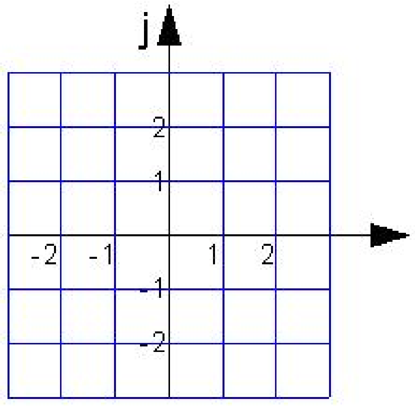
$$|z| = \frac{|50 + j4|}{|74|} = \frac{\sqrt{50^2 + 4^2}}{74} = \frac{\sqrt{2516}}{74} = 0{,}68$$

If one compares the above one can see that complex conjugation involves much more work!

Exercises

Question 1

In which direction points the complex pointer z = -2 + j2 ?



Question 2

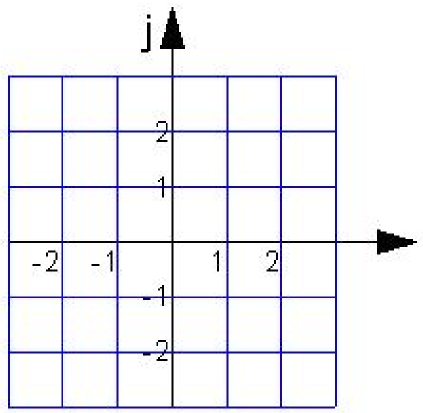
What is the sum of z_1 and z_2 if z_1 = 1 + j2 and z_2 = 2 - j ?

Question 3

How long is the pointer 3 + j4 ?

Question 4

Draw the pointer $z = z_1 - z_2$ if z_1 = 1 + j and z_2 = 2 + j ?



Question 5

How large is Im(z) if $z = z_1 + z_2$?
z_1 = 3(1+j) and z_2 = 2(1-j) .

Question 6

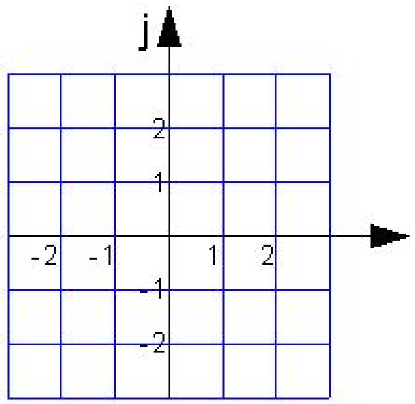
How large is |z| if $z = z_1 \cdot z_2$?
z_1 = 2 + j och z_2 = -(2 + j) .

Question 7

What is |3+j4|· |j2| ?

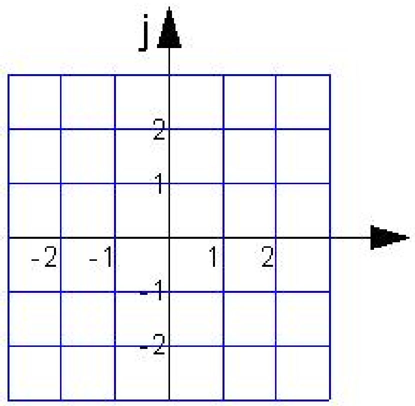
Question 8

Determine |z| and arg(z) if $z = z_1 \cdot z_2$ and z_1 = 1 + j and z_2 = -1 + j .



Question 9

what will be $z = z_1 \cdot z_2$ if z_1 = j and z_2 = 1 - j .



Question 10

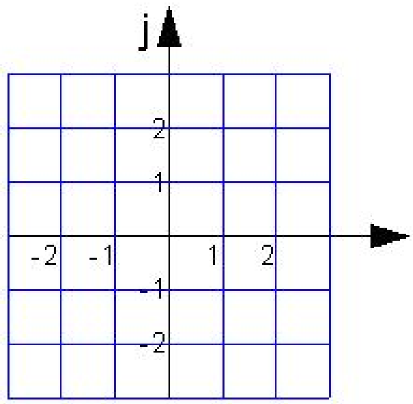
Vhat is $|z|$?

$$z = \frac{\frac{2}{j3}}{2 + \frac{1}{j3}}$$

Question 11

Calculate z.
$z_1 = 2 + j3$ and $z_2 = 1 + j$.

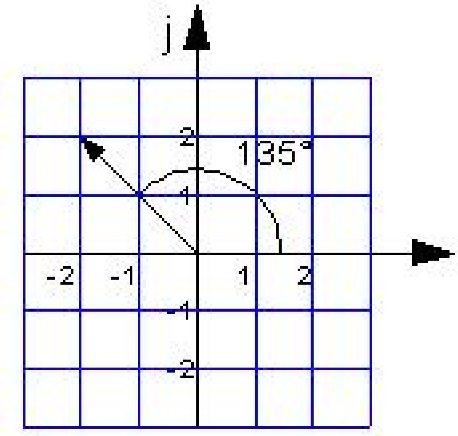
$$z = \frac{z_1}{z_2}$$



Refresher, complex numbers

Answers and solutions

Question 1



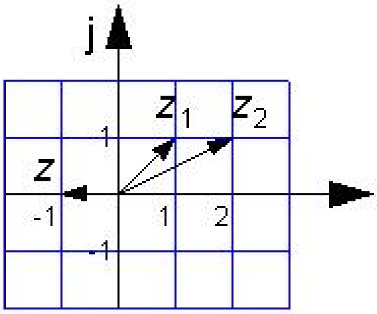
Question 2

$z = z_1 + z_2 = 1 + j2 + 2 - j = 3 + j$.

Question 3

$$\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$

Question 4



Question 5

$$\mathrm{Im}(z) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2) = 3 + (-2) = 1$$

Question 6

$$|z_1| = \sqrt{4+1} = \sqrt{5} \qquad |z_2| = \sqrt{4+1} = \sqrt{5}$$

$$|z| = |z_1| \cdot |z_2| = \sqrt{5} \cdot \sqrt{5} = 5$$

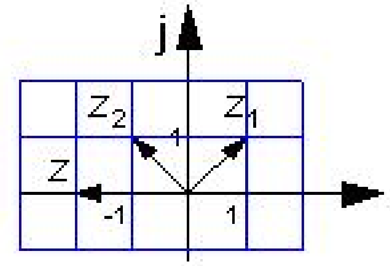
Question 7

$$\sqrt{9+16} \cdot 2 = 10 \quad \text{eller} \quad |j6-8| = \sqrt{36+64} = 10$$

Question 8

$$z = z_1 \cdot z_2 = (1+j)(-1+j) = -1+j-j+j^2 = -2$$

$$\arg(z) = \arctan\left(\frac{0}{-2}\right) + \pi = \pi$$



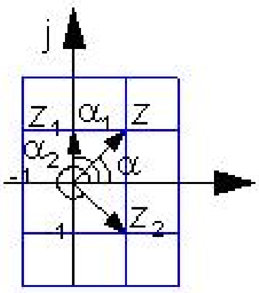
Question 9

$$z = z_1 \cdot z_2 = j(1-j) = j - j^2 = 1 + j$$

eller

$$z_1 = 1(\cos\tfrac{\pi}{2} + j\sin\tfrac{\pi}{2}) \qquad z_2 = \sqrt{2}(\cos\tfrac{7\pi}{4} + j\sin\tfrac{7\pi}{4})$$

$$z = z_1 \cdot z_2 = \sqrt{2}(\cos\tfrac{\pi}{4} + j\sin\tfrac{\pi}{4})$$



Question 10

$$z = \frac{\frac{2}{j3}}{2 + \frac{1}{j3}} \cdot \frac{j3}{j3} = \frac{2}{j6+1}$$

$$\left|z\right| = \frac{\left|2\right|}{\left|j6+1\right|} = \frac{2}{\sqrt{36+1}} = 0{,}329$$

Question 11

Algebraic

$$z = \frac{z_1}{z_2} = \frac{2 + j3}{1 + j} = \frac{(2 + j3)}{(1 + j)} \cdot \frac{(1 - j)}{(1 - j)} =$$

$$= \frac{2 + j - 3\,j^2}{1+1} = \frac{5 + j}{2} = 2{,}5 + 0{,}5\,j$$

Polar

$$\left|z\right| = \frac{\left|z_1\right|}{\left|z_2\right|} \qquad \arg(z) = \arg(z_1) - \arg(z_2)$$

$$z = \left|z\right| \cdot \left(\cos(\arg(z)) + j \cdot \sin(\arg(z)) \right)$$