KTH Teknikvetenskap

SF1624 Algebra och geometri
Solutions for Examn 15.03.13

DEL A

1. The plane H is given by the equation $3x - 2y + 5z + 1 = 0$.
   a) Determine a line N that is perpendicular to H. **(2 p)**
   b) Determine a line L that does not intersect H. **(2 p)**

Solution. a) We know that the vector $\vec{n} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$ is orthogonal to H. Consequently any line of the form $\{P + t \cdot \vec{n} \mid \text{tal } t\}$ is orthogonal to H, for any point P. For instance, we can chose P as the origo, and we have that the line $\begin{bmatrix} 3t \\ -2t \\ 5t \end{bmatrix}$ (parameter t) is orthogonal to H.

b) We chose two points Q and R in the plane H, and construct the vector $\vec{v} = R - Q$. Then any line of the form $\{P + t\vec{v} \mid \text{tal } t\}$ is parallell with the plane H. When we then chose a point P not in the plane, we get a line that does not interesect it. We chose the points

$$Q = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},$$

that gives $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. And we chose the point $P = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This gives us the line

$$\{ \begin{bmatrix} -t \\ t \\ t \end{bmatrix} \mid \text{scalars } t \}.$$

Answer.

2. Let $\vec{e_1}, \vec{e_2}, \vec{e_3}$ be the standard basis of $\mathbb{R}^3$. The linear map $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ is determined by

$$F(\vec{e_1}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, F(\vec{e_2}) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{och} \quad F(\vec{e_3}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(a) Determine $F(\vec{v})$ when $\vec{v} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$. **(1 p)**

(b) Determine the dimension of the kernel $\operatorname{Ker}(F)$, and the dimension of the image $\operatorname{Im}(F)$. **(2 p)**

(c) Determine a basis for the kernel $\operatorname{Ker}(F)$. **(1 p)**

Solution.

(a) The standard matrix of F is $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix}$, so

$$F(\vec{v}) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(a) We Gauss-Jordan eliminate the standard matrix of F to

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}.$$

Thus the rank of the standard matrix is 2, which also equals the dimension of the image, $\dim \operatorname{Im}(F) = 2$. The dimension of the kernel is then $\dim \mathbb{R}^3 = \dim \operatorname{Im}(F) + \dim \operatorname{Ker}(F)$, that is $\dim \operatorname{Ker}(F) = 1$.

(b) As $\dim \operatorname{Ker}(F) = 1$ we have that any non-zero vector $\vec{v} \neq \vec{0}$ such that $F(\vec{v}) = 0$, will be a basis. From (a) it follows that $\{ \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \}$ is a basis.

3. (a) What is an eigenvector? **(1 p)**

(b) Determine which of the following vectors

$$\vec{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{och} \quad \vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

that are eigenvectors for the matrix $A = \begin{bmatrix} 1 & 5 \\ 1 & 5 \end{bmatrix}$. **(2 p)**

(c) Determine eigenvalues and their corresponding eigenspaces of the matrix A. **(1 p)**

Solution.

(a) A vector $\vec{x}$ is an eigenvector for a matrix A means that $\vec{x}$ is non-zero and parallel with $A\vec{x}$, that is there exists a scalar λ such that $A\vec{x} = \lambda\vec{x}$.

(b) We multiply the vectors with the matrix A, and check wheter they are parallel with the vectors.

We get

$$A\vec{x} = \begin{bmatrix} 1 & 5 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 5 \cdot 5 \\ 1 \cdot 1 + 5 \cdot 5 \end{bmatrix} = \begin{bmatrix} 26 \\ 26 \end{bmatrix} \quad \text{is not parallel with } \vec{x},$$

$$A\vec{y} = \begin{bmatrix} 1 & 5 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 10 + 5 \cdot (-2) \\ 1 \cdot 10 + 5 \cdot (-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{is parallel with } \vec{y},$$

$$A\vec{z} = \begin{bmatrix} 1 & 5 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 5 \cdot 1 \\ 1 \cdot 1 + 5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} \quad \text{is parallel with } \vec{z}$$

and

$$A\vec{w} = \begin{bmatrix} 1 & 5 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 5 \cdot (-1) \\ 1 \cdot 1 + 5 \cdot (-1) \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix} \quad \text{is not parallel } \vec{y}.$$

We see that $\vec{y}$ and $\vec{z}$ are eigenvectors, whereas $\vec{x}$ are $\vec{w}$ not.

(c) From the calculations in (a) we see that $\vec{y}$ is an eigenvector with eigenvalue 0 and that $\vec{z}$ is an eigenvector with eigenvalue 6. There are at most two eigenvalues of A, so we have found them all. Their corresponding eigenspaces are the lines spanned by $\vec{y}$ and $\vec{z}$.

Answer.

(b) Both $\vec{y}$ and $\vec{z}$ are eigenvectors of A.

(a) The eigenvalue 0 has eigenspace $\mathrm{Span}\{\vec{y}\}$ and 6 has eigenspace $\mathrm{Span}\{\vec{z}\}$.

DEL B

4. In $\mathbb{R}^4$ we have, for each number a, the three following vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 2 \\ a \\ 1 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 11 \end{bmatrix}.$$

We let $V = \mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ denote their linear span.

(a) Determine for which numbers a the vector space V has dimension three. **(2 p)**

(b) Let $a = 1$, and determine a basis for the orthogonal complement $V^\perp$. **(2 p)**

Solution.

a) The vector space V has dimension three if and only if the vectors $\vec{v}_1$,$\vec{v}_2$, $\vec{v}_3$ are linearly independent. This again is equivalent with the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & a & 2 \\ 0 & 1 & 3 \\ -1 & 3 & 11 \end{bmatrix}$$

having rank three. The rank of the matrix we read off after doing Gauss-Jordan elimination:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & a & 2 \\ 0 & 1 & 3 \\ -1 & 3 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & a-4 & -6 \\ 0 & 1 & 3 \\ 0 & 5 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & a-4 & -6 \\ 0 & 5 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 6-3a \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix A has rank 3 if and only if $6 - 3a \neq 0$, that is $a \neq 2$. So the vector space V has dimension 3 if $a \neq 2$.

b) The orthogonal complement $V^\perp$ is all vectors $\vec{u} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ that are orthogonal against the three basis vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} \quad \text{och} \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 11 \end{bmatrix}.$$

That means that $\vec{v}_1 \cdot \vec{u} = 0, \quad \vec{v}_2 \cdot \vec{u} = 0$ and $\vec{v}_3 \cdot \vec{u} = 0$ simultaneously. If we write out the expresseions we get the homogeneous system of linear equations

$$\begin{cases} x + 2y - w = 0 \\ 2x + y + z + 3w = 0 \\ 4x + 2y + 3z + 11w = 0. \end{cases}$$

Elementary row operations on the augemented matrix of the system gives

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 1 & 1 & 3 \\ 4 & 2 & 3 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -3 & 1 & 5 \\ 0 & -6 & 3 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -3 & 1 & 5 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

We let $w = t$, and get that $z = -5t$, $y = 0$ and $x = t$. In other words

$$V^{\perp} = \{t \begin{bmatrix} 1 \\ 0 \\ -5 \\ 1 \end{bmatrix} \mid \text{scalars} \quad t\}.$$

And it follows that the vector $\begin{bmatrix} 1 \\ 0 \\ -5 \\ 1 \end{bmatrix}$ is a basis of $V^{\perp}$.

Answer.

5. (a) Define what is meant with the *coordinate vector* of a vector with respect to a basis. **(1 p)**

(b) We have the following vectors in $\mathbb{R}^2$:

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{och} \quad \vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Determine a basis $\mathcal{B}$ for $\mathbb{R}^2$ such that the coordinate vector of $\vec{v}$ is $\vec{w}$ and the coordinate vector of $\vec{w}$ is $\vec{v}$. **(3 p)**

Solution. a) Let $\beta = \{\vec{e}_1, \ldots, \vec{e}_n\}$ be a basis for a vector space V, and let $\vec{x}$ be a vector. Then the vector $\vec{x}$ can be expressed as

$$\vec{x} = \sum_{i=1}^{n} a_i \vec{e}_i,$$

for some scalars $a_1, \ldots, a_n$. The ordered sequence of scalars

$$[\vec{x}]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

is called the coordinate vector of $\vec{x}$ with respect to the basis β.

b) Let $\{\vec{e}, \vec{f}\}$ be the sought basis of $\mathbb{R}^2$. The criteria are that $\vec{v} = \vec{e} - \vec{f}$ and that $\vec{w} = \vec{2} + 2\vec{f}$. We can write these criteria as

$$\begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \vec{e} \\ \vec{f} \end{bmatrix}.$$

By inverting the 2×2-matrix, we read off the relations that

$$\frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix} = \begin{bmatrix} \vec{e} \\ \vec{f} \end{bmatrix}.$$

In other words we have that

$$\vec{e} = \frac{2}{3}\vec{v} + \frac{1}{3}\vec{w} = \frac{2}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\vec{f} = -\frac{1}{3}\vec{v} + \frac{1}{3}\vec{w} = -\frac{1}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Answer.

6. Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{n} = \begin{bmatrix} c \\ d \end{bmatrix}$ be two non-zero vectors in $\mathbb{R}^2$, where $ac + bd = 0$. Let L be the linear span of $\vec{v}$.

(a) Why is $\beta = \{\vec{v}, \vec{n}\}$ a basis of $\mathbb{R}^2$? **(1 p)**

(b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be reflection through L. Determine the matrix representation B of T with respect to β. **(1 p)**

(c) Let P be the change of basis matrix from the standard basis to β. Determine $P^{-1}BP$. **(2 p)**

Solution.

a) The two vectors form a basis if and only if they are linearly independent. Non-zero vectors $\vec{v}$ and $\vec{n}$ are orthogonal since their scalar product $\vec{v} \cdot \vec{n} = ac + bd = 0$. This implies that $\vec{v}$ and $\vec{n}$ form a basis for $\mathbb{R}^2$.

b) The reflection maps the vector $\vec{v}$ to $\vec{v}$ and $\vec{n}$ to $-\vec{n}$. As $T(\vec{v}) = \vec{v} = 1 \cdot \vec{v} + 0 \cdot \vec{n}$ and $T(\vec{n}) = -\vec{n} = 0 \cdot \vec{v} - 1 \cdot \vec{n}$ we get that the matrix representation is $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

c) The matrix $Q = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is the change of basis from β to the standard basis. It follows that $P = Q^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$. Then we get that

$$P^{-1}BP = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{ad-bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$\frac{1}{ad-bc} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} d & -c \\ b & -a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad+bc & -2ac \\ 2bd & -bc-ad \end{bmatrix}.$$

Note: If $b \neq 0$ and we substitute $d = -ac/b$, we can simplify

$$P^{-1}BP = \frac{1}{a^2+b^2} \begin{bmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{bmatrix}.$$

c') Alternatively: As $P^{-1}BP$ is the standard matrix, we can compute it directly. Let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ be a vector in $\mathbb{R}^2$. We have that

$$\vec{u} = \operatorname{proj}_L(\vec{u}) + (\vec{u} - \operatorname{proj}_L(\vec{u})).$$

This gives that $T(\vec{u}) = 2\operatorname{proj}_L(\vec{u}) - \vec{u}$. The line L is spanned by $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, and we get that

$$T(\vec{u}) = 2\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v} - \vec{u} = 2\frac{ax+by}{a^2+b^2} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2a^2x+2aby}{a^2+b^2} \\ \frac{2abx+2b^2y}{a^2+b^2} \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{a^2-b^2}{a^2+b^2} & \frac{2ab}{a^2+b^2} \\ \frac{2ab}{a^2+b^2} & \frac{b^2-a^2}{a^2+b^2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

It follows that $\begin{bmatrix} \frac{a^2-b^2}{a^2+b^2} & \frac{2ab}{a^2+b^2} \\ \frac{2ab}{a^2+b^2} & \frac{b^2-a^2}{a^2+b^2} \end{bmatrix}$ and the standard matrix is therefore

$$P^{-1}BP = \begin{bmatrix} \frac{a^2-b^2}{a^2+b^2} & \frac{2ab}{a^2+b^2} \\ \frac{2ab}{a^2+b^2} & \frac{b^2-a^2}{a^2+b^2} \end{bmatrix}.$$

DEL C

7. The number sequence $\{f_0, f_1, f_2, f_3, \ldots,\}$ satisfies the following recursive formulae

$$f_{n+2} = 2f_{n+1} + 8f_n, \qquad (*)$$

for all integers $n \geq 0$. The two first terms in the sequence are known, $f_0 = a$ and $f_1 = b$. Express f_{n+1} as a closed formula in a and b. (Tip: Denote by $F(n+1) = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$ and write the equation (*) in matrix form). **(4 p)**

Solution. We have that $F(n+1) = AF(n)$, where

$$A = \begin{bmatrix} 2 & 8 \\ 1 & 0 \end{bmatrix}.$$

It follows that $F(n+1) = A^n F(1)$. If we consider the matrix A as a linear transformation on $\mathbb{R}^2$, we have that $A = PDP^{-1}$ where P is the change of basis matrix from a basis of eigenvectors to the standard basis, and where D is a diagonal matrix with eigenvalues on the diagonal. We will use this to calculate A^n.

The characteristic polynomial of A is $c(\lambda) = (\lambda - 2)\lambda - 8$. The roots are given by

$$0 = c(\lambda) = \lambda^2 - 2\lambda + 1 - 9 = (\lambda - 1)^2 - 9.$$

In other words $\lambda = -2$ and $\lambda = 4$. We then determine the corresponding eigenspaces. With $\lambda = -2$ the eigenspace is given by the equation $x+2y = 0$. A basis being $\vec{e} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. The eigenspace corresponding to the eigenvalue $\lambda = 4$ is given by the equation $x - 4y = 0$. A basis being $\vec{f} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. This gives us the change of basis matrices

$$P = \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}.$$

We have that

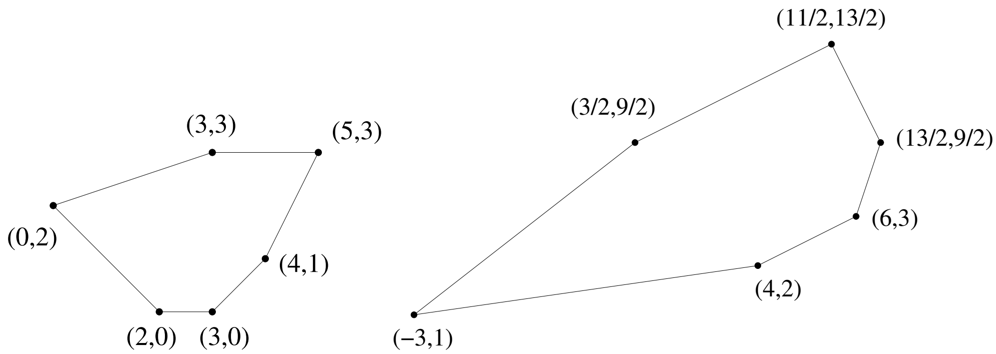
$$\begin{aligned} A^n = PD^nP = P \begin{bmatrix} (-2)^n & 0 \\ 0 & 4^n \end{bmatrix} P^{-1} \\ &= \frac{2^n}{6} \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix} \\ &= \frac{2^{n-1}}{3} \begin{bmatrix} (-1)^n \cdot 2 & 4 \cdot 2^n \\ (-1)^{n+1} & 2^n \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix} \\ &= \frac{2^{n-1}}{3} \begin{bmatrix} (-1)^n \cdot 2 + 4 \cdot 2^n & (-1)^{n+1} \cdot 8 + 8 \cdot 2^n \\ (-1)^{n+1} + 2^n & (-1)^n \cdot 4 + 2^{n+1} \end{bmatrix}. \end{aligned}$$

We have that $F(n+1) = A^n F(1)$, which gives

$$\begin{aligned} f_{n+1} &= \frac{2^{n-1}}{3}\Big(\big((-1)^n \cdot 2 + 4 \cdot 2^n\big)b + \big((-1)^{n+1} \cdot 8 + 8 \cdot 2^n\big)a\Big) \\ &= \frac{2^n}{3}((-1)^n + 2^{n+1})b + \frac{2^{n+2}}{3}((-1)^{n+1} + 2^n)a. \end{aligned}$$

Answer.

8. We have the two following figures (Every point is given by coordinates in a usual Cartesian system).



(a) Determine the matrix of a linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that transforms the figure on the left hand side, to the figure on the right. **(2 p)**

(b) Determine the area of the figure on the right hand side. **(2 p)**

Solution.

(a) We first figure out which points (corners) in the leftmost figure that are mapped to which point in the figure on the right hand side. We name the following points in the left figure: $\vec{p} := \left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$, $\vec{q} := \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ and $\vec{r} := \left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]$.

As $\vec{q} = \frac{2}{3}\vec{r}$ and T is linear, we must have $T(\vec{q}) = \frac{2}{3}T(\vec{r})$. The only points in the figure on the right satisfying this relation are $(4,2)$ and $(6,3)$. Thus we need to send $T(\vec{q}) = \left[\begin{smallmatrix}4\\2\end{smallmatrix}\right]$ and $T(\vec{r}) = \left[\begin{smallmatrix}6\\3\end{smallmatrix}\right]$. In the left figure $\vec{p}$ is neighbour to $\vec{q}$, so we need that $T(\vec{p})$ is neighbour to$T(\vec{q})$ and therefore we must send $T(\vec{p}) = \left[\begin{smallmatrix}-3\\1\end{smallmatrix}\right]$.

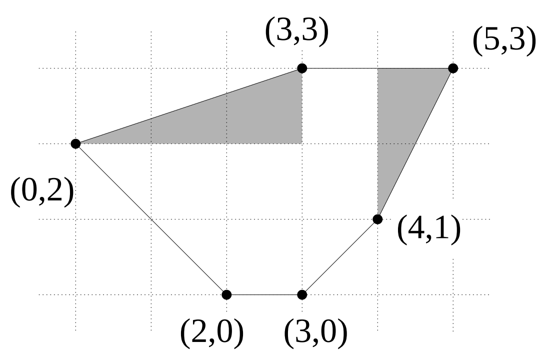
The equations $T(\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]) = \left[\begin{smallmatrix}4\\2\end{smallmatrix}\right]$ and $T(\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]) = \left[\begin{smallmatrix}-3\\1\end{smallmatrix}\right]$ together determine the matrix of T as

$$A = \frac{1}{2}\begin{bmatrix}4 & -3\\ 2 & 1\end{bmatrix}.$$

We check our calculations:

$$A\begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}3/2\\9/2\end{bmatrix},\ A\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}11/2\\13/2\end{bmatrix},\ A\begin{bmatrix}4\\1\end{bmatrix} = \begin{bmatrix}13/2\\9/2\end{bmatrix}.$$

(b) We can easily compute the area of the leftmost figure by drawing som helpful line segments.



The shadowed triangles have areas $3/2$ and 1, and the remaining area is $5+\frac{3}{2}$. The total area of the leftmost figure is then $\frac{3}{2}+1+5+\frac{3}{2}=9$. The determinant of the matrix A representing the linear map T is $(1/2)^2\cdot(4\cdot 1-(-3)\cdot 2)=5/2$, so the area of the rightmost figure is $9\cdot\frac{5}{2}=45/2$.

9. If A, B, C and D are quadratic matrices of the same size, we can build the bigger quadratic *block matrix*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Assume that A is invertible, and that the matrices A and C commute, that is $AC = CA$. Show that **(4 p)**

$$\det(M) = \det(AD - CB).$$

(You can freely use the fact that if B or C is the zero-matrix, then we have that $\det(M) = \det(AD)$.)

Solution. Since the matrix A^{-1} is assumed to exist, we can construct the matrix

$$\begin{bmatrix} I & 0 \\ -A^{-1}C & I \end{bmatrix},$$

where I is the identity matrix, and 0 is the zero matrix - both of the same size as A, B, C och D. The constructed matrix has obviously determinant 1. This means that the matrix product,

$$\begin{bmatrix} I & 0 \\ -A^{-1}C & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

has the determinant equal $\det(M)$. When we compute the product we get the matrix

$$\begin{bmatrix} A & B \\ -A^{-1}CA + C & -A^{-1}CB + D \end{bmatrix}.$$

We now use that $CA = AC$. The block down to the left then becomes $-A^{-1}CA + C = -A^{-1}AC + C = -C + C = 0$. We then use that we know that the determinant of matrices where one block is zero, that is

$$\det(\begin{bmatrix} A & B \\ 0 & -A^{-1}CB + D \end{bmatrix}) = \det(A)\det(-A^{-1}CB + D).$$

In other words the sougth determinant is $\det(M) = \det(A)\det(-A^{-1}CB + D)$. From the course we have that the determinant preserves product, so

$$\det(A)\det(-A^{-1}CB + D) = \det(A(-A^{-1}CB + D)) = \det(-CB + AD).$$

And as $-CB + AD = AD - CB$, we have proven the statement.