1. We will investigate the function \( f \) given by \( f(x) = xe^{1/x} \).

   A. Find the domain of definition of \( f \).
   B. Compute the four limits \( \lim_{x \to \pm \infty} f(x) \) and \( \lim_{x \to 0^\pm} f(x) \)
   C. Find all local extreme values of \( f \).
   D. Sketch, using the above, the graph \( y = f(x) \)

**Solution.**

A. The domain of definition consists of all real \( x \) for which \( xe^{1/x} \) is defined, i.e. all \( x \neq 0 \).

B. These are standard limits:

\[
\lim_{x \to \infty} f(x) = \infty, \quad \lim_{x \to -\infty} f(x) = -\infty, \quad \lim_{x \to 0^+} f(x) = \infty, \quad \lim_{x \to 0^-} f(x) = 0.
\]

C. We differentiate and obtain:

\[
f'(x) = e^{1/x} + xe^{1/x} \left( -\frac{1}{x^2} \right) = e^{1/x} \left( 1 - \frac{1}{x} \right)
\]

that exists for all \( x \neq 0 \) and is equal to 0 iff \( x = 1 \). We have:

If \( x < 0 \) then \( f'(x) > 0 \). Consequently \( f \) is strictly increasing on this interval. Hence there are no extreme values when \( x < 0 \)

If \( 0 < x < 1 \) then \( f'(x) < 0 \). Consequently \( f \) is strictly decreasing on this interval.

If \( x = 1 \) then \( f'(x) = 0 \).

If \( x > 1 \) then \( f'(x) > 0 \). Consequently \( f \) is strictly increasing on this interval.

From the above it follows that \( f \) has exactly one local extreme value, a local minimum when \( x = 1 \), and \( f(1) = e \).

D. Now we can sketch the graph:
Answer: Se lösningen.
2. Compute the integral
\[ \int_{\pi/4}^{\pi/2} \cos \sqrt{x} \, dx \]
by doing the following:
A. Perform the substitution \( \sqrt{x} = t \) (don’t forget to change the interval of integration).
B. Compute, using integration by parts, the integral you obtain in A.

Solution. A. Using the substitution \( \sqrt{x} = t \), or \( x = t^2 \), with \( dx = 2t \, dt \) and interval of integration between \( \pi/2 \) and \( \pi \), we get
\[ \int_{\pi/4}^{\pi/2} \cos \sqrt{x} \, dx = \int_{\pi/2}^{\pi} 2t \cos t \, dt. \]

B. Using integration by parts on the integral from problem A we get
\[ \int_{\pi/2}^{\pi} 2t \cos t \, dt = [2t \sin t]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} 2 \sin t \, dt = -\pi - 2. \]

Answer: A. \( \int_{\pi/2}^{\pi} 2t \cos t \, dt \).
B. \(-\pi - 2\).
3. A tin can in the shape of a cylinder, with lid and bottom plate, containing 1 litre, is to be manufactured. Compute the height of the cylinder and the radius of its circular bottom plate in order to minimize the total surface area of the can.

Solution. Let \( r \) be the radius of the bottom plate and \( h \) the height of the cylinder. The volume of the cylinder is \( \pi r^2 h \) and with this equal to 1 we get \( \pi r^2 h = 1 \), i.e. \( h = \frac{1}{\pi r^2} \).

The area of surface of the cylinder, to be minimized, is \( 2\pi r^2 + 2\pi rh \). If we substitute \( h = \frac{1}{\pi r^2} \) into this we see that we need to minimize the function

\[
A(r) = 2\pi r^2 + \frac{2}{r}
\]

där \( r > 0 \). We differentiate and obtain

\[
A'(r) = 4\pi r - \frac{2}{r^2}
\]

that exists for all \( r > 0 \). We see that \( A'(r) = 0 \iff 2r = \frac{1}{\pi r^2} = h \). If we study the derivative we see that we have a local and global minimum when \( 2r = h \). Argument for this:

- If \( 0 < r < \frac{1}{(2\pi)^{1/3}} \) then \( A'(r) < 0 \) and consequently \( A \) is strictly decreasing.
- If \( r = \frac{1}{(2\pi)^{1/3}} \) then \( A'(r) = 0 \).
- If \( r > \frac{1}{(2\pi)^{1/3}} \) then \( A'(r) > 0 \) and consequently \( A \) is strictly increasing.

The surface area is therefore minimized when \( r = \frac{1}{(2\pi)^{1/3}} \) and \( h = 2r \).

\[\square\]

Answer: \( r = \frac{1}{(2\pi)^{1/3}} \) and \( h = \frac{2}{(2\pi)^{1/3}} \).
4. We study the differential equation $y''(t) + y(t) = \sin t$.

A. Solve the differential equation.

B. Does there exist a bounded solution to the differential equation?

**Solution.** A. The solution to the differential equation is $y = y_h + y_p$ where $y_h$ is the general solution to homogeneous equation $y'' + y = 0$, and $y_p$ is any particular solution.

First we find $y_h$. The characteristic equation $r^2 + 1 = 0$ has solutions $\pm i$, and so

$$y_h(t) = A \cos t + B \sin t,$$

where $A$ and $B$ are arbitrary constants.

Then we find $y_p$. Normally we would look for a particular solution of the form $a \cos t + b \sin t$ but this is part of the homogeneous solution and will not work. Instead we look for

$$y_p = t(a \cos t + b \sin t).$$

Then

$$y'_p = a \cos t + b \sin t + t(-a \sin t + b \cos t)$$

and

$$y''_p = -a \sin t + b \cos t - a \sin t + b \cos t + t(-a \cos t - b \sin t).$$

We see that $y''_p + y_p = \sin t \iff a = -1/2$ and $b = 0$.

We have a particular solution

$$y_p = -\frac{t}{2} \cos t.$$

Putting it all together we see that the full solution to the given differential equation is

$$y(t) = A \cos t + B \sin t - \frac{t}{2} \cos t, \quad A, B \text{ arbitrary constants}.$$ 

For $t = n2\pi$ we have $y(t) = A - n\pi$ that tends to $-\infty$ when the integer $n \to \infty$, independent of the choice of constants $A$ and $B$. Therefore there is no bounded solution.

**Answer:** A. $y(t) = A \cos t + B \sin t - \frac{t}{2} \cos t$, $A, B$ arbitrary constants.

B. No.
5. Find the Taylor polynomial of degree 2 about the point \( x = 100 \) to the function \( f(x) = \sqrt{x} \) and use this Taylor polynomial to compute an approximate value of \( \sqrt{104} \). Also, decide whether the error of your approximation is less than \( 10^{-4} \) in absolute value.

**Solution.** We differentiate and obtain

\[
\begin{align*}
 f'(x) &= \frac{1}{2\sqrt{x}}, \\
 f''(x) &= -\frac{1}{4x\sqrt{x}}, \\
 f'''(x) &= \frac{3}{8x^2\sqrt{x}}
\end{align*}
\]

existing for all \( x > 0 \). The Taylor polynomial of degree 2 to \( f \) about \( x = 100 \) is therefore

\[
p(x) = 10 + \frac{1}{20}(x - 100) - \frac{1}{8000}(x - 100)^2.
\]

If we use this for approximating \( \sqrt{104} \) we obtain

\[
\sqrt{104} = f(104) \approx p(104) = 10 + \frac{1}{20}(104 - 100) - \frac{1}{8000}(104 - 100)^2 = 10.198.
\]

The absolute value of the error in the approximation is for some \( c \) between 100 and 104:

\[
\left| \frac{3/(8c^2\sqrt{c})}{3!} \cdot 4^3 \right| \leq \frac{4}{100000} \leq 10^{-4}
\]

**Answer:** \( p(x) = 10 + \frac{1}{20}(x - 100) - \frac{1}{8000}(x - 100)^2. \)

\( \sqrt{104} \approx 10.198 \) with an error less than \( 10^{-4} \).
6. Is the improper integral

\[ \int_1^\infty \frac{dx}{x^2 + x} \]

convergent or divergent? If it is convergent, compute the integral.

**Hint:** For \( x \geq 1 \) we have \( \frac{1}{x^2} \geq \frac{1}{x^2 + x} = \frac{1}{x} - \frac{1}{x+1} \).

**Solution.** Since

\[ 0 \leq \frac{1}{x^2 + x} \leq \frac{1}{x^2}, \quad \text{for } x \geq 1, \]

we have

\[ 0 \leq \int_1^\infty \frac{dx}{x^2 + x} \leq \int_1^\infty \frac{dx}{x^2} = 1 \]

and it follows that our integral is convergent. We compute it:

\[
\int_1^\infty \frac{dx}{x^2 + x} = \int_1^\infty \left( \frac{1}{x} - \frac{1}{x+1} \right) \, dx \\
= \lim_{R \to \infty} \int_1^R \left( \frac{1}{x} - \frac{1}{x+1} \right) \, dx \\
= \lim_{R \to \infty} \left[ \ln x - \ln(x+1) \right]_1^R \\
= \lim_{R \to \infty} \left[ \ln \frac{x}{x+1} \right]_1^R \\
= \ln 2.
\]

**Answer:** The integral is convergent and its value is \( \ln 2 \).
7. A. Give the definition of what it means for a function $f$ to be continuous at a point $a$.
   
   B. Give the definition of what it means for a function $f$ to be differentiable at a point $a$.
   
   C. Show that a function $f$ differentiable at a point $a$ also must be continuous at $a$.
   
   D. Give an example showing that a function that is continuous at a point does not have to be differentiable at that point.

*Solution.* A. The function $f$ is continuous at $a$ if $f$ is defined at $a$ and has a limit when $x$ approaches $a$ and \( \lim_{x \to a} f(x) = f(a) \).

   B. $f$ is differentiable at $a$ if the limit
   \[
   \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
   \]
   exists as a finite number. This limit is then called the derivative of $f$ at $a$, written $f'(a)$.

   C. Suppose $f$ is differentiable at $a$. We must show that in that case $\lim_{x \to a} f(x) = f(a)$ or equivalently $\lim_{h \to 0} (f(a + h) - f(a)) = 0$. We have
   \[
   \lim_{h \to 0} (f(a + h) - f(a)) = \lim_{h \to 0} \left( \frac{f(a + h) - f(a)}{h} \cdot h \right) = h' f'(a) \cdot 0 = 0.
   
   The proof is complete.

   D. Let $f(x) = |x|$. Clearly $f$ is continuous at the origin, since $f(0) = 0$ and $\lim_{x \to 0} f(x) = 0$. The function $f$ is not differentiable at the origin since
   \[
   \lim_{h \to 0} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}
   \]
   and this limit does not exist (if $h$ is positive and tends to zero we get 1 but if $h$ is negative and tends to zero we get $-1$).

*Answer:* See the solution.
8. A hole with radius 1 is drilled through the center of a ball with radius 2. How great a part (in percent) of the volume of the ball remains?

Solution. Let’s choose coordinates so that the origin is the center of the ball and the ball is obtained as a solid of revolution of the curve $x^2 + y^2 = 4$ around the $x$-axis. We may assume that the hole is drilled so that the $x$-axis is the line of symmetry of the drilling cylinder. In that case points of intersections in the $xy$-plane between the cylinder and the ball are $(\pm \sqrt{3}, \pm 1)$. The drilled part then consists of a cylinder with radius 1 and height $2\sqrt{3}$ plus two solids of revolution at each end of the cylinder.

The volume of the cylinder is $2\pi\sqrt{3}$. The two solids of revolution are obtained when $x^2 + y^2 = 4$ is rotated around the $x$-axis, on the intervals $[\sqrt{3}, 2]$, and $[-2, -\sqrt{3}]$. The volume of these are

$$2\pi \int_{\sqrt{3}}^{2} (4 - x^2) \, dx = 2\pi \left( \frac{16}{3} - 3\sqrt{3} \right).$$

Since the volume of the ball is $32\pi/3$ we see that the percentage of the part that has been removed is

$$\frac{2\pi\sqrt{3} + 2\pi \left( \frac{16}{3} - 3\sqrt{3} \right)}{32\pi/3} = \frac{32\pi/3 - 4\pi\sqrt{3}}{32\pi/3} \approx 0.35.$$

Approximately 35 percent of the volume of the ball has been removed and hence the remaining part is approximately 65 percent.

Answer: Appox. 65 percent.
9. Show that the function
\[ f(x) = x \left( \frac{\pi}{2} - \arctan x \right) \]
is increasing.

Solution. First we observe that the function is defined for all \( x \). We differentiate and obtain
\[ f'(x) = \frac{\pi}{2} - \arctan x - \frac{x}{1 + x^2} \]
existing for all \( x \). We differentiate a second time and get
\[ f''(x) = -\frac{1}{1 + x^2} - \frac{1 + x^2 - 2x^2}{(1 + x^2)^2} = \frac{-2}{(1 + x^2)^2} \]
existing for all \( x \) and negative for all \( x \).

The fact that \( f''(x) < 0 \) for all \( x \) implies that \( f'(x) \) is strictly decreasing for all \( x \). Since \( \lim_{x \to \infty} f'(x) = 0 \) this implies that \( f'(x) \) is positive for all \( x \). Consequently \( f \) is strictly increasing.

Answer: Se lösningen.