KTH Teknikvetenskap

SF1624 Algebra och geometri
Solutions for Examn 2015.06.10

DEL A

1. We have the following points in three space:

$$A = (-1, 0, 1),\ B = (1, 1, 2) \quad \text{och} \quad C = (0, 0, 2).$$

(a) Give a parametric representation of the line l passing through B and C. **(1 p)**

(b) Determine an equation (normal form) for the plane π containing A, and orthogonal against l. **(1 p)**

(c) Determine the distance between point A and the line l. **(2 p)**

Solution.

(a) The line l passing through B and C has directional vector $\vec{BC} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$. The line passes through e.g. the point B, which gives the parametric representation

$$\begin{pmatrix} x \\ y \\ x \end{pmatrix} = t \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

(a) As the sought plane is orthogonal l, the directional vector $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ is also a normal vector for the plane. The plane also passes through A. An equation is then

$$(-1) \cdot \big(x - (-1)\big) + (-1) \cdot (y - 0) + 0 \cdot (z - 1) = 0, \text{ d.v.s. } x + y + 1 = 0.$$

(b) We have that l is orthogonal to the plane π. It follows that the distance we are seeking equals the distance between A and P, where P is the intersection point of π and l. We have that

$$l \cap \pi = \{ \begin{pmatrix} -t + 1 \\ -t + 1 \\ 2 \end{pmatrix} \text{ så att } -t + 1 - t + 1 + 1 = 0 \}.$$

It follows that $t = \frac{3}{2}$, and that $P = \begin{pmatrix} -1/2 \\ -1/2 \\ 2 \end{pmatrix}$. The sought distance is

$$d(A, P) = \|\vec{AP}\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{3}{2}}.$$

2. For each number a we have the matrix

$$A = \begin{bmatrix} 1 & 1 & -a \\ -1 & 1 & 0 \\ 2 & 2a+2 & -2a-4 \end{bmatrix}.$$

(a) For which values of a is the matrix A invertible? **(2 p)**
(b) Let $a = 3$, and determine the inverse of A. **(2 p)**

Solution.
(a) By adding multiples of the first row to the second and the third, we get

$$\begin{aligned} \det \begin{bmatrix} 1 & 1 & -a \\ -1 & 1 & 0 \\ 2 & 2a+2 & -2a-4 \end{bmatrix} &= \det \begin{bmatrix} 1 & 1 & -a \\ 0 & 2 & -a \\ 0 & 2a & -4 \end{bmatrix} \\ &= 1 \det \begin{bmatrix} 2 & -a \\ 2a & -4 \end{bmatrix} \\ &= 2(-4) - 2a(-a) \\ &= 2(a^2 - 4). \end{aligned}$$

The determinant $\det(A)$ is therefore zero if and only if $a = \pm 2$, so the matrix A is invertible for all other values of a.

(b) When $a = 3$, we have the matrix

$$A = \begin{bmatrix} 1 & 1 & -3 \\ -1 & 1 & 0 \\ 2 & 8 & -10 \end{bmatrix}.$$

Prosecuting elementary row operations give

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 8 & -10 & 0 & 0 & 1 \end{array}\right] &\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 2 & -3 & 1 & 1 & 0 \\ 0 & 6 & -4 & -2 & 0 & 1 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 2 & -3 & 1 & 1 & 0 \\ 0 & 0 & 5 & -5 & -3 & 1 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & -9/5 & 3/5 \\ 0 & 2 & 0 & -2 & -4/5 & 3/5 \\ 0 & 0 & 1 & -1 & -3/5 & 1/5 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -7/5 & 3/10 \\ 0 & 1 & 0 & -1 & -2/5 & 3/10 \\ 0 & 0 & 1 & -1 & -3/5 & 1/5 \end{array}\right] \end{aligned}$$

Our candidate for the inverse is

$$A^{-1} = \begin{bmatrix} -1 & -7/5 & 3/10 \\ -1 & -2/5 & 3/10 \\ -1 & -3/5 & 2/10 \end{bmatrix}.$$

We control our answer

$$\begin{bmatrix} 1 & 1 & -3 \\ -1 & 1 & 0 \\ 2 & 8 & -10 \end{bmatrix} \cdot \begin{bmatrix} -1 & -7/5 & 3/10 \\ -1 & -2/5 & 3/10 \\ -1 & -3/5 & 2/10 \end{bmatrix} = \begin{bmatrix} -1-1+3=1 & \frac{-7-2+9}{5}=0 & \frac{3+3-6}{10}=0 \\ 1-1=0 & \frac{7-2}{5}=1 & \frac{-14-16+30}{5}=0 \\ \frac{3+3-6}{10}=0 & \frac{-3+3}{10}=0 & \frac{6+24-20}{10}=1 \end{bmatrix}.$$

Answer.

3. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map

$$T(x, y, z) = (x + 2y + z, 2x + y - z, -3x - y + 2z).$$

(a) Determina a matrix representation of the map T. **(1 p)**
(b) Determine a basis for the kernel, $\ker(T)$. **(1 p)**
(c) Determine the dimension of the image of T. **(1 p)**
(d) Let $P = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Determine another point Q such that $T(P) = T(Q)$. **(1 p)**

Solution.

(a) We have that

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y + z \\ 2x + y - z \\ -3x - y + 2z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ -3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and it follows that the matrix representation of T is

$$M_T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ -3 & -1 & 2 \end{bmatrix}.$$

(b) Gauss-Jordan elimination transforms M_T to row echolon form

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ -3 & -1 & 2 \end{bmatrix} &\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & 5 & 5 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

We read off the solutions to the system

$$M_T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

for real numbers t. A basis for the kernel is then given by, for instance, the vector

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

(c) The dimension of the image equals the rank of the matrix M_T, which equals the number of leading ones. In this case the rank is two.

(d) We have that $P + Q'$, with Q' in the kernel of T have the same image as $T(P)$. The kernel of T is $\begin{bmatrix} t & -t & t \end{bmatrix}^T$. So, we can for instance choose

$$Q = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

Answer.

DEL B

4. We have the matrix

$$A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}.$$

(a) Determine one eigenvalue that has two linearly independent eigenvectors. **(2 p)**

(b) Determine all eigenvalues, and determine wheter the matrix A is diagonalizable. **(2 p)**

Solution.

(a) The matrix A has rank 1, clearly as

$$\begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 5 & 5 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, the kernel is of dimension two. Therefore $\lambda = 0$ is one eigenvalue having two linearly independent eigenvectors.

(b) The characteristic polynomial of A is

$$0 = \det(\lambda I - A) = \det \begin{bmatrix} \lambda - 5 & -5 & -5 \\ -5 & \lambda - 5 & -5 \\ -5 & -5 & \lambda - 5 \end{bmatrix}.$$

If we add the second and the third row to the first row, we get

$$\begin{aligned} 0 &= \det \begin{bmatrix} \lambda - 5 & -5 & -5 \\ -5 & \lambda - 5 & -5 \\ -5 & -5 & \lambda - 5 \end{bmatrix} \\ &= \det \begin{bmatrix} \lambda - 15 & \lambda - 15 & \lambda - 15 \\ -5 & \lambda - 5 & -5 \\ -5 & -5 & \lambda - 5 \end{bmatrix} \\ &= (\lambda - 15) \det \begin{bmatrix} 1 & 1 & 1 \\ -5 & \lambda - 5 & -5 \\ -5 & -5 & \lambda - 5 \end{bmatrix} \\ &= (\lambda - 15) \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \\ &= (\lambda - 15)\lambda^2. \end{aligned}$$

We now know all the roots of the characteristic polyomial, $\lambda = 0$ and $\lambda = 15$. As the dimensions of their corresponding eigenspaces equals their algebraic multiplicities (2 and 1, respectively) we get that the matrix is diagonalizable.

5. Let V be the linear span of the vectors $\begin{bmatrix}-2\\-2\\0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\\2\\2\end{bmatrix}$, and let $V^\perp$ denote its orthogonal complement.
   (a) Determine a basis for $V^\perp$. **(2 p)**
   (b) Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the reflection through V, i.e. $T(\vec{x}) = \vec{x}$ if $\vec{x}$ is in V, and $T(\vec{x}) = -\vec{x}$ if $\vec{x}$ is in $V^\perp$. Determine $T(\begin{bmatrix}1\\1\\1\\1\end{bmatrix})$. **(2 p)**

Solution.

(a) The vector space $V^\perp$ consists of all vectors $\begin{bmatrix}x & y & z & w\end{bmatrix}^T$ that are orthogonal against the two vectors $\vec{v}_1 = \begin{bmatrix}-2 & -2 & 0 & 1\end{bmatrix}^T$ and $\vec{v}_2 = \begin{bmatrix}1 & 0 & 2 & 2\end{bmatrix}^T$. Written in matrix form that means that $\begin{bmatrix}x & y & z & w\end{bmatrix}^T$ satisfies

$$\begin{bmatrix}1 & 0 & 2 & 2\\ -2 & -2 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

By elementary row operations we get that

$$\begin{bmatrix}1 & 0 & 2 & 2\\ -2 & -2 & 0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2 & 2\\ 0 & -2 & 4 & 5\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2 & 2\\ 0 & 1 & -2 & -\frac{5}{2}\end{bmatrix},$$

and we read off the solutions as

$$\begin{bmatrix}x\\y\\z\\w\end{bmatrix} = \begin{bmatrix}-2s-2t\\ 2s+\frac{5}{2}t\\ s\\ t\end{bmatrix} = s\begin{bmatrix}-2\\2\\1\\0\end{bmatrix} + \frac{t}{2}\begin{bmatrix}-4\\-5\\0\\2\end{bmatrix}$$

with parameters s and t. A basis for $V^\perp$ can therefore be chosen as the two vectors

$$\vec{v}_3 = \begin{bmatrix}-2\\2\\1\\0\end{bmatrix} \quad \text{and} \quad \vec{v}_4 = \begin{bmatrix}-4\\5\\0\\2\end{bmatrix}.$$

(b) We normalize the vectors $\vec{v}_1$ and $\vec{v}_2$, and then get an ON-basis for the vector space V,

$$\vec{n}_1 = \frac{1}{3}\begin{bmatrix}-2\\-2\\0\\1\end{bmatrix} \quad \text{och} \quad \vec{n}_2 = \frac{1}{3}\begin{bmatrix}1\\0\\2\\2\end{bmatrix}.$$

We let $\vec{x} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$. We have that

$$\begin{aligned}
\operatorname{proj}_V(\vec{x}) &= (\vec{n}_1 \cdot \vec{x})\vec{n}_1 + (\vec{n}_2 \cdot \vec{x})\vec{n}_2 \\
&= \frac{1}{3}(-3)\vec{n}_1 + \frac{5}{3}\vec{n}_2 \\
&= \frac{1}{9}\begin{bmatrix} 6+5 \\ 6+0 \\ 0+10 \\ -3+10 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 11 \\ 6 \\ 10 \\ 7 \end{bmatrix}.
\end{aligned}$$

Then we get that $\vec{x} - \operatorname{proj}_V(\vec{x}) = \frac{1}{9}\begin{bmatrix} -2 \\ 3 \\ -1 \\ 2 \end{bmatrix}$. It follows that

$$\begin{aligned}
T(\vec{x}) &= T(\operatorname{proj}_V(\vec{x})) + T(\vec{x} - \operatorname{proj}_V(\vec{x})) \\
&= \operatorname{proj}_V(\vec{x}) - \vec{x} + \operatorname{proj}_V(\vec{x}) \\
&= \frac{1}{9}\begin{bmatrix} 11 \\ 6 \\ 10 \\ 7 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} +2 \\ 3 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 13 \\ 3 \\ 11 \\ 5 \end{bmatrix}.
\end{aligned}$$

(b') *Alternatively:* We have the basis $\{\vec{v}_1, \vec{v}_2\}$ for V, and a basis $\{\vec{v}_3, \vec{v}_4\}$ for $V^\perp$. We determine the coordinate matrix for $\vec{x} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$ with respect to the basis $\{\vec{v}_1, \dots, \vec{v}_4\}$ of $\mathbb{R}^4$. This is given as the solution of the system

$$\left[\begin{array}{cccc|c} -2 & 1 & -2 & -4 & 1 \\ -2 & 0 & 2 & 5 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 & 1 \end{array}\right].$$

By applying elementary row operations we get

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 4 & 2 & 9 & 3 \\ 0 & 5 & -2 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 9 & 1 \\ 0 & 0 & -\frac{9}{2} & 0 & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{3}{9} \\ 0 & 1 & 0 & 0 & \frac{5}{9} \\ 0 & 0 & 1 & 0 & -\frac{1}{9} \\ 0 & 0 & 0 & 1 & \frac{1}{9} \end{array}\right].$$

So,

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = -\frac{3}{9}\vec{v}_1 + \frac{5}{9}\vec{v}_2 - \frac{1}{9}\vec{v}_3 + \frac{1}{9}\vec{v}_4.$$

And in particular we get that

$$T(\vec{x}) = -\frac{3}{9}\vec{v}_1 + \frac{5}{9}\vec{v}_2 + \frac{1}{9}\vec{v}_3 - \frac{1}{9}\vec{v}_4 = \frac{1}{9}\begin{bmatrix} 13 \\ 3 \\ 11 \\ 5 \end{bmatrix}.$$

Answer.

6. Determine a symmetric matrix A that satisfies the following.
   (a) The eigenspace corresponding to the eigenvalue $\lambda = 2$ is $\begin{bmatrix} 2t & t & -t \end{bmatrix}^T$, where t is a parameter.
   (b) The eigenspace corresponding to the eigenvalue $\lambda = 4$ has dimension two. **(4 p)**

Solution. The first condition (a) gives that $v = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue 2. As the matrix A is symmetric, its different eigenspaces are orthogonal. So, the two dimensional eigenspace corresponding to the eigenvalue 4, is orthogonal to $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. In other words the eigenspace E_4 is given as the plane $2x + y - z = 0$. We choose an orthogonal basis $\{\vec{u}, \vec{w}\}$ for the plane;

$$u := \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{och} \quad w := \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

This is not necessary, but does simplify our calculations that follow. I the basis $\{v, u, w\}$ the matrix representation of our linear map is the diagonal matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

The matrix we are looking for is the matrix representation in the standard basis. That means that

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}^{-1}$$

We note, before we continue, that

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Consequently the sought inverse matrix is given as the following product

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

We now proceed by caculating the matrix A,

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{2}{3} & 0 & \frac{4}{3} \\ \frac{1}{3} & 2 & \frac{-4}{3} \\ \frac{-1}{3} & 2 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{8}{3} & \frac{-2}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{11}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{11}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 8 & -2 & 2 \\ -2 & 11 & 1 \\ 2 & 1 & 11 \end{bmatrix}.$$

DEL C

7. Determine the line L that passes through the point $P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and intersects both the lines

(4 p)

$$L_1 = \{ \begin{bmatrix} 3t \\ t \\ t+1 \end{bmatrix} \mid \text{tal } t\} \quad \text{and} \quad L_2 = \{ \begin{bmatrix} s \\ s+4 \\ 2s \end{bmatrix} \mid \text{tal } s\}.$$

Solution. A directional vector from the point P to a point Q on the line L_1 is $Q - P = \begin{bmatrix} 3t-1 & t-1 & t \end{bmatrix}^T$. And similarly we get the directiona vector $\begin{bmatrix} s-1 & s+3 & 2s-1 \end{bmatrix}^T$ from P to a point on L_2. We need to determine the numbers s and t such that the vectors $\begin{bmatrix} 3t-1 \\ t-1 \\ t \end{bmatrix}$ och $\begin{bmatrix} s-1 \\ s+3 \\ 2s-1 \end{bmatrix}$ are parellel. The vectors being parallel means that there exists a number λ solving the system

$$\begin{cases} 3t\lambda - \lambda &= s-1 \\ t\lambda - \lambda &= s+3 \\ t\lambda &= 2s-1. \end{cases}$$

We subtract the second equation from the first, and get that $2t\lambda = -4$. So $t\lambda = -2$. The third equation gives that $-2 = 2s - 1$, so $s = \frac{-1}{2}$. We use that information in the second equation $-2 - \lambda = \frac{-1}{2} + 3$, and we obtain that $\lambda = \frac{-9}{2}$. So $t = \frac{4}{9}$. One verifies that $\lambda = \frac{-9}{2}$, $t = \frac{4}{9}$ and $s = \frac{-1}{2}$ also satisfies the first equation.

We then have that $\begin{bmatrix} s-1 \\ s+3 \\ 2s-1 \end{bmatrix}$, with $s = \frac{-1}{2}$, is a directional vector for L. This vector is $\begin{bmatrix} \frac{-3}{2} \\ \frac{5}{2} \\ -2 \end{bmatrix}$, and then also $\begin{bmatrix} -3 \\ 5 \\ -4 \end{bmatrix}$ is a directional vector L.

The line L is $\begin{bmatrix} -3t+1 \\ 5t+1 \\ -4t+1 \end{bmatrix}$, where t is a parameter.

Answer.

8. Let $V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n$ be a collection of subspaces in $\mathbb{R}^n$ where V_k has dimension k for each k, and where V_{k-1} is a subspace in V_k for each $k \geq 2$. Such a collection is called a *flag*. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map that *stabilizes* the flag. That means that for each $k = 1, 2, \ldots, n$ and for each vector $v \in \mathbb{R}^n$ the implication $v \in V_k \Rightarrow T(v) \in V_k$ holds. Let $v_1, v_2, \ldots, v_n$ be vectors in $\mathbb{R}^n$ such that $\operatorname{span}\{v_1, v_2 \ldots, v_k\} = V_k$ for each k. Show that the matrix representing T with respect to the basis $v_1, v_2 \ldots, v_n$ is upper triangular.
(4 p)

Solution. Let B denote the matrix representation of T with respect to the basis $v_1, v_2, \ldots, v_n$. For each $k = 1, 2, \ldots, n$ we have that v_k belongs to V_k, as $\operatorname{span}\{v_1, v_2, \ldots, v_k\} = V_k$. As T stabilizes the flag, we get that even $T(v_k)$ belongs to V_k. So $T(v_k)$ can be written as a linear combination $b_{1,k}v_1 + b_{2,k}v_2 + \cdots + b_{k,k}v_k$ where $b_{1,k}, b_{2,k}, \ldots, b_{k,k}$ are real numbers. With respect to the basis $v_1, v_2, \ldots, v_n$ we have that v_k and $T(v_k)$ have the coordinate vectors

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{respectively} \quad \begin{bmatrix} b_{1,k} \\ b_{2,k} \\ \vdots \\ b_{k,k} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where the singleton 1 occurs on row k. We conclude from this that

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ 0 & b_{2,2} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n,n} \end{pmatrix}$$

, which is upper triangular.

9. The matrix
$$A = \frac{1}{10}\begin{bmatrix} 3 & 1 & 4 \\ 2 & 8 & 0 \\ 5 & 1 & 6 \end{bmatrix}$$
has eigenvectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ with corresponding eigenvalues
$$\lambda_1 = 1, \quad \lambda_2 = \frac{7+\sqrt{57}}{20}, \quad \text{och} \quad \lambda_3 = \frac{7-\sqrt{57}}{20}.$$
Let $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a vector with positive coefficients $a \geq 0, b \geq 0$ and $c \geq 0$ such that $a+b+c=1$. Determine the point A^nX, when $n \to \infty$. **(4 p)**

Solution. We have three different eigenvalues, and know thereby that the eigenvectors $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ form a basis for $\mathbb{R}^3$. Write X as a linear combination $X = \alpha_1\vec{x}_1 + \alpha_2\vec{x}_2 + \alpha_3\vec{x}_3$. From the linearity of A^n we obtain that
$$\begin{aligned} A^nX &= \alpha_1 A^n\vec{x}_1 + \alpha_2 A^n\vec{x}_2 + \alpha_3 A^n\vec{x}_3 \\ &= \alpha_1\lambda_1^n\vec{x}_1 + \alpha_2\lambda_2^n\vec{x}_2 + \alpha_3\lambda_3^n\vec{x}_3. \end{aligned}$$
We have that $|7+\sqrt{57}| < 20$ and $|7-\sqrt{57}| < 20$. This means that λ_2 and λ_3 have absolute values strictly less than 1, and when $n \to \infty$ then $\lambda_2^n \to 0$ and $\lambda_3^n \to 0$. The implication is that $A^nX \to \alpha_1\vec{x}_1$ when $n \to \infty$. We need now to determine the line spanned by the eigenvector $\vec{x}_1$. The eigenvector $\vec{x}_1$ is one, non-trivial, solution to the homogeneous system given by the matrix
$$A - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -7 & 1 & 4 \\ 2 & -2 & 0 \\ 5 & 1 & -4 \end{bmatrix}.$$
Gauss-Jordan elimination gives
$$\begin{bmatrix} -7 & 1 & 4 \\ 2 & -2 & 0 \\ 5 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ -7 & 1 & 4 \\ 5 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & -6 & 4 \\ 0 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$
In other words, the eigenspace corresponding to the eigenvalue $\lambda_1 = 1$ are the vectors $\begin{bmatrix} \frac{2}{3}t \\ \frac{2}{3}t \\ t \end{bmatrix}$ where t is a parameter.

We then note that every column in the matrix A sum up to one. That implies that if we have a vector $X = \begin{bmatrix} a & b & c \end{bmatrix}^T$ such that $a+b+c=1$, then also the coefficients of AX sum up to one;
$$AX = \frac{1}{10}\begin{bmatrix} 3a+b+4c \\ 2a+8b \\ 5a+b+6c \end{bmatrix},$$

and we have that

$$\frac{1}{10}(3a+b+4c+2a+8b+5a+b+6c)=\frac{1}{10}10(a+b+c)=1.$$

In other words the matrix A maps the plane $x+y+z=1$ onto itself. We use now these two properties: One is that A^nX converges towards the line $\mathrm{Span}(\vec{x}_1)$, and the other property is that the coefficients of A^nX are positive and sum up to one.

A point on the line $\mathrm{Span}(\vec{x}_1)$ is of the form $\begin{bmatrix}\frac{2}{3}t\\ \frac{2}{3}t\\ t\end{bmatrix}$, and the condition that the coefficients are positive and sum up to one gives that $t=\frac{3}{7}$. We therefore get that the vector A^nX will converge towards

$$\begin{bmatrix}\frac{2}{7}\\ \frac{2}{7}\\ \frac{3}{7}\end{bmatrix}.$$