Department of Mathematics

SF1625
Calculus 1
Year 2015/2016

Module 2
Differentiation, linear approximation, the mean value theorem

This module is about chapter 2 in Calculus by Adams och Essex. There will be 3 lectures, 2 tutorials and 1 seminar.

Important concepts. The most important concept is that of **the derivative**. Its meaning is stated in a precise definition. Most often though you do not use the definition to compute derivatives, but rather the **rules of differentiation**: the product rule, the quotient rule, the chain rule.

If you know the derivatives of some basic functions and the rules differentiation, you can differentiate a large class of functions. You have to practice this a lot, so you get really good at it.

It is important to understand the interpretation of the derivative as a measure of the rate of change of the function at a point. You can use this to approximate the function at points nearby. This is called **linear approximation** and is very useful. An example of this is the **tangent**.

To prove how you can use derivatives to study where a function increases and decreases you need **the mean value theorem.** More about this will follow in modules 3 and 4.

Observe that **not all functions are differentiable** in the sense of the condition in the definition. There are continuous functions that are not differentiable. Be sure to know examples of this. An important theorem says that differentiable functions are continuous.

Recommended exercises from Calculus. Ch 2.1: 5, 7. Ch 2.2: 1, 3, 11, 26, 27, 40, 41, 42, 43, 44, 45, 47. Ch 2.3: 1, 7, 11, 17, 25, 33, 35, 47. Ch 2.4: 3, 5, 11, 18, 23, 30, 31, 37. Ch 2.5: 13, 15, 23, 29, 31, 35, 45, 62. Ch 2.6: 3, 9. Ch 2.7: 1, 3, 11, 13, 23, 29. Ch 2.8: 5, 13, 21, 27. Ch 2.9: 3, 9, 13. Ch 2.9: 3, 9, 13. Ch 2.11: 5, 7, 13, 16, 17, 18, 19.

CAN YOU SOLVE THESE EXERCISES?

Exercise 1. Let $f(x) = \dfrac{\sin x}{1 + \cos x}$.

A. Find the domain of definition of f.
B. At what points ís f continuous?
C. Compute $f'(x)$.
D. At what points ís f differentiable?

Exercise 2. Let $g(x) = x \cos^2 x$.

A. Find the domain of definition of g.
B. At what points ís g continuous?
C. Compute $g'(x)$.
D. At what points ís g differentiable?

Exercise 3. Låt $h(t) = |1 + t| + (1 + 3t^2)^{19}$.

A. Find the domain of definition of h.
B. At what points ís h continuous?
C. Compute $h'(t)$.
D. At what points ís h differentiable?

Exercise 4. Differentiate with respect to x and state where these functions are differentiable.

A. $\dfrac{ax + b}{cx + d}$.
B. $\sqrt{1 - x}$.
C. $\sqrt{1 + x^2}$.
D. $|\sin x|$
E. $\cos(\sin x^2)$

Exercise 5. Find equations for the tangent and the normal at $(2, 3)$ tto the curve $y = x^3 - x - 3$.

Exercise 6. Find an equation for the tangent at $(4, 2)$ to the curve $y = \sqrt{x}$. Can you find an approximate value of $\sqrt{4.2}$?

Exercise 7. Find an equation of the tangent to $y = \tan x$ at the point with x-coordinate $-\pi/6$.

Exercise 8. On what intervals is $f(x) = x^3 - 3x^2 + 1$ increasing? Decreasing?

Exercise 9. On what intervals is $f(x) = x - \tan x$ increasing? Decreasing?

Exercise 10. The function s is given by

$$s(x) = \begin{cases} x^2 + 1, & x < 0 \\ x + 2, & 0 \leq x < 2 \\ x^2, & x \geq 2 \end{cases}$$

A.At what points is s differentiable?

B. Compute if possible $s'(1)$ and find an equation for the tangent at the point on the graph where $x = 1$.

Exercise 11. Find an equation for the tangent to $x^3 + y^3 + y + x = 0$ at $(-1, 1)$. Hint: implicit differentiation.

Exercise 12. Let $f(x) = \cos x^2$. Compute $f^{(n)}(x)$ for $n = 1, 2, 3$.

FACIT OCH LÖSNINGSTIPS

1. A. Alla $x \neq (2n + 1)\pi$, n godtyckligt heltal. Dvs $x \neq \pm\pi,\ \pm 3\pi,\ \pm 5\pi, \ldots$

1. B. Samma som ovan.

1. C. $f'(x) = \dfrac{\cos x(1 + \cos x) - \sin^2 x}{(1 + \cos x)^2}$ är varken udda eller jämn.

1. D. Samma som A och B.

2. A. Alla x

2. B. Samma svar som i 2A.

2. C. $g'(x) = \cos^2 x - 2x \cos x \sin x$

2. D. Samma svar som A och B

3. A. Definitionsmängden är alla reella tal t.

3. B. Funktionen är kontinuerlig överallt.

3. C. För $t > -1$ är $h'(t) = 1 + 114(1 + 3t^2)^{18}$. För $t < -1$ är $h'(t) = -1 + 114(1 + 3t^2)^{18}$

3. D. Funktionen är deriverbar överallt utom i punkten $t = -1$.

4. A. $\dfrac{ad - bc}{(cx + d)^2}$ definierat för $x \neq -d/c$

4. B. $-\dfrac{1}{2\sqrt{1 - x}}$ definierat för $x < 1$

4. C. $\dfrac{x}{\sqrt{1 + x^2}}$ definierat för alla x

4. D. $\cos x$ om $2n\pi < x < (2n + 1)\pi$ och $-\cos x$ om $(2n + 1)\pi < x < (2n + 2)\pi$, n godtyckligt heltal. Definierat för alla $x \neq n\pi$, n heltal.

4. E. $(\sin(\sin x^2)) \cdot (\cos x^2) \cdot 2x$, definierat för alla x

5. Tangent: $y-3=11(x-2)$. Normal: $y-3=-\dfrac{1}{11}\,(x-2)$

6. $y-2=\dfrac{1}{4}\,(x-4)$. Vi får $\sqrt{4.2}\approx 2.05$

7. $y+\dfrac{1}{\sqrt{3}}=\dfrac{4}{3}\left(x+\dfrac{\pi}{6}\right)$

8. Strängt växande på intervallen $x\geq 2$ och $x\leq 0$. Strängt avtagande på intervallet $0\leq x\leq 2$.

9. Funktionen är strängt avtagande på intervallen $\dfrac{(2n+1)\pi}{2}<x<\dfrac{(2n+3)\pi}{2}$ för godtyckligt heltal n. Funktionen är inte växande på något intervall. (Tips: $\tan x$ är inte definierat när $\cos x=0$).

10. A. Funktionen är deriverar överallt utom i punkterna $x=0$ och $x=2$.

10. B. $s'(1)=1$ och den sökta tangentens ekvation är $y=x+2$.

11. $y-1=\dfrac{1}{2}(x+1)$ (Tips: implicit derivering)

12. $f'(x)=-2x\sin x^2$,

$f''(x)=-2\sin x^2-4x^2\cos x^2$,

$f'''(x)=-12x\cos x^2+8x^3\sin x^2$