KTH Teknikvetenskap

SF1625 Calculus in one variable
Lösningsförslag till tentamen 2015-01-12

DEL A

1. We will investigate the function f given by $f(x) = xe^{1/x}$.
   A. Find the domain of definition of f.
   B. Compute the four limits $\lim_{x\to\pm\infty} f(x)$ and $\lim_{x\to 0^{\pm}} f(x)$
   C. Find all local extreme values of f.
   D. Sketch, using the above, the graph $y = f(x)$

Solution. A. The domain of definition consists of all real x for which $xe^{1/x}$ is defined, i.e. all $x \neq 0$.

B. These are standard limits:

$$\lim_{x\to\infty} f(x) = \infty, \quad \lim_{x\to-\infty} f(x) = -\infty, \quad \lim_{x\to 0^+} f(x) = \infty, \quad \lim_{x\to 0^-} f(x) = 0.$$

C. We differentiate and obtain:

$$f'(x) = e^{1/x} + xe^{1/x}\left(-\frac{1}{x^2}\right) = e^{1/x}\left(1 - \frac{1}{x}\right)$$

that exists for all $x \neq 0$ and is equal to 0 iff $x = 1$. We have:

If $x < 0$ then $f'(x) > 0$. Consequently f is strictly increasing on this interval. Hence there are no extreme values when $x < 0$

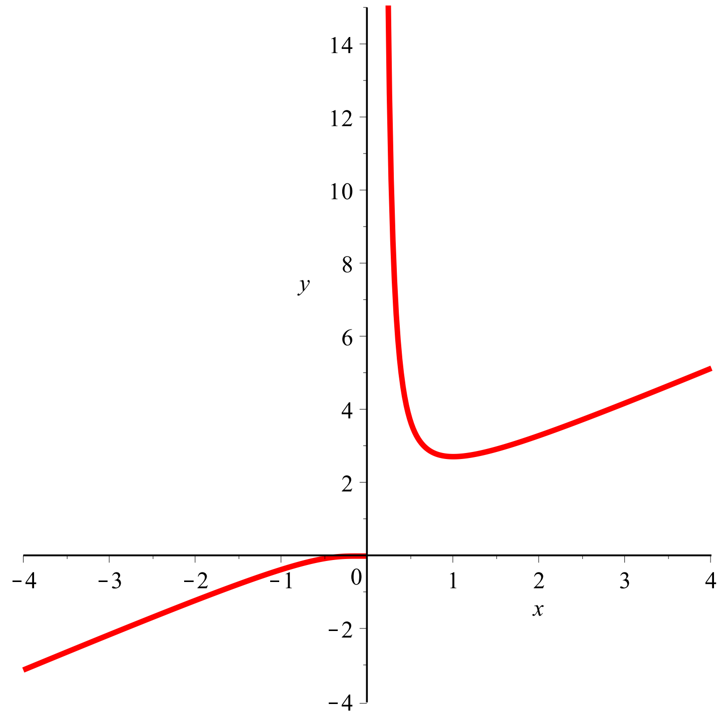
If $0 < x < 1$ then $f'(x) < 0$. Consequently f is strictly decreasing on this interval.

If $x = 1$ then $f'(x) = 0$.

If $x > 1$ then $f'(x) > 0$. Consequently f is strictly increasing on this iterval.

From the above it follows that f has exactly one local extreme value, a local minimum when $x = 1$, and $f(1) = e$.

D. Now we cab sketch the graph:



□

Answer: Se lösningen.

2. Compute the integral

$$\int_{\pi^2/4}^{\pi^2} \cos\sqrt{x}\,dx$$

by doing the following:
A. Perform the substitution $\sqrt{x} = t$ (don't forget to change the interval of integration).
B. Compute, using integration by parts, the integral you obtain in A.

Solution. A. Using the substitution $\sqrt{x} = t$, or $x = t^2$, with $dx = 2t\,dt$ and interval of integration between $\pi/2$ and π, we get

$$\int_{\pi^2/4}^{\pi^2} \cos\sqrt{x}\,dx = \int_{\pi/2}^{\pi} 2t\cos t\,dt.$$

B. Using integration by parts on the integral from problem A we get

$$\int_{\pi/2}^{\pi} 2t\cos t\,dt = [2t\sin t]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} 2\sin t = -\pi - 2.$$

□

Answer: A. $\int_{\pi/2}^{\pi} 2t\cos t\,dt$.
B. $-\pi - 2$

3. A tin can in the shape of a cylinder, with lid and bottom plate, containing 1 litre, is to be manufactured. Compute the height of the cylinder and the radius of its circular bottom plate in order to minimize the total surface area of the can.

Solution. Let r be the radius of the bottom plate and h the hight of the cylinder. The Volume of the cylinder is $\pi r^2 h$ and with this equal to 1 we get $\pi r^2 h = 1$, i. e. $h = 1/\pi r^2$.

The area of surface of the cylindern, to be minimized, is $2\pi r^2 + 2\pi r h$. If we substitute $h = 1/\pi r^2$ into this we see that we need to minimize the function

$$A(r) = 2\pi r^2 + \frac{2}{r}$$

där $r > 0$. We differentiate and obtain

$$A'(r) = 4\pi r - \frac{2}{r^2}$$

that exists for all $r > 0$. We see that $A'(r) = 0 \iff 2r = 1/\pi r^2 = h$. If we study the derivative we see that we have a local and global minimum when $2r = h$. Argument for this:

If $0 < r < 1/(2\pi)^{1/3}$ then $A'(r) < 0$ and consequently A is strictly decreasing.

If $r = 1/(2\pi)^{1/3}$ then $A'(r) = 0$.

If $r > 1/(2\pi)^{1/3}$ then $A'(r) > 0$ and consequently A is strictly increasing.

The surface area is therefore minimized when $r = 1/(2\pi)^{1/3}$ and $h = 2r$.

□

Answer: $r = 1/(2\pi)^{1/3}$ and $h = 2/(2\pi)^{1/3}$.

DEL B

4. We study the differential equation $y''(t) + y(t) = \sin t$.
   A. Solve the differential equation.
   B. Does there exist a bounded solution to the differential equation?

Solution. A. The solution to the differential equation is $y = y_h + y_p$ where y_h is the general solution to homogeneous equation $y'' + y = 0$, and y_p is any particular solution.

First we find y_h. The characteristic equation $r^2 + 1 = 0$ has solutions $\pm i$, and so

$$y_h(t) = A\cos t + B\sin t,$$

where A and B are arbitrary constants.

Then we find y_p. Normally we would look for a particular solution of the form $a\cos t + b\sin t$ but this is part of the homogeneous solution and will not work. Instead we look for

$$y_p = t(a\cos t + b\sin t).$$

Then

$$y_p' = a\cos t + b\sin t + t(-a\sin t + b\cos t)$$

and

$$y_p'' = -a\sin t + b\cos t - a\sin t + b\cos t + t(-a\cos t - b\sin t).$$

We see that $y_p'' + y_p = \sin t \iff a = -1/2$ and $b = 0$.
We have a particular solution

$$y_p = -\frac{t}{2}\cos t.$$

Putting it all together we see that the full solution to the given differential equation is

$$y(t) = A\cos t + B\sin t - \frac{t}{2}\cos t, \quad A, B \text{ arbitrary costants.}$$

For $t = n2\pi$ we have $y(t) = A - n\pi$ that tends to $-\infty$ when the integer $n \to \infty$, independent of the choice of constants A and B. Therefore there is no bounded solution.

□

Answer: A. $y(t) = A\cos t + B\sin t - \frac{t}{2}\cos t,$ A, B arbitrary constants.
B. No.

5. Find the Taylor polynomial of degree 2 about the point $x = 100$ to the function $f(x) = \sqrt{x}$ and use this Taylor polynomial to compute an approximate value of $\sqrt{104}$. Also, decide wether the error of your approximation is less than 10^{-4} in absolute value.

Solution. We differentiate and obtain

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f''(x) = -\frac{1}{4x\sqrt{x}}, \quad f'''(x) = \frac{3}{8x^2\sqrt{x}}$$

existing for all $x > 0$. The Taylor polynomial of degree 2 to f about $x = 100$ is therefore

$$p(x) = 10 + \frac{1}{20}(x - 100) - \frac{1}{8000}(x - 100)^2.$$

If we use this for approximating $\sqrt{104}$ we obtain

$$\sqrt{104} = f(104) \approx p(104) = 10 + \frac{1}{20}(104 - 100) - \frac{1}{8000}(104 - 100)^2 = 10.198.$$

The absolute value of the error in the approximation is for some c between 100 and 104:

$$\left| \frac{3/(8c^2\sqrt{c})}{3!} 4^3 \right| \le \frac{4}{100000} \le 10^{-4}$$

$\square$

Answer: $p(x) = 10 + \frac{1}{20}(x - 100) - \frac{1}{8000}(x - 100)^2$.
$\sqrt{104} \approx 10.198$ with an error less than 10^{-4}

6. Is the improper integral

$$\int_1^\infty \frac{dx}{x^2+x}$$

convergent or divergent? If it is convergent, compute the integral.

Hint: For $x \geq 1$ we have $\dfrac{1}{x^2} \geq \dfrac{1}{x^2+x} = \dfrac{1}{x} - \dfrac{1}{x+1}$.

Solution. Since

$$0 \leq \frac{1}{x^2+x} \leq \frac{1}{x^2}, \quad \text{för } x \geq 1,$$

we have

$$0 \leq \int_1^\infty \frac{dx}{x^2+x} \leq \int_1^\infty \frac{dx}{x^2} = 1$$

and it follows that our integral is convergent. We compute it:

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^2+x} &= \int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right) dx \\
&= \lim_{R\to\infty} \int_1^R \left(\frac{1}{x} - \frac{1}{x+1}\right) dx \\
&= \lim_{R\to\infty} [\ln x - \ln(x+1)]_1^R \\
&= \lim_{R\to\infty} \left[\ln \frac{x}{x+1}\right]_1^R \\
&= \ln 2.
\end{aligned}$$

□

Answer: The integral is convergent and its value is $\ln 2$.

DEL C

7. A. Give the definition of what it means for a function f to be continuous at a point a.
B. Give the definition of what it means for a function f to be differentiable at a point a.
C. Show that a function f differentiable at a point a also must be continuous at a.
D. Give an example showing that a function that is continuous at a point does not have to be differentiable at that point.

Solution. A. The function f is continuous at a if f is defined at a and has a limit when x approaches a and $\lim_{x\to a} f(x) = f(a)$.

B. f is differentiable at a if the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists as a finite number. This limit is then called the derivative of f at a, written $f'(a)$.

C. Suppose f is differentiable at a. We must show that in that case $\lim_{x\to a} f(x) = f(a)$ or equivalently $\lim_{h\to 0}(f(a+h) - f(a)) = 0$. We have

$$\lim_{h\to 0}(f(a+h) - f(a)) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \cdot h = f'(a) \cdot 0 = 0.$$

The proof is complete.

D. Let $f(x) = |x|$. Clearly f is continuous at the origin, since $f(0) = 0$ and $\lim_{x\to 0} f(x) = 0$. The function f is not differentiable at the origin since

$$\lim_{h\to 0} \frac{|0+h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

and this limit does not exist (if h is positive and tends to zero we get 1 but if h is negative and tends to zero we get -1).

□

Answer: See the solution.

8. A hole with radius 1 is drilled through the center of a ball with radius 2. How great a part (in percent) of the volume of the ball remains?

Solution. Let's choose coordinates so that the origin is the center of the ball and the ball is obtained as a solid of revolution of the curve $x^2 + y^2 = 4$ around the x-axis. We may assume that the hole is drilled so that the x-axis is the line of symmetry of the drilling cylinder. In that case points of intersections in the xy-plane between the cylinder and the ball are $(\pm\sqrt{3}, \pm 1)$. The drilled part then consists of a cylinder with radius 1 and height $2\sqrt{3}$ plus two solids of revolution at each end of the cylinder.

The volume of the cylinder is $2\pi\sqrt{3}$. The two solids of revolution are obtained when $x^2 + y^2 = 4$ is rotated around the x-axis, on the intervals $[\sqrt{3}, 2]$, and $[-2, -\sqrt{3}]$. The volume of these are

$$2\pi \int_{\sqrt{3}}^{2} (4 - x^2)\, dx = 2\pi \left(\frac{16}{3} - 3\sqrt{3} \right).$$

Since the volume of the ball is $32\pi/3$ we see that the percentage of the part that has been removed is

$$\frac{2\pi\sqrt{3} + 2\pi\left(\frac{16}{3} - 3\sqrt{3}\right)}{32\pi/3} = \frac{32\pi/3 - 4\pi\sqrt{3}}{32\pi/3} \approx 0.35.$$

Approximately 35 percent of the volume of the ball has been removed and hence the remaining part is approximately 65 percent.

□

Answer: Appox. 65 percent.

9. Show that the function
$$f(x) = x\left(\frac{\pi}{2} - \arctan x\right)$$
is increasing.

Solution. First we observe that the function is defined for all x. We differentiate and obtain
$$f'(x) = \frac{\pi}{2} - \arctan x - \frac{x}{1+x^2}$$
existing for all x. We differentiate a second time and get
$$f''(x) = -\frac{1}{1+x^2} - \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{-2}{(1+x^2)^2}$$
existing for all x and negative for all x.

The fact that $f''(x) < 0$ for all x implies that $f'(x)$ is strictly decreasing for all x. Since $\lim_{x\to\infty} f'(x) = 0$ this implies that $f'(x)$ is positive for all x. Consequently f is strictly increasing.

□

Answer: Se lösningen.