Exponential function

Exponential processes with time constants are very common in virtually all physical applications.

Instead of formally solve the underlying differential equations engineers usually use "fast formulas" and "rules of thumb".

Here are the most common …

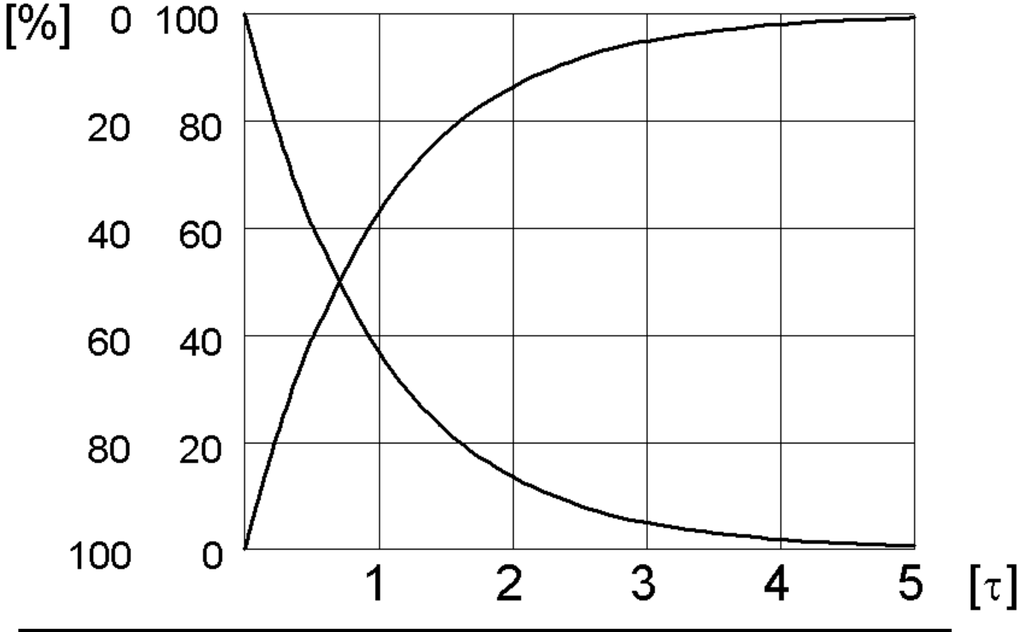
Exponential function

Rising curve

$$x(t) = 1 - e^{-\frac{t}{\tau}}$$

Descending curve

$$x(t) = e^{-\frac{t}{\tau}}$$



You can use this "normalized" chart for reading an estimate of what happens at an exponential process with a time constant..

Normalized chart 0…100% och 0…5 τ

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Exponential function

Rule of thumb
for 1τ and for 5 τ.

Rising curve

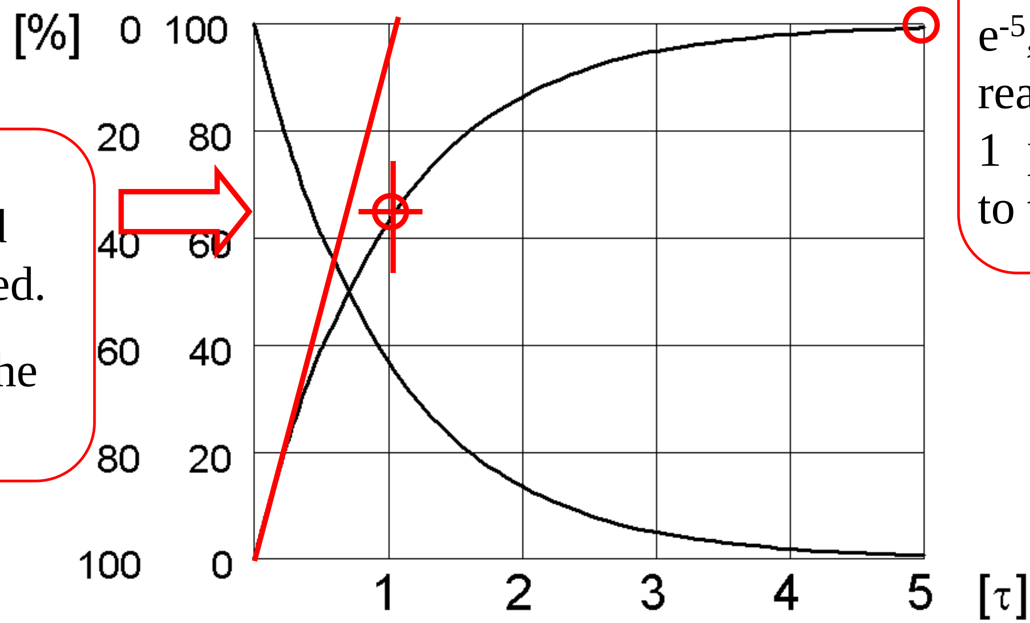
$$x(t) = 1 - e^{-\frac{t}{\tau}}$$

Descending curve

$$x(t) = e^{-\frac{t}{\tau}}$$

At time $t = \tau$ has $1\text{-}e^{-1}$, **63%** of end value been reached.

37% remains to the end value.



At time t = **5**·τ has $1\text{-}e^{-5}$, of end value been reached. Less than 1 per mille remains to the end value.

- One therefore considers that the final value is reached after **5 time constants**.

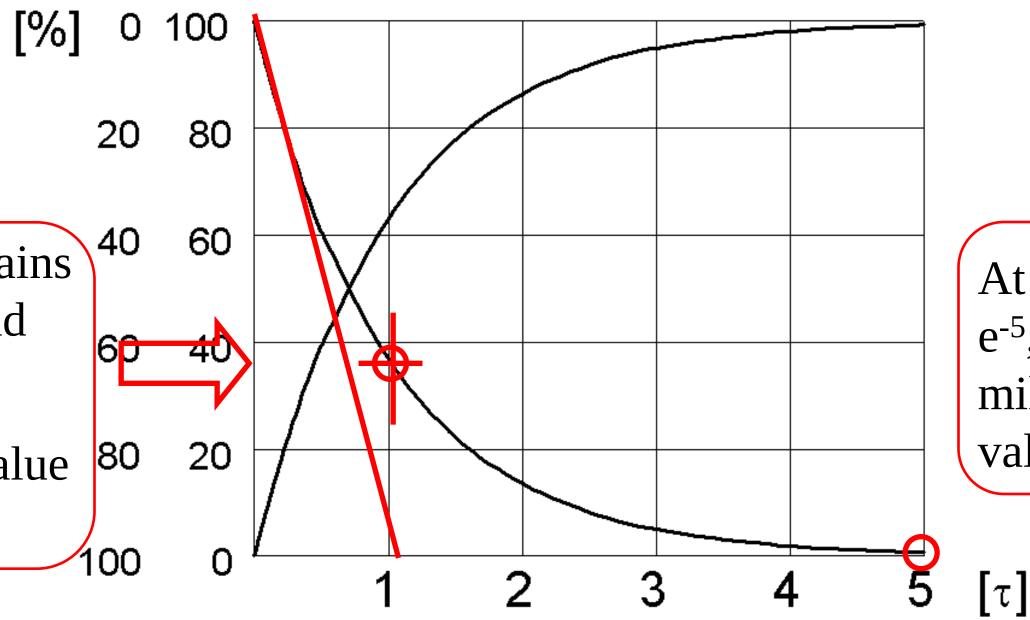
Exponential function

Rising curve

$$x(t) = 1 - e^{-\frac{t}{\tau}}$$

Descending curve

$$x(t) = e^{-\frac{t}{\tau}}$$



At time $t = \tau$ remains e^{-1}, 37%, to the end value.

67% of the end value has been reached.

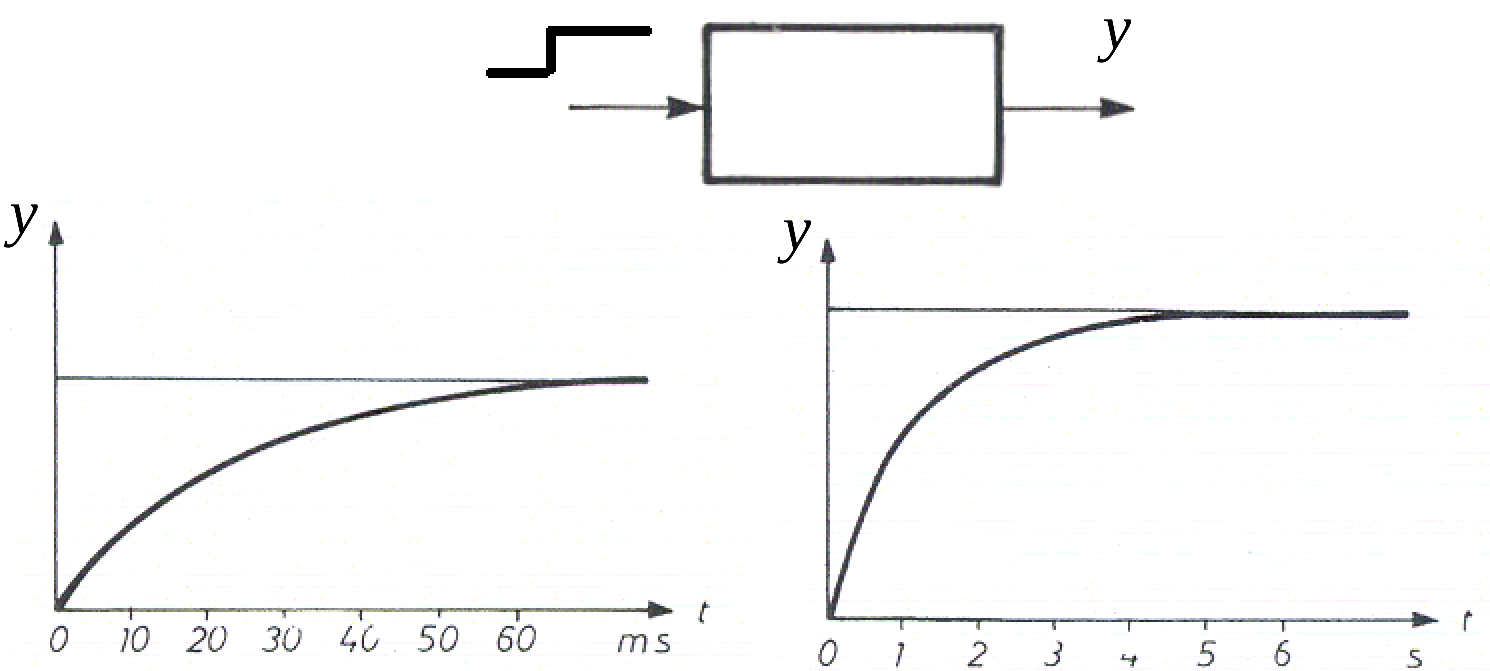
At time $t = \mathbf{5}\cdot\tau$ it is e^{-5}, less than 1 per mille, left to the end value.

- One therefore considers that the final value is reached after **5 time constants**.

Ex. Quick estimate of the time constant

The figure shows the "step response" for two processes with a "time constant". How big is the time constant *T* for the two processes?

Test signal: **step (** = turn on the power)



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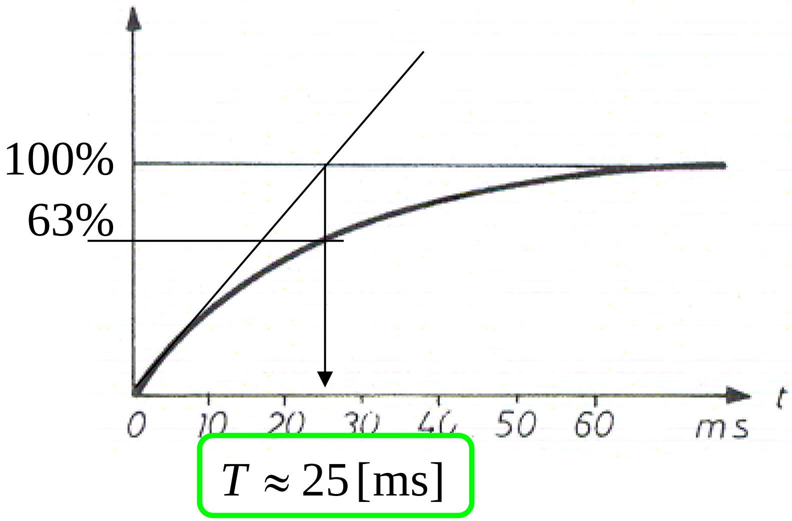
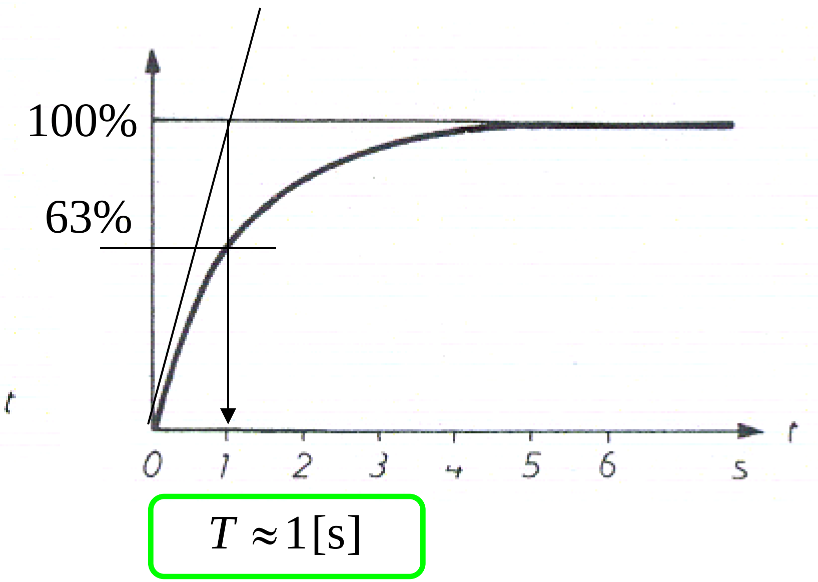
Ex. Quick estimate of the time constant





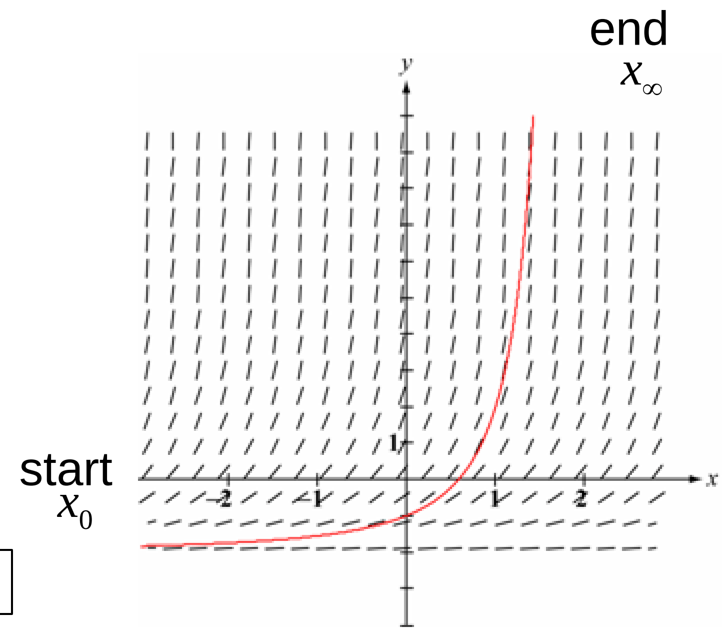
Time constant is where the tangent cross the asymptot, or at 63% of the end value.

Differential equations describes a family of curves

Time constant indicates the curve slope.

Differential equations describes a family of curves.

If we know that the curve is an exponetial then we *also* need to know the startvalue $\boldsymbol{x_0}$ and the end value $\boldsymbol{x_\infty}$ in order to ”choose” the **correct curve**.



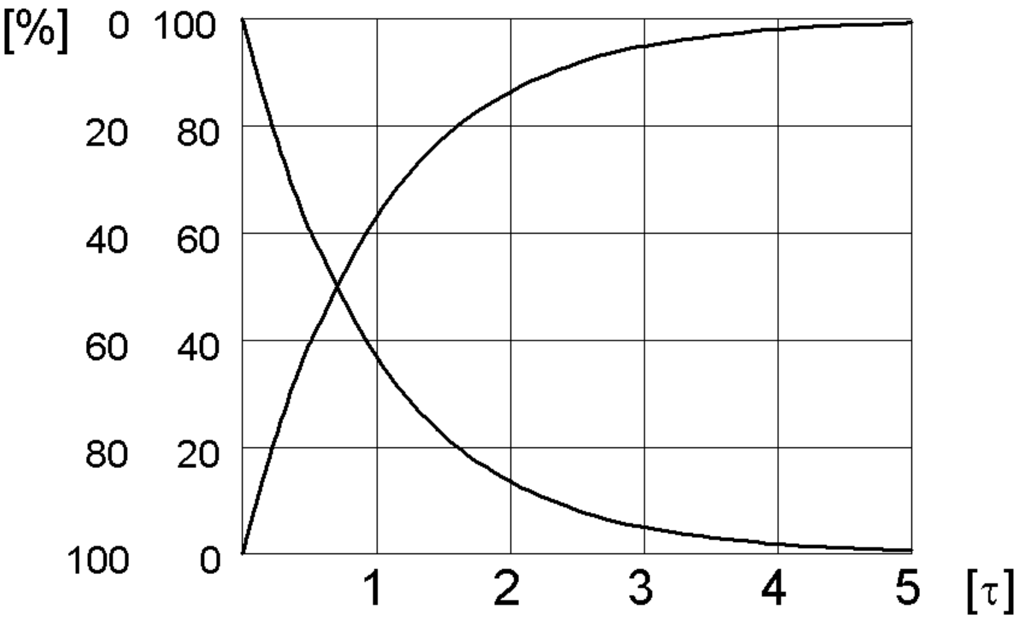
Quick Formula for exponential

- Rising process

$$x(t) = 1 - e^{-\frac{t}{\tau}}$$

- Falling process

$$x(t) = e^{-\frac{t}{\tau}}$$



The Quick Formula directly provides the equation for a rising/falling exponential process:

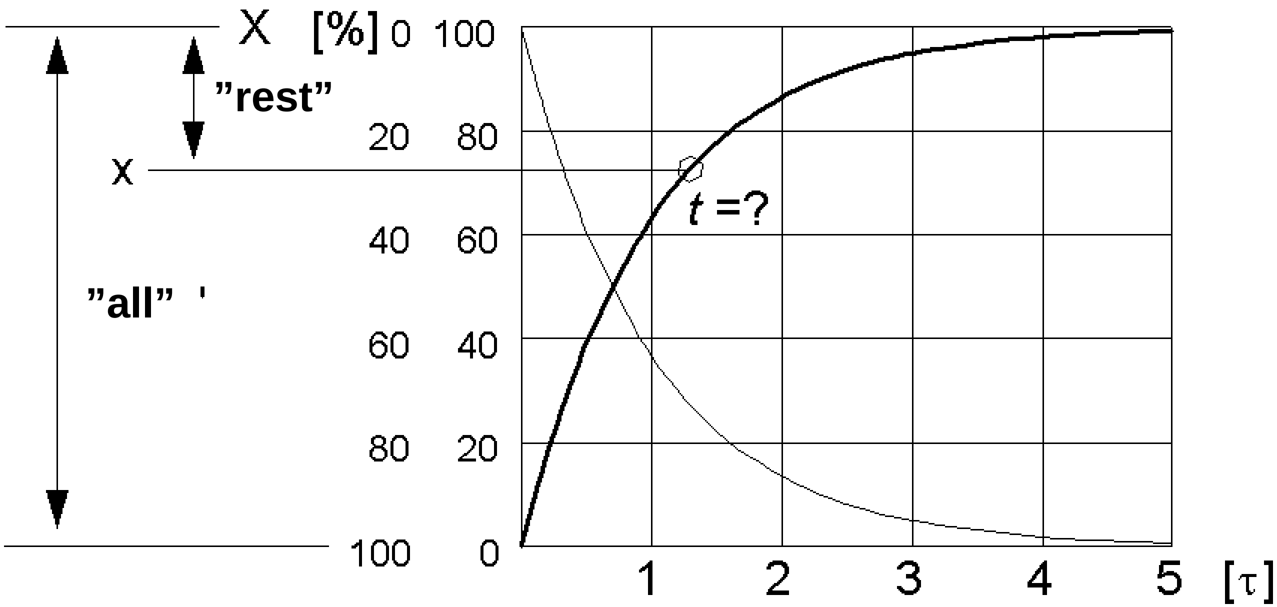
x_0 = process start value
x_∞ = process end value
τ = process time constant

$$x(t) = x_\infty - (x_\infty - x_0)e^{-\frac{t}{\tau}}$$

”All” by ”the rest”

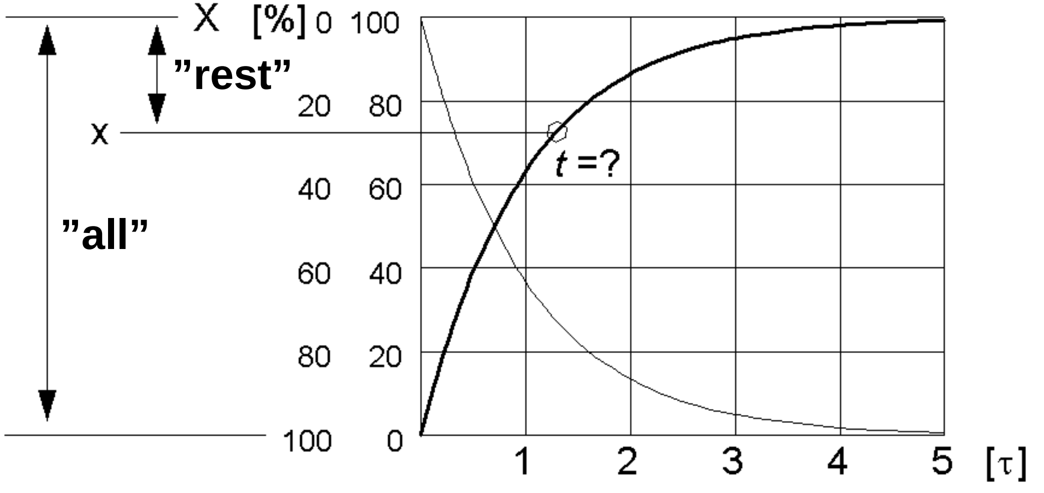
A common question at exponential progression is:
How long *t* will it take to reach *x* ?

- Rising process



”All” by ”the rest”

- Rising process



$$x = X(1 - e^{-\frac{t}{\tau}}) \quad \Rightarrow \quad \frac{x}{X} = 1 - e^{-\frac{t}{\tau}} \quad \Rightarrow \quad \ln\left(1 - \frac{x}{X}\right) = -\frac{t}{\tau} \quad \Rightarrow \quad t = -\tau \cdot \ln\frac{X - x}{X}$$

$$\boxed{t} = \tau \cdot \ln\frac{X}{X - x} = \boxed{\tau \cdot \ln\frac{\text{"all"}}{\text{"rest"}}}$$

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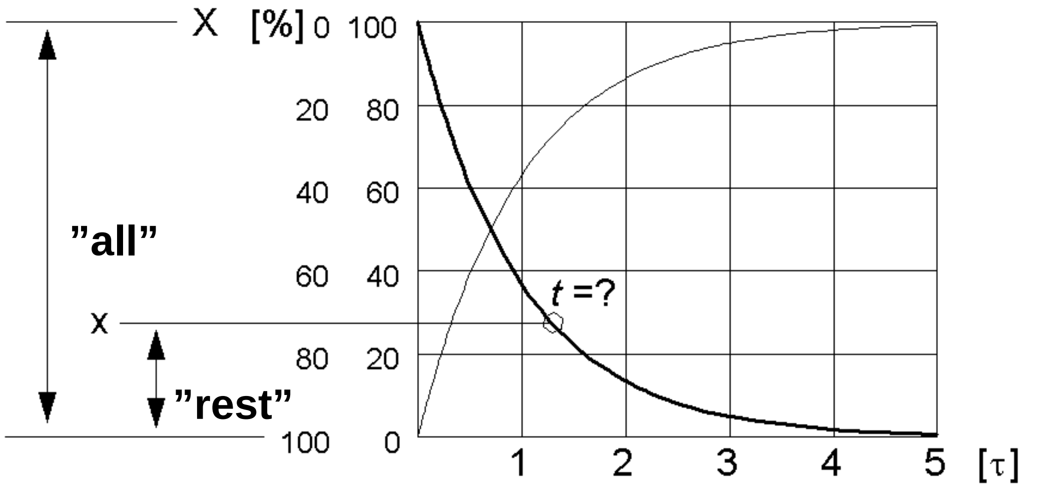
”All” by ”the rest”

- Falling process



$$x = X \cdot e^{-\frac{t}{\tau}} \quad \Rightarrow \quad \frac{x}{X} = e^{-\frac{t}{\tau}} \quad \Rightarrow \quad \ln\frac{x}{X} = -\frac{t}{\tau} \quad \Rightarrow \quad t = -\tau \cdot \ln\frac{x}{X}$$

$$\boxed{t} = \tau \cdot \ln\frac{X}{x} = \boxed{\tau \cdot \ln\frac{\text{"all"}}{\text{"rest"}}}$$

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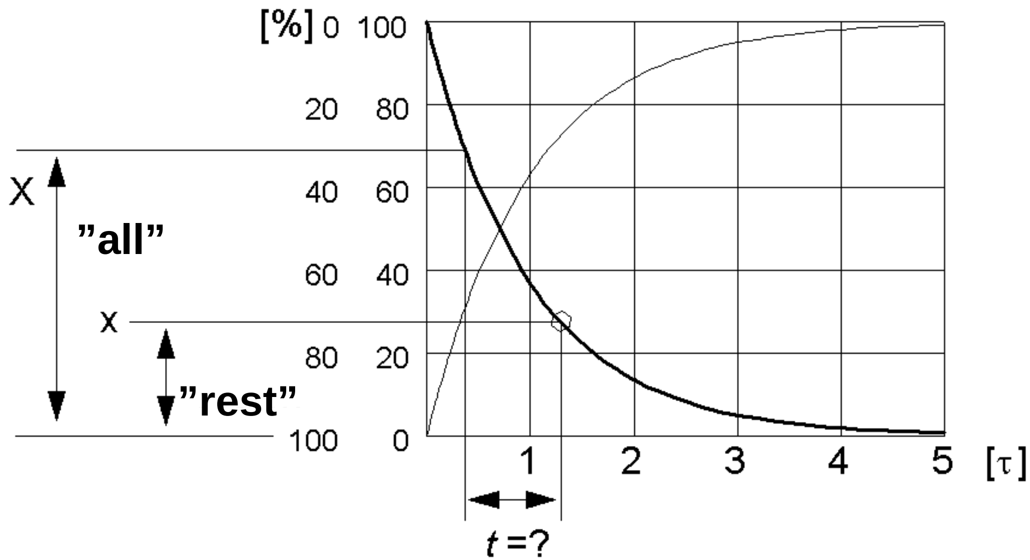
”All” by ”the rest”

- **Part of the process**



$$t = \tau \cdot \ln \frac{"all"}{"rest"}$$

Always apply to exponential progression with time constant, just redefine “all”!

Ex. measurement of the time constant

a) For a particular process with a "time constant" it was measured that it took 12 seconds for the output to reach 50% of its final value at a step-shaped signal change. What is the process time constant?

b) For another process took 10 minutes to reach 90% of the final value. What was the process time constant?

Ex. measurement of the time constant

a) 12 sekonds for 50% $T = ?$

$$t = T \cdot \ln \frac{\text{"all"}}{\text{"rest"}} \quad \Rightarrow \quad 12 = T \cdot \ln \frac{100-0}{100-50} \quad \Rightarrow \quad T = \frac{12}{\ln 2} = 17{,}3\,[\mathrm{s}]$$

b) 10 minutes for 90% $T = ?$

$$t = T \cdot \ln \frac{\text{"all"}}{\text{"rest"}} \quad \Rightarrow \quad 10 = T \cdot \ln \frac{100-0}{100-90} \quad \Rightarrow \quad T = \frac{10}{\ln 10} = 4{,}34\,[\mathrm{min}]$$