Theorem:

• Given \( n \) data points \((y_i, t_i)\) to interpolate and fixed end conditions (either prescribed 1st derivative, or zero second derivative), a piecewise cubic interpolant minimizes the energy:

\[
E(f) = \int_0^n f''(t)^2 \, dt
\]

• This means: A cubic spline curve has the least square acceleration.

• Related to elastic energy: Hooke’s elastic energy of a straight line is given by:

\[
E(f) = \int_0^n \lambda \|k2[f](t)\|^2 \, dt
\]

• I.e.: cubic spline interpolation approximates elastic beams.
Proof: Cubics Minimize Acceleration

Cubic spline: \( c(t) \)

Another \( C^2 \) interpolating curve: \( a(t) \)

Residual: \( d(t) = a(t) - c(t) \).

Energy functional:

\[
E(a) = \int_0^1 a''(t)^2 \, dt \\
= \int_0^1 [c''(t) + d''(t)]^2 \, dt \\
= \int_0^1 c''(t)^2 \, dt + 2 \int_0^1 c''(t)d''(t) \, dt + \int_0^1 d''(t)^2 \, dt
\]
Proof: Cubics Minimize Acceleration

Cubic spline: \( c(t) \)

Another \( C^2 \) interpolating curve: \( a(t) \)

Residual: \( d(t) = a(t) - c(t) \).

Energy functional:

\[
E(a) = \int_0^n c''(t)^2 \, dt + 2 \int_0^n c''(t)d''(t) \, dt + \int_0^n d''(t)^2 \, dt
\]

Integration by parts:

\[
\int_0^n c''(t)d''(t) \, dt = [c''(t)d'(t)]_0^n - \int_0^n c'''(t)d'(t) \, dt
\]

\[
= [c''(t)(a'(t) - c'(t))]_0^n - \int_0^n c'''(t)d'(t) \, dt
\]

\[
= c''(n)(a'(n) - c'(n)) - c''(0)(a'(0) - c'(0)) - \int_0^n c'''(t)d'(t) \, dt
\]

0 for \( c''(n) = 0 \) 0 if identical first order end cond. 0 for \( c''(n) = 0 \) 0 if identical first order end cond.
Proof: Cubics Minimize Acceleration

Cubic spline: \( c(t) \)

Another \( C^2 \) interpolating curve: \( a(t) \)

Residual: \( d(t) = a(t) - c(t) \).

Energy functional:

\[
E(a) = \int_0^b c''(t)^2 \, dt + 2 \int_0^b c''(t)d''(t) \, dt + \int_0^b d''(t)^2 \, dt
\]

Integration by parts:

\[
b \int_a^b a(t)b'(t) \, dt = \left[ a(t)b(t) \right]_{t=a}^{t=b} - \int_a^b a'(x)b(x) \, dx
\]

Middle term (cont.):

\[
\int_0^b c''(t)d''(t) \, dt = \int_0^b c'''(t) \, d'(t) \, dt
\]

\[
= \sum_{i=0}^{n-1} c'''(i + 0.5) \left[ d(t) \right]_{t=i+1}^{t=i+1}
\]

\[
= 0 \text{ (interpolation)}
\]

\[
= 0
\]
Proof: Cubics Minimize Acceleration

Cubic spline: \( c(t) \)

Another \( C^2 \) interpolating curve: \( a(t) \)

Residual: \( d(t) = a(t) - c(t) \).

Energy functional:

\[
E(a) = \int_0^n c''(t)^2 \, dt + \int_0^n d''(t)^2 \, dt
\]

Positive additional energy:

Any function that differs in second derivative from \( c \) will have higher energy.

\( \Rightarrow \) \( c \) is a minimal function in terms of \( E \).