# Notes on quantum critical points

**John Hertz, June 2015**

In these notes I will describe how quantum critical problem can be mapped into corresponding classical ones with one extra dimension, with possible complications duke to anisotropy in the space-time coordinates. I will start with the example of the Ising model in a transverse field (which, however, does not have any such anisotropy). But to do this efficiently it will be useful first to rederive the $\phi^4$ model we used to study classical problems near $d = 4$ in a different way from the one we employed there.

## 0.1 Another path to the LGW/$\phi^4$ model for the classical Ising model

We start with the Ising partition function

$$Z = \sum_{\{s\}} \exp\left(\sum_{ij} J_{ij} s_i s_j\right) \tag{1}$$

Now we use the Gaussian integral identity

$$\mathrm{e}^{\frac{1}{2}a^2} = \int \frac{dx}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2 + ax} \tag{2}$$

(to prove it just complete the square in the exponent on the right hand side and shift the integration variable) to write

$$Z = \frac{1}{\sqrt{\det J}} \int \prod_i \frac{d\phi_i}{\sqrt{2\pi}} \sum_{\{s\}} \exp\left(-\tfrac{1}{2}\sum_{ij} \phi_i [J^{-1}]_{ij} \phi_j + \sum_i \phi_i s_i\right). \tag{3}$$

To save a little space, we abbreviate

$$\frac{1}{\sqrt{\det J}} \prod_i \frac{d\phi_i}{\sqrt{2\pi}} = D\phi, \tag{4}$$

so

$$Z = \int D\phi \sum_{\{s\}} \exp\left(-\tfrac{1}{2}\sum_{ij} \phi_i [J^{-1}]_{ij} \phi_j + \sum_i \phi_i s_i\right). \tag{5}$$

The point of doing all this was to make the sum on $\{s\}$ simple: It factorizes, and each factor is simply $2\cosh\phi_i$:

$$Z = \int D\phi \exp\left(-\tfrac{1}{2}\sum_{ij} \phi_i [J^{-1}]_{ij} \phi_j + \sum_i \log 2\cosh\phi_i\right). \tag{6}$$

Now it is simple to get our Landau-Ginzburg-Wilson model. In the quadratic term, we go over to $k$ space, expand $J^{-1}(k)$ to second order, and then go back to real space. In the log cosh term, we just expand to 4th order. Everything else is, by familiar arguments, irrelevant near $d = 4$. Setting various unimportant constants equal to 1 and taking the continuum limit, we get the familiar form

$$Z = \int D\phi \exp(-H_{\text{eff}}) \tag{7}$$

with

$$H_{\text{eff}} = \int dx \left[\tfrac{1}{2} t\phi^2(x) + \tfrac{1}{2}(\nabla\phi)^2 + u\phi^4(x)\right]. \tag{8}$$

## 0.2 Ising model in a transverse field

Now add a field in the $x$ direction:

$$H = -\tfrac{1}{2}\sum_{ij} J_{ij}\sigma_i^z\sigma_j^z - \sum_i h_i\sigma_i^x. \tag{9}$$

(The $\sigma_i^a$ are Pauli matrices.) Now the two terms in $H$ do not commute, so in computing $\mathrm{Tr}\, \mathrm{e}^{-\beta H}$ we have to slice (imaginary) time up into into little steps and compute a time-ordered average:

$$Z = Z_0 \left\langle \mathrm{T}\exp\left[\tfrac{1}{2}\int_0^\beta d\tau \sum_{ij} J_{ij}\sigma_i^z(\tau)\sigma_j^z(\tau)\right]\right\rangle_0, \tag{10}$$

where the average is under the free Hamiltonian, i.e., the transverse field terms in (9)., and $Z_0$ is the partition function for that independent-spin system. The only thing that is different from what you learned in first-year quantum mechanics is that the "time" variable runs from 0 to $-i\beta$.

We now apply the Gaussian integral identity (2) to (10):

$$\begin{aligned} Z &= Z_0 \int D\phi \left\{ \exp\left[-\tfrac{1}{2}\int_0^\beta d\tau \sum_{ij}\phi_i(\tau)[J^{-1}]_{ij}\phi_j(\tau)\right] \right. \\ &\times \left. \left\langle \mathrm{T}\exp\left(\int_0^\beta \sum_i d\tau \phi_i(\tau)\sigma_i^z(\tau)\right)\right\rangle_0 \right\}, \end{aligned} \tag{11}$$

where

$$D\phi \propto \prod_{i,\tau} d\phi_i(\tau). \tag{12}$$

(Implicitly, one is to imagine doing this for discretized time, taking the continuous-time time limit in the end.)

Again, the time-ordered term factorizes:

$$Z = Z_0 \int D\phi \exp\left[-\tfrac{1}{2}\int_0^\beta d\tau \sum_{ij}\phi_i(\tau)[J^{-1}]_{ij}\phi_j(\tau) + \sum_i \log z_0[\phi_i]\right], \tag{13}$$

with

$$z_0[\phi] = \left\langle \mathrm{T} \exp \left( \int_0^\beta \sum_i d\tau \phi_i(\tau) \sigma^z(\tau) \right) \right\rangle_0 . \tag{14}$$

The (functional) derivatives of $\log z_0[\phi]$ with respect to $\phi(\tau_1)$, $\phi(\tau_2)$, etc., give the time-ordered cumulants of $\sigma^z$. The first non vanishing one, of second order, is just the dynamical susceptibility

$$\frac{\delta \log z_0[\phi]}{\delta\phi(\tau_1)\delta\phi(\tau_2)} = \frac{\delta\langle\sigma^z(\tau_1)\rangle}{\delta\phi(\tau_2)} = \chi_0(\tau_1, \tau_2), \tag{15}$$

the susceptibility of a single spin in a constant field $h\hat{\mathbf{x}}$ to a perturbing field in the $\hat{\mathbf{z}}$ direction. We can figure out simple what this is, in the frequency domain, by simple arguments. First, the static limit: The transverse field $\phi$ just rotates the spin from the $\hat{\mathbf{x}}$ direction by an angle $\approx \phi/h$, so $\langle\sigma_z\rangle \approx 1/h$, i.e. for $\omega \to 0$, $\chi_0 \to 1/h$. Next, for nonzero $\omega$, we know that $\chi$ will be resonant at $\pm$ the Larmor precession frequency, which is (also) $h$. This gives

$$\chi_0(\omega) = \frac{h}{h^2 + \omega^2}, \tag{16}$$

where the + sign in the denominator is because we are using imaginary time.

Now putting this into (13) and expanding in $k$ and $\omega$ to second order (and setting uninteresting constants equal to 1 as before), we get a quadratic piece of the exponent of the form

$$S_0[\phi] = \tfrac{1}{2} \sum_k \sum_\omega (\tilde{t} + k^2 + \omega^2)|\phi_{k,\omega}|^2, \tag{17}$$

where we have also Fourier transformed with respect to time: the sum on $\omega$ is over $\omega_n = 2\pi n/\beta$ (we impose periodic Kubo-Martin-Schwinger boundary conditions on $\phi(\tau)$.) Back in space-(imaginary)time (and in the continuum limit), this is

$$S_0[\phi] = \tfrac{1}{2} \int d^d x \int_0^\beta d\tau \left[\tilde{t}\phi^2(x,\tau) + (\nabla\phi)^2 + (\partial_\tau\phi)^2\right]. \tag{18}$$

It is worth remarking that although this model exhibits decaying correlations in space and (imaginary) time, back in real time it describes propagating modes with $\omega^2 = \tilde{t} + k^2$, i.e., $\sqrt{\tilde{t}}$ is an *energy gap* or "mass gap" ($\tilde{t} = m^2$).

As usual, of all the higher-order terms in the expansion in powers of $\phi$, we will only be concerned with the quartic one, and that only in the static limit. I won't evaluate it explicitly, just calling it "$u$", as we did in the classical case. Everything else is irrelevant near the upper critical dimensionality. So finally we get an "effective Hamiltonian" for this problem of

$$S[\phi] = \int d^d x \int_0^\beta d\tau \left[\tfrac{1}{2}\tilde{t}\phi^2(x,\tau) + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}(\partial_\tau\phi)^2 + u\phi^4(x,\tau)\right]. \tag{19}$$

We see that in the zero-temperature limit ($\beta \to \infty$) this is just like the classical problem with one more dimension. Thus the upper critical dimensionality for this problem is 3.

At finite $\beta$, the system is finite in the extra dimension. Thus we can use what we learned about finite-size scaling to conclude that at nonzero temperature, as long as the correlation length (= correlation time, in the units we are using) in the quantum zero-T problem is less than $\beta$, the system looks essentially like the zero-T problem. However, in the immediate vicinity of the critical point, where $\xi > \beta$, the system crosses over to a classical one, with critical exponents characteristic of the $d$ spatial dimensions.

## 0.3 Some other quantum LGW models

It is possible to make the same kind of arguments for other quantum systems. In these cases, the assumptions we have made here about regularity of the higher-order terms in the expansion in $\phi$ are not always valid, so the real story is more involved than what I will tell here. Nevertheless, looking at these problems at this level gives some insight into their long-wavelength, low-frequency fluctuations.

We consider first a superconductor or antiferromagnet. In this case, $\phi$ has to be, respectively, 2- or 3-dimensional. Furthermore, the order parameter dynamics are relaxational, not propagating (i.e., first order in $\omega$, not second). Thus, in this case we have, instead of (17),

$$S_0[\vec{\phi}] = \tfrac{1}{2} \sum_k \sum_\omega (\tilde{t} + k^2 + |\omega|) |\vec{\phi}_{k,\omega}|^2, \tag{20}$$

The fact that $\omega$ and $k$ do not occur with the same power means that when we renormalise, we have to do so differently (i.e., by different powers of the rescaling factor $b$) in space and time. Elementary dimensional considerations (valid at or above the upper critical dimensionality) show that if $x \to x/b$ we have to take $\tau \to \tau/b^2$ to keep both the $k^2$ and $|\omega|$ terms marginal, i.e., we have a dynamical critical exponent of $z = 2$. As a result, we find

$$u \to u b^{4-d-z} \tag{21}$$

instead of the classical result

$$u \to u b^{4-d}. \tag{22}$$

For $z = 1$ we just recover the shift of the critical dimensionality by 1 found above for the transverse-field Ising model, but for the superconductor or antiferromagnet, with $z = 2$, the critical dimensionality is reduced to 2.

Another interesting case is the itinerant-electron ferromagnet. In this case,

$$S_0[\vec{\phi}] = \tfrac{1}{2} \sum_k \sum_\omega \left( \tilde{t} + k^2 + \frac{|\omega|}{v_F k} \right) |\vec{\phi}_{k,\omega}|^2, \tag{23}$$

where $v_F$ is the Fermi velocity. The $\omega/v_F k$ dependence comes from expanding the free-electron susceptibility

$$\chi_0(k,\omega) = \sum_p \frac{f_p - f_{p+k}}{\omega - \epsilon_p + \epsilon_{p+k}} \tag{24}$$

for small $\omega/k$. The frequency-dependent term reflects the way free electrons and holes of total momentum $k$ get our of phase. Note also that the characteristic relaxation rate $\omega_k \propto v_F k(\tilde{t} + k^2)$, which goes to zero as $k \to 0$, reflecting the conservation of total magnetisation. At the critical point $\tilde{t} \to 0$, so $\omega_k \propto k^3$, implying $z = 3$. (However, the form of the inverse static susceptibility assumed here, $\tilde{t} + k^2$, does not hold for $d < 3$, so we can't say anything just from this about the critical dimensionality. We can only conclude that $d = 3$ is safely in the mean-field region.

Finally, a model used for itinerant magnets with (weak) impurities: For such a system, the impurities cause the long-wavelength, slow dynamics to be diffusive, and the quadratic part of $S$ becomes

$$S_0[\vec{\phi}] = \tfrac{1}{2} \sum_k \sum_\omega \left( \tilde{t} + k^2 + \frac{|\omega|}{Dk^2} \right) |\vec{\phi}_{k,\omega}|^2, \tag{25}$$

where $D$ is the diffusion constant for the noninteracting electrons. Here, by the same arguments as above, we have $z = 4$. This ignores the effects of randomness that we considered earlier in the classical problem. The treatment there suggest that we can ignore such effects when $\alpha < 0$ in the classical problem.