KTH Engineering Sciences

Exam

DN2255 – Numerical Solutions of Differential Equations

8–13, May 27, 2014 in L52

Closed books examination. Read all the questions before starting work. Ask if you are uncertain. Answers MUST be motivated well and communicated in a clear way. Do not leave room for interpretation. Define variables. Explain figures and the notation you use. Paginate and write your name on EVERY page handed in. A total of 20 out of max 40 points guarantees a "pass". GOOD LUCK!

P1. (4 p)

Consider the following Riemann problem

$$\mathbf{u}_t + \begin{bmatrix} 0 & 9 \\ 1 & 0 \end{bmatrix} \mathbf{u}_x = 0, \quad \mathbf{u} = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \quad \mathbf{u}(x,0) = \begin{cases} \mathbf{u}_L \text{ if } x < 0 \\ \mathbf{u}_R \text{ if } x > 0. \end{cases}$$

where $\mathbf{u}_L = [1 \;\; 0]^T$ and $\mathbf{u}_R = [4 \;\; 0]^T$. Solve and write down the solution to the problem. Sketch the solution as a function of x at some time, say $t = 1$, i.e. sketch $u^1(x,1)$ and $u^2(x,1)$.

P2. (8 p)

We want to show that the following initial-boundary value strip problem

$$u_t = \alpha u_{xx} + bu, \quad u(x,0) = f(x)$$
$$u(0,t) = 0 \text{ and } \; u(1,t) + hu_x(1,t) = 0$$

where α, b and h are real constants and $\alpha, h > 0$, is well-posed in L_2-norm.

(a) (1 p) Explain why Fourier analysis is not immediately applicable to show well-posedness of the problem.

(b) (4 p) Instead, use the energy method to show well-posedness.

(c) (3 p) Now let $b = 0$ and change the boundary conditions to periodic boundary conditions, i.e. $u(0,t) = u(1,t)$. Discretize the problem as

$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{(\Delta x)^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \quad u_j^n = u_{j+N}^n.$$

Use von Neumann analysis to determine for which Δx and Δt the scheme is stable.

P3. (9 p) The linear advection equation $u_t + au_x = 0$, a real, is discretized by the *forward time and centered space* (FTCS) method,

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n).$$

(a) (2) Draw the domain of dependence of the solution to the advection equation. In the same figure, draw the domain of dependence of the FTCS-method (make a proper sketch of the grid) when the

i) CFL condition is fulfilled

ii) CFL condition is NOT fulfilled

(b) (4 p) Derive the modified equation for this method.

(c) (2 p) How can we use the modified equation to draw the conclusion that this discretization is unstable for all fixed $\Delta t/\Delta x$?

(If you did not manage to derive the modified equation in (a) you can instead show that the discretization is unstable by von Neumann analysis.)

(d) (1 p) Explain why the FTSC-method can be unstable even if the CFL conditions is fulfilled.

P4. (7 p)

Consider the Riemann problem

$$u_t + f(u)_x = 0, \qquad f(u) = e^u, \tag{1}$$

$$u(x,0) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases}$$

(a) (4 p) Solve the Riemann problem, i.e. given arbitrary values of u_L and u_R, provide an expression for $u(x,t)$ for $t > 0$ in terms of x, t, u_L and u_R. Specify carefully all quantities you use to express the solution. (Hint: sometimes it is a shock solution, sometimes a rarefaction solution, where the solution is on the form $u(x,t) = v(x/t)$ for some function $v(\xi)$.) Sketch the characteristics/shock/rarefaction fan for the solutions.

(b) (3 p) The numerical flux function for Godunov's method solving a non-linear scalar equation is given by $F_{j+1/2}^n = f(\hat{Q})$ where $\hat{Q}$ denotes the solution to the Riemann problem centred in $x_{j+1/2}$ at $t > t^n$. Determine $\hat{Q}$ for this problem, Eq. (1).

P5. (2 p)

The Gudonov method can be constructed by following the *Reconstruction-Evolve-Average (REA)* algorithm. Discuss how the reconstruction should be modified to achieve better than first order accuracy but still TVD stable (no oscillations). You need to explain the concept of slope limiters.

P6. (10 p)

Consider the initial-boundary value problem on the strip $0 \leq x \leq 1$, $t > 0$,

$$\mathbf{q}_t + \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \mathbf{q}_x = \mathbf{0}, \quad \mathbf{q} = \begin{pmatrix} u \\ v \end{pmatrix}$$

with boundary conditions

$$v(0,t) = 0, \quad u(1,t) + \alpha v(1,t) = 0 \quad (\alpha \text{ real})$$

and initial conditions

$$u(x,0) = e^{-500(x-0.5)^2} \quad \text{(a short pulse at x=0.5, details not important)},$$
$$v(x,0) = \beta u(x,0) \quad (\beta \text{ real}).$$

(a) (3 p) Diagonalize the system and find the characteristic speeds and characteristic variables w_1 and w_2. Write down the expressions for w_1 and w_2 in terms of u and v.

(b) (4 p) Show that the boundary conditions give a well-posed problem *except* for one value of α. Which? Why?

(c) (3 p) Find two values of β that make the solution a single travelling pulse. Sketch the solution for these values of β (either for w_1 and w_2 **or** u and v).