KTH Teknikvetenskap

SF1625 Calculus in one variable
Lösningsförslag till tentamen 2015-04-07

DEL A

1. Let $f(x) = \arcsin x + 2\sqrt{1-x^2}$.
   A. Find the domain of definition of f.
   B. Find the maximum and minimum values of f.

(If you do not recall the derivative of $arcsinx$ you can find it by implicit differentiation of the relation $\sin(\arcsin x) = x$. If you recall the derivative you do not have to do this.)

Solution. A. The domain of definition is the largest set of real numbers x for which $f(x)$ is defined and real.

Since both $\arcsin x$ and the root expression is defined and real for $-1 \leq x \leq 1$ and for no other x, this is the domain of definition of f.

B. Since f is continuous on the closed and bounded interval $-1 \leq x \leq 1$ we know that f must have a maximum and minimum value, and these are attained at points that are critical (the derivative is zero) or singular (the derivative does not exist) or end points of the interval. The endpoints are ± 1. We differentiate and obtain:

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{2\cdot(-2x)}{2\sqrt{1-x^2}} = \frac{1-2x}{\sqrt{1-x^2}},$$

that exists for all x such that $-1 < x < 1$. It is clear that $f'(x) = 0 \iff x = 1/2$. This is the only critical point. Singular points (that are not end points of the interval) do not exist.

It follows that the maximum and minimum values must be attained at -1, 1 or $1/2$. Since

$$f(-1) = -\frac{\pi}{2}, \quad f(1) = \frac{\pi}{2} \quad \text{och} \quad f(1/2) = \frac{\pi}{6} + \sqrt{3},$$

we see that the maximum is $f(1/2) = \frac{\pi}{6} + \sqrt{3}$ and the minimum is $f(-1) = -\frac{\pi}{2}$.

$\square$

Answer: A. The domain of definition is $-1 \leq x \leq 1$.
B. maximum is $f(1/2) = \frac{\pi}{6} + \sqrt{3}$ and the minimum is $f(-1) = -\frac{\pi}{2}$.

2. Is it true that $\displaystyle\int_{-1}^{1} e^{-|x|}\,dx < 2$?

Solution. This can be solved in a number of different ways. Here are three:

1. Since $e^{-|x|} \leq 1$ with equality only when $x = 0$ we know that $\displaystyle\int_{-1}^{1} e^{-|x|}\,dx < \int_{-1}^{1} 1 \cdot dx = 2$.

2. We have $\displaystyle\int_{-1}^{1} e^{-|x|}\,dx = \int_{-1}^{0} e^{x}\,dx + \int_{0}^{1} e^{-x}\,dx = [e^x]_{-1}^{0} + [-e^{-x}]_0^1 = 2 - \frac{2}{e} < 2$.

3.The interval of integration is symmetrical around the origin and $e^{-|x|}$ is an even function (i.e. $e^{-|-x|} = e^{-|x|}$). It follows that

$$\int_{-1}^{1} e^{-|x|}\,dx = 2\int_{0}^{1} e^{-|x|}\,dx = 2\int_{0}^{1} e^{-x}\,dx = 2 - \frac{2}{e} < 2$$

□

Answer: It is true that $\displaystyle\int_{-1}^{1} e^{-|x|}\,dx < 2$.

3. Simple harmonic motion is described by the differetial equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

where $y(t)$ is the deviation from the equilibrium position at time t och ω is a constant.
A. Solve the differential equation if $\omega = 4$.
B. Find the solution to the differential equation (still with $\omega = 4$) that satisfies the conditions $y(0) = -6$ and $y'(0) = 32$.
C. Find the period and the amplitude of your solution.

Solution. A. When $\omega = 4$ we get $y'' + 16y = 0$. The characteristic equation $r^2 + 16 = 0$ has solution $r = 4i$, and so the differential equation has the solution

$$y(t) = a\cos 4t + b\sin 4t, \quad \text{for arbitrary constants } a, b.$$

B. We choose the constants so that the conditions are fulfilled. We see that $y(0) = -6$ yields $a = -6$ and $y'(0) = 32$ yields $4b = 32$ i.e. $b = 8$. The solution we obtain is

$$y(t) = -6\cos 4t + 8\sin 4t.$$

C. Since $\sin t$ and $\cos t$ are periodic with period 2π we find that $y(t) = -6\cos 4t + 8\sin 4t$ is periodic with period $\pi/2$. The amplitude is the maximum of $|-6\cos 4t + 8\sin 4t|$, i.e. 10. $\square$

Answer: A. $y(t) = a\cos 4t + b\sin 4t$, a, b arbitrary constants.
B. $y(t) = -6\cos 4t + 8\sin 4t$.
C. Period $\pi/2$ Amplitude 10

Del B

4. A. Compute the area of the domain between the curves $y = 1$ och $y = \dfrac{x^2+4x+4}{x^2+4x+3}$, for x n the i interval $0 \le x \le R$.

B. Decide whether the domain between the curves $y = 1$ och $y = \dfrac{x^2+4x+4}{x^2+4x+3}$, for x in the interval $0 \le x < \infty$, has finite area. If the area is finite, compute it.

Solution. A. The area is given by

$$\int_0^R \left(\frac{x^2+4x+4}{x^2+4x+3} - 1 \right) dx = \int_0^R \frac{1}{x^2+4x+3}\, dx.$$

We use partial fractions. Since the zeros of the denominator are -1 och -3 we can write the integrand as $A/(x+1) + B/(x+3)$ where we must choose $A = 1/2$ and $B = -1/2$. The area is

$$\begin{aligned} \int_0^R \frac{1}{x^2+4x+3}\, dx &= \int_0^R \left(\frac{1/2}{x+1} - \frac{1/2}{x+3} \right) dx \\ &= \frac{1}{2} \left[\ln|x+1| - \ln|x+3| \right]_0^R \\ &= \frac{1}{2} \left(\ln \frac{R+1}{R+3} + \ln 3 \right). \end{aligned}$$

B. A. The area is given by

$$\begin{aligned} \lim_{R\to\infty} \int_0^R \left(\frac{x^2+4x+4}{x^2+4x+3} - 1 \right) dx &= \lim_{R\to\infty} \int_0^R \frac{1}{x^2+4x+3}\, dx \\ &= \lim_{R\to\infty} \frac{1}{2} \left(\ln \frac{R+1}{R+3} + \ln 3 \right) \\ &= \frac{\ln 3}{2}. \end{aligned}$$

The area is $\dfrac{\ln 3}{2}$. The area is indeed finite.

□

Answer: A. $\frac{1}{2}\left(\ln \frac{R+1}{R+3} + \ln 3\right)$. B. $\frac{\ln 3}{2}$.

5. Let $f(x) = 1-(x-1)^2$, $0 \leq x \leq 2$. Make a simple sketch of the function graph $y = f(x)$ and find the point (x_0, y_0) on the graph making the area of the triangle with corners at $(0,0)$, $(x_0, 0)$ and (x_0, y_0) maximal.

Solution. This is a graph of a quadratic polynomial. The highest point on the graph is $(1,1)$ and the x-intercepts are 0 and 2.

The triangeln with corners at $(0,0)$, $(x,0)$ and (x,y) has area $\dfrac{xy}{2}$ which is equal to $\dfrac{x(1-(x-1)^2)}{2}$ since (x,y) is a point on the graph. We shall maximize the function

$$A(x) = \frac{x(1-(x-1)^2)}{2}, \quad \text{when} \quad 0 \leq x \leq 2.$$

The function A is continuous on the closed and bounded interval, hence a maximum must exist. It is obtained either at a critical point or at one of the endpoints (singular points do not exist as A is a polynomial). We differentiate and obtain

$$A'(x) = \frac{1-(1-x)^2 + 2x(1-x)}{2} = \frac{4x-3x^2}{2}.$$

We see that $A'(x) = 0 \iff x = 0$ or $x = 4/3$. We have three points where the maximum can possibly be obtained , namely $x = 0$, $x = 4/3$ och $x = 2$. Max is obtained at one of them. Since

$$A(0) = 0, \quad A(4/3) = \frac{16}{27} \quad \text{and} \quad A(2) = 0,$$

the maximal area is $16/27$, and this is obtained when $x = 4/3$ and $y = 8/9$. The sought for point on the graph is therefore $(4/3, 8/9)$.

$\square$

Answer: $(4/3, 8/9)$.

6. Compute, using for instance substitution or integration by parts, the integral

$$\int_0^1 x\sqrt{1-x}\,dx$$

Solution. This integral can be computed in many different ways. Here are two:

1. Using integration by parts, we get

$$\begin{aligned}\int_0^1 x\sqrt{1-x}\,dx &= \left[\frac{-x(1-x)^{3/2}}{3/2}\right]_0^1 - \int_0^1 \frac{-(1-x)^{3/2}}{3/2}\,dx \\ &= \left[\frac{-(1-x)^{5/2}}{(3/2)(5/2)}\right]_0^1 \\ &= \frac{4}{15}.\end{aligned}$$

2. Using the substitution $\sqrt{1-x} = u$ we get (observe that $x = 1-u^2$, $dx = -2udu$ and the new interval of integration is from 1 to 0):

$$\begin{aligned}\int_0^1 x\sqrt{1-x}\,dx &= \int_1^0 (1-u^2)u(-2u)\,du \\ &= \int_0^1 2u^2(1-u^2)\,du = \cdots = \frac{4}{15}\end{aligned}$$

□

Answer: $4/15$

DEL C

7. A. Formulate the mean-value theorem of differential calculus.
   B. Use it to show that a function with derivative equal to zero in an open interval must be constant on that interval.

   Solution. See the text book, theorem 11 och theorem 13 of chapter 2.8.

   □

   Answer:

8. Let f be three times differentiable on the interval $-1 < x < 2$, and suppose $f(0) = f'(0) = 0$, $f''(0) = 6$ and $|f'''(x)| \leq 1$ for all x in the interval. Show that
$$1 - \frac{1}{24} \leq \int_0^1 f(x)\,dx \leq 1 + \frac{1}{24}.$$

Solution. Using Taylor's formula and the given information on f we get that for x between -1 and 2:

$$f(x) = 3x^2 + E(x), \quad \text{where } |E(x)| \leq \frac{x^3}{6}.$$

It follows that

$$\int_0^1 f(x)\,dx = \int_0^1 (3x^2 + E(x))\,dx = 1 + \int_0^1 E(x)\,dx.$$

Since

$$\left|\int_0^1 E(x)\,dx\right| \leq \int_0^1 |E(x)|\,dx \leq \int_0^1 \frac{x^3}{6}\,dx = \frac{1}{24},$$

we must have

$$1 - \frac{1}{24} \leq \int_0^1 f(x)\,dx \leq 1 + \frac{1}{24}.$$

$\square$

Answer:

9. Is there a solution $y(t)$ to the differential equation $y''(t) + y(t) = e^t$ such that $y(t)/t^2$ is bounded as $t \to 0$? Find such a solution, if such a solution exists. Can there be more than one?

Solution. The solution to the differential equation has the form $y(t) = y_h(t) + y_p(t)$ where y_h is the solution to the corresponding homogeneous equation and y_p is some particular solution.

We start by finding y_h. The characteristic equation $r^2 + 1 = 0$ has solution $r = \pm i$ and so $y_h(t) = a\cos t + b\sin t$.

Now we find y_p. We see that if $y_p(t) = ce^t$ for some constant c, then $y_p''(t) + y_p(t) = e^t$ if and only if $c = 1/2$. We obtain $y_p(t) = \frac{1}{2}e^t$.

The solution to the given differential equation is $y(t) = y_h(t) + y_p(t)$, i.e.

$$y(t) = a\cos t + b\sin t\frac{1}{2}e^t, \quad a, b \text{ arbitrary constants.}$$

Let us now examine the quiotient $y(t)/t^2$ when t is close to 0. Using Taylor expansion around the origin we get

$$y(t) = a(1 - \frac{t^2}{2}) + bt + \frac{1}{2}(1 + t + \frac{t^2}{2}) + R(t)t^3$$

where $R(t)$ is some function bounded near the origin.

We see that

$$\lim_{t\to 0}\frac{y(t)}{t^2} = \lim_{t\to 0}\frac{a - \frac{at^2}{2} + bt + \frac{1}{2} + \frac{t}{2} + \frac{t^2}{4} + R(t)}{t^2}$$

can only be finite if $a = b = -1/2$. For all other choices of a and b the limit is $\pm\infty$. Therefore the only solution $y(t)$ to the differential equation that makes $y(t)/t^2$ bounded in a neighborhood of the origin is

$$y(t) = -\frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}e^t.$$

For this solution we get that

$$\frac{y(t)}{t^2} = \frac{\frac{t^2}{2} + R(t)t^3}{t^2} = \frac{1}{2} + R(t)t$$

where $R(t)$ is bounded near the origin

The conclusion is that there exists exactly one solution $y(t)$ to the differential equation $y''(t) + y(t) = e^t$ that is bounded in a neighborhood of the origin and that solution is

$$y(t) = -\frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}e^t.$$

□