Cubics Minimize Acceleration

Theorem:

- Given n data points (y_i, t_i) to interpolate and fixed end conditions (either prescribed 1st derivative, or zero second derivative), a piecewise cubic interpolant minimizes the energy: $E(f) = \int_0^n f''(t)^2 \, dt$
- This means: A cubic spline curve has the least square acceleration.
- Related to elastic energy: Hooke's elastic energy of a straight line is given by: $E(\mathbf{f}) = \int_0^n \lambda \left\| \boldsymbol{\kappa} 2[\mathbf{f}](t) \right\|^2 dt$
- I.e.: cubic spline interpolation approximates elastic beams.

Proof: Cubics Minimize Acceleration

Cubic spline: $c(t)$

Another C^2 interpolating curve: $a(t)$

Residual: $d(t) = a(t) - c(t)$.

Energy functional:

$$E(a) = \int_0^n a''(t)^2\, dt$$

$$= \int_0^n \left[c''(t) + d''(t)\right]^2 dt$$

$$= \int_0^n c''(t)^2\, dt + 2\int_0^n c''(t)d''(t)\, dt + \int_0^n d''(t)^2\, dt$$

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Residual: $d(t) = a(t) - c(t)$.

Integration by parts:

$$\int_a^b a(t)b'(t)dt = \left[a(t)b(t)\right]_{t=a}^{t=b} - \int_a^b a'(x)b(x)dx$$

Energy functional:

$$E(a) = \int_0^n c''(t)^2 dt + 2\int_0^n c''(t)d''(t)dt + \int_0^n d''(t)^2 dt$$

Integration by parts:

$$\begin{aligned}\int_0^n c''(t)d''(t)dt &= \left[c''(t)d'(t)\right]_0^n - \int_0^n c'''(t)d'(t)dt \\ &= \left[c''(t)\left(a'(t)-c'(t)\right)\right]_{t=0}^{t=n} - \int_0^n c'''(t)d'(t)dt \\ &= \underbrace{c''(n)}_{\substack{0\text{ for}\\ c''(n)=0}}\underbrace{\left(a'(n)-c'(n)\right)}_{\substack{0\text{ if identical first}\\ \text{order end cond.}}} - \underbrace{c''(0)}_{\substack{0\text{ for}\\ c''(n)=0}}\underbrace{\left(a'(0)-c'(0)\right)}_{\substack{0\text{ if identical first}\\ \text{order end cond.}}} - \int_0^n c'''(t)d'(t)dt\end{aligned}$$

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Integration by parts:

$$\int_a^b a(t)b'(t)dt = \left[a(t)b(t)\right]_{t=a}^{t=b} - \int_a^b a'(x)b(x)dt$$

Energy functional:

$$E(a) = \int_0^n c''(t)^2 dt + 2\int_0^n c''(t)d''(t)dt + \int_0^n d''(t)^2 dt$$

Middle term (cont.):

$$\int_0^n c''(t)d''(t)dt = \int_0^n \underbrace{c'''(t)}_{\text{piecewise const.}} d'(t)dt$$

$$= \sum_{i=0}^{n-1} c'''(i+0.5) \underbrace{\left[d(t)\right]_{t=i}^{t=i+1}}_{=0\,(\text{interpolation})}$$

$$= 0$$

Proof: Cubics Minimize Acceleration

Cubic spline: $c(t)$

Another C^2 interpolating curve: $a(t)$

Residual: $d(t) = a(t) - c(t)$.

Energy functional:

$$E(a) = \underbrace{\int_0^n c''(t)^2 dt}_{\text{cubic spline}} + \underbrace{\int_0^n d''(t)^2 dt}_{\text{additional energy: positive}}$$

Positive additional energy:

Any function that differs in second derivative from c will have higher energy.

$\Rightarrow$ c is a minimal function in terms of E.