Reglerteknik Allmän Kurs

Del 2

Lösningar till Exempelsamling

Läsår 2015/16

Reglerteknik AK

Hints

Answers

Solutions

Dictionary

AUTOMATIC CONTROL
COMMUNICATION SYSTEMS
LINKÖPINGS UNIVERSITET

Hints

This version: August 2013

2 Dynamic Systems

2.1 Start with $J\ddot{\theta} = -f\dot{\theta} + M$ and try to write M as a function of θ and u using Kirchhoff's voltage law.

2.2 What is the relationship between the response of the system and the pole locations?

2.3 Separate the pure delay and the dynamic response. Use the final value theorem to find the steady state gain and calculate the time constant by estimating the time to reach 63% of the final value (neglecting the time delay).

2.4 Identify the coefficients ω_0 and ζ in the system description

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

2.9 See Glad&Ljung.

2.11 a) Consider what you can control, what is uncontrollable and what is desired.

b) Consider the relationship between the signals.

2.12 a) Use mass balance and assume that the densities are equal.

b) Consider the change in mass and change in component A.

c) Assume that all the other independent variables ($q_1,\ q_2,\ c_{\mathrm{A},2}$) are constant.

2.13 a) Use mass and component balance.

3 Feedback Systems

3.1 a) Consider the three blocks; tank, valve, and PID. What is the input and the output from each block? Connect the blocks and consider v as a disturbance.

For the tank model use the fact that the flow into the tank is $x - v$ and the amount of liquid changes as $\dot{h} \cdot A$.

b) Consider the final value and the time constant.

d) Put $F(s) = K$ and express the closed loop poles as a function of K.

e) Use the final value theorem.

f) Put $F(s) = \frac{K_P s + K_I}{s}$ in the expression for the error, and use the final value theorem.

3.2 a) Use the expression for the poles from Problem **??**.

b) Put $F(s) = K_P + K_D s$ in the expression for the closed loop system from Problem **??**. The relative damping is defined in Glad&Ljung.

3.3 Use Newton's force equation $F = ma$ to derive the transfer function for the astronaut.

3.4 Start with deriving an expression for the transfer function from the disturbance f_c to the error e.

a) Use $F(s) = K$ and the final value theorem.

b) Use $F(s) = K_1 + K_2/s$ and the final value theorem.

3.6 a) The characteristic equation is

$$s(s+1)(s+3) + K(s+2) = 0$$

which gives $P(s) = s(s+1)(s+2)$ and $Q(s) = s + 2$.

b) Characteristic equation:

$$s(s^2 + 2s + 2) + K = 0$$

$P(s) = s(s^2 + 2s + 2)$, $Q(s) = 1$.

c) Characteristic equation:

$$s(s-1)(s+6) + K(s+1) = 0$$

$P(s) = s(s-1)(s+6)$, $Q(s) = s + 1$.

3.7 Derive the general closed loop transfer function by first deriving the transfer function for the inner loop.

a) Let $\alpha = 0$. The characteristic equation is then

$$s(s+2) + 4K = 0$$

Compute the poles explicitly as a function of K.

b) The characteristic equation is

$$s(s+2) + 4K(1+s) = 0$$

c) Characteristic equation:

$$s(s+2) + 4K(1+s/3) = 0$$

d) Characteristic equation:

$$s^2 + 2s + 4 + 4\alpha s = 0$$

3.8 a) Derive the transfer function from ω_{ref} to ω.

b) The characteristic equation is

$$(s+10)(s+4)(s-3) + 10K(s+1) = 0$$

3.9 a) Derive the closed loop transfer function by first deriving the transfer function for the inner loop. The characteristic function is

$$s((s+1)(s+10) + K_1) + K_2 = 0$$

We get two principally different root loci when there are complex starting points, and when all starting points are equal. Treat the cases separately.

3.10

a) The characteristic equation is

$$(s+1)(s-1)(s+5)+K=0$$

b) Characteristic equation:

$$(s+1)(s-1)(s+5)+K(1+0.5s)=0$$

3.11

a) Characteristic equation:

$$s^3+2s^2+a(s^2+2s+6)=0$$

b) First check for which a the system is stable and the steady state requirement is fulfilled. Then use that “sinusoid in” gives “sinusoid out” after transients.

3.12 Check the root locus to find which K-values gives a stable/unstable system, more/less oscillative system.

3.13 Investigate the starting points and end points of the root locus.

3.14 Find the open loop transfer function and use the Nyquist criterion.

3.15 Since $G(s)$ has no poles in the RHP, the closed loop system is stable if the Nyquist path of KG_o does not encircle -1. Note that the K will only modify the distance to the origin, not the shape of the curve.

3.16 Study the amplitude and phase of $G(i\omega)$.

3.17

a) Draw the complete Nyquist path and use the Nyquist criterion. (Note that $G_o(-i\omega)$ is the mirror image of $G_o(i\omega)$, mirrored in the real axis.)

b) Use the final value theorem, and that $G_o(0)$ is known from the Nyquist path.

c) Apply the Nyquist criterion to $\frac{K}{s}G(s)$

3.18 The system oscillates when the open-loop gain is equal to -1 (check $Ke^{-i\omega T}G(i\omega)$).

3.19 Try to find the ω that gives $\arg F(i\omega)G(i\omega)=-180°$.

3.20 The Nyquist curve for small ω determines if the system may have an integrator or not. Also check if the system is unstable for some K (from the Nyquist diagram).

3.25 Check the steady state error, the relative damping, etc.

3.26

a) To compute the closed loop transfer function combine

$$\theta(s)=G(s)U(s)$$

and

$$U(s)=F(s)(\theta_{\text{ref}}(s)-\theta(s))$$

b) The control error can be computed using

$$E(s)=\frac{1}{1+F(s)G(s)}\theta_{\text{ref}}(s)$$

To find the steady-state error, use the final value theorem.

c) See b).

3.31

a) Consider what you can control, what is uncontrollable and what is desired.

b) Use mass and energy balance for both the tank and the heating system.

3.32

a) Compute the poles of the system.

b) Note that the system has negative sign in the numerator.

3.33 Check the pole for the closed loop system.

4 Frequency Description

4.1 Determine the angular frequency ω of the signals using the figure. Use the relationship saying that when $u(t) = A\sin\omega t$ the output becomes

$$y(t) = |G(\mathrm{i}\omega)|\, A\sin(\omega t + \arg G(\mathrm{i}\omega))$$

to determine $|G(\mathrm{i}\omega)|$ and $\arg G(\mathrm{i}\omega)$.

4.2 a) For $K = 0.5$ the open loop system is given by

$$G_{\mathrm{o}}(s) = F(s)G_{\mathrm{r}}(s)G_{\mathrm{s}}(s) = \frac{0.05(1+s/0.02)}{s(1+s/0.01)(1+s/0.05)(1+s/0.1)}$$

Use the rules in Glad&Ljung to make the Bode plot.

b) What can be said about the phase and gain margin when the output of the closed loop system oscillates with constant amplitude?

c) When the reference signal is $A\sin\alpha t$ the output signal becomes

$$y(t) = |G_{\mathrm{c}}(\mathrm{i}\omega)|\, A\sin(\alpha t + \arg G_{\mathrm{c}}(\mathrm{i}\omega))$$

The Bode plot of the open loop system can be used to compute $G_{\mathrm{c}}(\mathrm{i}\omega)$.

4.3 a) Check the behavior of $G(\mathrm{i}\omega)$ when $\omega \to 0$ and $\omega \to \infty$ respectively. See also if the absolute value and the argument decrease monotoneoulsy or not.

b) Translate the behavior of the amplitude and phase curves to a pole-zero diagram.

4.4 Check the final values of $y(t)$ against the static gain $G(0)$. Check also the overshoots of $y(t)$ against the height of the resonance peaks in $G(\mathrm{i}\omega)$. Check the frequency of the oscillation in $y(t)$ against the resonance frequency in $G(\mathrm{i}\omega)$.

4.5 Use MATLAB, in particular the command `bode`.

4.6 Recall that for stable, linear systems "a sinusoid in gives a sinusoid out" after initial transients.

4.7 Recall that for stable, linear systems "a sinusoid in gives a sinusoid out" after initial transients.

4.11 a) Use the rules in Glad&Ljung to make the Bode plot.

b) What can be said of the phase and gain margin when the output of the closed loop system oscillates with constant amplitude?

4.12 a) What is the stability criterion in the Bode plot?

b) What is the current phase margin? Is a lead really necessary?

5 Compensation

5.1 Try a lead-lag compensator. A table of phase advance versus "the N-parameter" is found in Glad&Ljung.

5.2 a) Glad&Ljung gives a good description of how Bode plots can be drawn by hand.

b) A proportional controller does not affect the phase curve.

c) Try lead compensator.

5.3 a) Draw asymptotic Bode plot (see 5.1) by hand or use MATLAB.

b) Start with calculating the controller and then use the final value theorem.

c) Try a lag compensator.

5.4 Start with drawing a Bode plot for the open loop transfer function. The final value theorem is a good tool in this exercise.

5.5 Check for signs of dominating poles, pure integrations, resonance frequencies...

5.6 Draw asymptotic Bode plot using the guidelines in Glad&Ljung. See the discussion on lead-lag compensators in Glad&Ljung.

5.7 Use values of $|G(i\omega)|$ and $\arg G(i\omega)$ to plot the Nyquist curve $G(s)$.

5.8 The time delay alters the phase curve but not the amplitude curve. Use the Nyquist stability criterion.

5.9 Check steady state level and rise time. Modify Figure **??** using $G_A(s)$ and adapt a lead-lag compensator. You can use two lead compensators to acchieve a big phase advance.

5.10 Start by adjusting Figure **??** to obtain the Bode plot of G.

5.11 b) It is possible to derive limits on K using either the Bode plot or the Nyquist curve.

c) Use the final value theorem.

d) A time delay is described by the transfer function e^{-sT}.

5.12 Think of all possible phase curves, for example originating from time delays, and think about the corresponding Nyquist curves or Bode plots.

5.13 See "Introduktion till CSTB" and previous exercises in this section.

5.14 Try a lead-lag compensator.

6 Sensitivity and Robustness

6.1 The sensitivity function is the transfer function from v to y.

6.2 Derive the relative model error

$$G_\Delta(s) = \frac{G^0(s) - G(s)}{G(s)}$$

Make a simple plot of $G_c(i\omega)$ using the information in the problem formulation. Compare with the inverse of the relative model error.

6.3 Convert the condition that the amplitude of y is larger than the amplitude of v to the condition

$$|1 + G_o(i\omega)| < 1$$

What does this inequality say about the distance between the Nyquist curve and the origin?

6.4 Compute the transfer function of the closed loop system. Apply the robustness criterion using the given upper bound of the relative model error.

6.5 a) Derive the relative model order

$$G_\Delta(s) = \frac{G^0(s) - G(s)}{G(s)}$$

and plot $1/|G_\Delta(i\omega)|$.

b) Determine the level that $|G_c(i\omega)|$ cannot exceed.

6.6 a) Use the robustness criterion and check the condition for $\tilde{G}(i\omega)$ when $\omega \to \infty$.

b) The characteristic equation of the closed loop system is

$$s^2(2 + 25\alpha) + 5s(2 + 25\alpha) + 25 = 0$$

6.7 a) The characteristic equation becomes

$$s^2(s + 1) + \alpha(s^2 + s + 4) = 0$$

b) Derive the relative model error

$$G_\Delta(s) = \frac{G^0(s) - G(s)}{G(s)}$$

Check where the absolute value of the inverse of the relative model error intersects $|G_c(i\omega)|$ given in the figure. It is sufficient to check the low frequency asymptote.

c) What can be said about the necessity and sufficiency of the stability conditions in a) and b)?

6.8 Check where the absolute value of the relative model error intersects $|G_c(i\omega)|$ given in the figure.

6.9 Derive the closed loop equation relating $y(t)$, $r(t)$, $v(t)$, and $n(t)$ using $Y(s) = V(s) + G_o(s)(R(s) - N(s) - Y(s))$. Then use the fact that the sensitivity function $S(s)$ and the complementary sensitivity function $T(s)$ are related as $S(s) + T(s) = 1$. (Here $T(s)$ coincides with the closed loop system.)

6.10 a) The relative model error is given by

$$G_\Delta(s) = \frac{G^0(s) - G(s)}{G(s)}$$

b) Use MATLAB and results from previous exercises.

6.11 Recall that for stable, linear systems "a sinusoid in gives a sinusoid out" after initial transients.

6.12 Create the loop gain transfer function and use its Bode plot to check stability.

6.13 The sensitivity function is the transfer function from v to y.

7 Special Controller Structures

7.1 a) Derive the transfer function from w to θ_m which then implies the open loop transfer function

$$\Theta(s) = \frac{0.9}{(1+s/0.033)(1+s/0.33)(1+s)} W(s)$$

Draw the Bode plot using the rules from Glad&Ljung.

b) Draw the Bode plot using the rules from Glad&Ljung.

7.2 a) Use the relationship

$$H(s) = \frac{1}{As}\left(\frac{1}{1+s/2}U(s) - V(s)\right)$$

b) Derive the transfer function from V to H when both feedforward and feedback are used.

7.3 a) Use $Y(s) = (G_\mathrm{u}(s)F_\mathrm{f}(s) + G_\mathrm{v}(s))\,V(s)$.

b) Recall that for stable, linear systems “a sinusoid in gives a sinusoid out” after initial transients.

8 State Space Description

8.1 Define $x_1 = \theta$, $x_2 = \dot{\theta}$, and utilize the differential equation for the motor.

8.2 For the nonlinear equation $\dot{x}_2 = f_2(x_1, x_2, u)$, the linearized equation is given by

$$\begin{aligned}\dot{x}_2 = {} & f_2(x_{1,0}, x_{2,0}, u_0) \\ & + \frac{\partial f_2}{\partial x_1}(x_{1,0}, x_{2,0}, u_0) \cdot (x_1 - x_{1,0}) \\ & + \frac{\partial f_2}{\partial x_2}(x_{1,0}, x_{2,0}, u_0) \cdot (x_2 - x_{2,0}) \\ & + \frac{\partial f_2}{\partial u}(x_{1,0}, x_{2,0}, u_0) \cdot (u - u_0)\end{aligned}$$

8.3 Define $x_1 = y$, $x_2 = \theta$, and $x_3 = z$. Use block diagram algebra to find expressions for $s \cdot X_i(s)$, then use the inverse Laplace transform.

8.4 Use canonical forms.

8.5 Take the Laplace transform of $g(t)$.

8.6 $G(s) = C(sI - A)^{-1}B$.

8.7 $x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau$

8.8 a) Insert the control signals and take Laplace transforms. Use the final value theorem.

b) Examine the difference $u_1(t) - u_2(t)$ for arbitrarily small constant ϵ.

8.9 Check controllability.

8.10 The controllable subspace is spanned by the linearly independent columns of $\mathcal{S}$. The unobservable subspace is spanned by the null space of $\mathcal{O}$.

8.11 b) Compare what happens to the states as $t \to \infty$, to the transfer function poles.

c) Check if $\det \mathcal{S}$ and $\det \mathcal{O}$ are nonzero.

8.12 The system is minimal, compute the transfer function.

8.13 a) For small deviations around 0, $\sin(\phi) \approx \phi$, $\cos(\phi) \approx 1$. Take $\ddot{z}$ as input.

b) $\det \mathcal{S} = \frac{1}{\alpha}(1 - \frac{1}{\alpha})^2$

8.15 a) Combine mass balance with the given equation for reaction speed.

9 State Feedback

9.1 a) The closed loop system $\dot{x} = Ax + Bu$, $y = Cx$, $u = -Lx + y_{\text{ref}}$ has characteristic polynomial $\det(sI - A + BL) = 0$.

b) The observer poles are given by $\det(sI - A + KC) = 0$ and should be placed to the left of the closed loop poles.

9.2 a) Write the system in state space form by introducing three state variables corresponding to the outputs of the three left-most integrators in the figure ($\dot{z}$ = output). Design a state feedback controller $u = -Lx + y_{\text{ref}}$ and place the poles in -0.5.

c) Design an observer with poles to the left of the closed loop poles.

9.3 a) The constant l_0 can be found by using that $\dot{\theta} = \dot{\omega} = 0$ at steady state.

b) Introduce the integrated control error as an auxillary state.

9.4 Decompose the system into two subsystems, one controlled by u_1 and one by u_2, and check the controllability.

9.5 Is the system observable?

9.6 Is the system observable?

9.7
- Is the system controllable?
- $\tilde{x}(t)$ converges to zero as $p(t)e^{-\alpha t}$ if $(A - KC)$ has a double eigenvalue in $-\alpha$.
- Check the observability of the system.

9.9 Study the phase margin of the open loop system.

9.11 Use the initial value theorem.

9.12 Compute the transfer function from $u(t)$ to $z(t) = L\hat{x}(t)$, that is, the loop gain, and check the stability margin given a certain time delay T.

9.13
- The closed loop poles are given by $det(sI - A + BL) = 0$

9.17 a) The closed loop system $\dot{x} = Ax + Bu$, $u = -Lx$ has the characteristic polynomial $\det(sI - A + BL) = 0$.

b) Check oberservability.

c) Introduce a new state.

Answers

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1 Mathematics

1.1 a) A step has Laplace transform $\frac{A}{s}$.

b) A ramp has Laplace transform $\frac{A}{s^2}$.

c) $\frac{1}{s+2}$

d) $\frac{s}{s^2+25}$

e) $sU(s) - u(0)$

f) $sU(s)$. ($u(0) = 0$ is a common assumption in the course.)

g) $s^2U(s) - su(0) - \dot{u}(0)$

h) $s^2U(s)$. ($u(0) = \dot{u}(0) = 0$ is a common assumption in the course.)

i) A time delayed signal has Laplace transform, $e^{-sT}U(s)$.

1.2 a) $\lim_{t\to\infty} y(t) = 5/2$

b) $Y(s) = \frac{1}{s+2}U(s)$

1.3 The general solution is given by

$$y(t) = C_1e^{-2t} + (C_2 + C_3t)e^{-t} - \frac{3}{100}(\cos(2t) + 7\sin(2t))$$

1.4 a) $y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}, \quad t \geq 0$

b) $y(t) = 1 - 0.5e^{-t} + 0.5\sin t - 0.5\cos t$

1.5 a) $\sqrt{2}e^{\mathrm{i}\frac{\pi}{4}}$

b) $\frac{\sqrt{2}}{10}e^{-\mathrm{i}\frac{105}{180}\pi}$

c) $1 + \sqrt{3}\mathrm{i}$

d) -5

1.6

deciBel (dB_{20})	Definition		Amplification F
20	$20\log F = 20$	$\Rightarrow$	$F = 10^1 = 10$
-3	$20\log F = -3$	$\Rightarrow$	$F = 10^{-3/20} \approx 0.708 \approx \frac{1}{\sqrt{2}}$
0	$20\log F = 0$	$\Rightarrow$	$F = 10^0 = 1$
10	$20\log F = 10$	$\Rightarrow$	$F = 10^{0.5} = \sqrt{10} \approx 3.16$
-10	$20\log F = -10$	$\Rightarrow$	$F = 10^{-0.5} = \frac{1}{\sqrt{10}} \approx 0.316$

1.7 Multiplication of the two matrices gives the unit matrix.

1.8

$$\lambda_1 = 3 \qquad v_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\lambda_2 = -1 \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\lambda_3 = 4 \qquad v_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

1.9

$$T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

1.10 A basis for the null space is for example

$$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

A basis for the range space is

$$\begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix}$$

The rank of the matrix is hence 3.

1.11 a) $f(t) = 1 - e^{-t}$; 1.

b) $f(t) = -0.5e^{-t} + 0.5e^{t}$; ∞.

c) $f(t) = e^{-t} \cdot t$; 0.

1.12 $y^{(3)} + 2\ddot{y} + 2\dot{y} + y = u$

2 Dynamic Systems

2.1 a) Differential equation

$$\ddot{\theta} + \frac{1}{\tau} \cdot \dot{\theta} = k_0 \cdot u$$

where

$$\frac{1}{\tau} = \frac{R_a f + k_a k_v}{J R_a} \qquad k_0 = \frac{k_a}{J R_a}$$

b) Transfer function

$$G(s) = \frac{\theta(s)}{U(s)} = \frac{k_0}{s(s + 1/\tau)}$$

c) Step response

$$\theta(t) = k_0 \tau t - k_0 \tau^2 (1 - e^{-t/\tau})$$

2.2 (1) $K = 0.1$

(2) $K = 2.5$

(3) $K = 3$

(4) $K = 0.5$

2.3 $G(s) = \frac{10e^{-(L/V)s}}{1+3s}$

2.4 a) $a < 1$

b) $b = 2$

2.5 A–B, B–F, C–A, D–C, E–E, F–D.

2.6

System	T_r	T_s	M	poles
G_A	3.3	4.7	0%	$-1, -1$
G_B	1.2	13.6	52%	$-0.2 \pm i0.98$
G_C	10.6	14.6	0%	$-4.8, -0.2$
G_D	1.7	5.4	16%	$-0.5 \pm i0.87$
G_E	0.8	2.6	16%	$-1 \pm i1.73$

2.7 $\alpha > 0$ gives overshoot, $\alpha < 0$ gives undershoot.

2.8 $y(t) = \mathcal{L}^{-1}(G(s)\frac{1}{s})$

2.9 a) 1.5

b) $M \approx 26\%$

c) $T_r \approx 1.5$

d) $T_s \approx 7.8$

2.10 G_1–C, G_3–B, G_4–A, G_5–D.

2.11 a) The signals can be classified as

- ⋄ Disturbances signal: Acid process flow (unknown pH and flow)
- ⋄ Control signal: NaOH solution
- ⋄ Measured and controlled signal: The pH of the outflow

b) A block diagram where the control strategy is based on feedback could look like Figure 2.11a

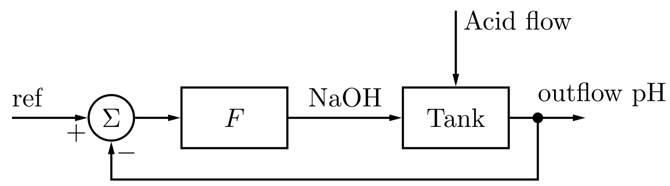

Figure 2.11a

2.12 a) $q^* = 1.5$ m^3/min and $c_A^* = 2.0$ kmol/m^3.

b) The model is nonlinear since the model is described by the following nonlinear equations

$$\frac{\mathrm{d}(\rho V)}{\mathrm{d}t} = \rho(q_{in} - q_{out})$$
$$\frac{\mathrm{d}(V c_{\mathrm{A}})}{\mathrm{d}t} = q_1 c_{\mathrm{A},1} + q_2 c_{\mathrm{A},2} - q c_{\mathrm{A}}$$

c) $k_0 = 2.0$ kmol/m^3, $k_1 = 0.13$ kmol/m^3, and $\tau = \frac{1}{1.5} = 0.67$ min.

2.13 a) The model is given by

$$y_i = \frac{\alpha x_i}{1 + (\alpha - 1)x_i}$$
$$\frac{\mathrm{d}M_i}{\mathrm{d}t} = L_{i-1} + V_{i+1} - L_i - V_i$$
$$M_i \frac{\mathrm{d}x_i}{\mathrm{d}t} = L_{i-1}\left(x_{i-1} - x_i\right) + V_{i+1}\left(y_{i+1} - x_i\right) + V_i\left(x_i - y_i\right)$$

b) The linearized model is given by

$$M_i^* \frac{\mathrm{d}x_{i\Delta}}{\mathrm{d}t} = L_{i-1}^* x_{i-1,\Delta} + V_{i+1}^* y_{i+1,\Delta} - L_i^* x_{i\Delta} - V_i^* y_{i\Delta}$$
$$+ x_{i-1}^* L_{i-1,\Delta} + y_{i+1}^* V_{i+1,\Delta} - x_1^* L_{i\Delta} - y_i^* V_{i\Delta}$$
$$y_{i\Delta} = \frac{\alpha}{\left(1 + (\alpha - 1)x_i^*\right)^2} x_{i\Delta}$$

3 Feedback Systems

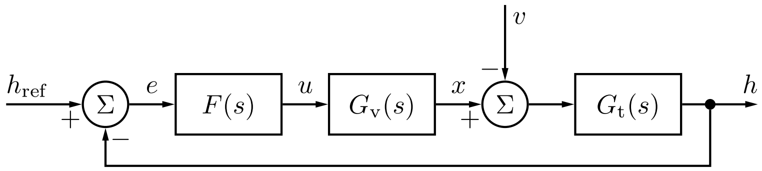

Figure 3.1a

3.1 a) Transfer function of the tank $G_t(s) = \frac{1}{s}$. Block diagram see Figure 3.1a.

b) $k_v = 2$, $T = 5$

c) $\frac{H(s)}{H_{ref}(s)} = \frac{G_t(s)G_v(s)F(s)}{1+G_t(s)G_v(s)F(s)}$, $\frac{H(s)}{V(s)} = -\frac{G_t(s)}{1+G_t(s)G_v(s)F(s)}$

d) $K < 0.05$

e) $\frac{1}{2K}$

f) 0

3.2 a) $-0.1 \pm \sqrt{0.39}i$

b) $K_D > 1.7$

3.3 $K_2 < 1$ and $K_1 = 200/K_2^2$.

3.4 a) $-a/K$

b) 0

3.5 a) For small values of K_P the step response is slow, well damped and the steady state error is large. For increasing K_P the step response becomes faster but more oscillatory, while the error is reduced. For large K_P the amplitude of the oscillations increases, that is, the closed loop system becomes unstable.

b) The integrator in the regulator eliminates the steady state error. A too small value of K_I gives a large settling time while a too large value gives an oscillatory (finally unstable) closed loop system.

c) Using the (approximate) derivative of the error in the regulator increases the damping of the closed loop system. Increasing K_D too much, however, gives that an oscillation with higher frequency appears in the step response and finally (approximately when $K_D > 65$) the closed loop system becomes unstable.

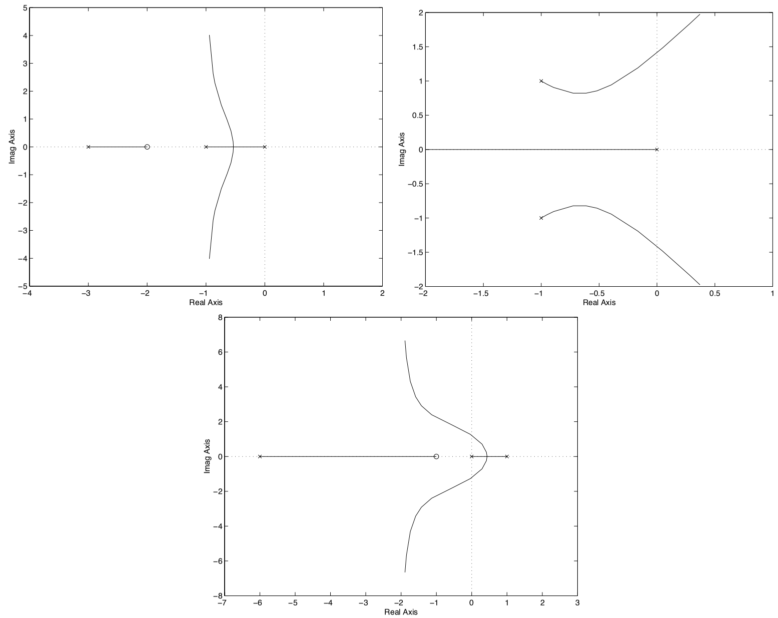

Figure 3.6a

3.6 Root loci are shown in Figure 3.6a.

b) Intersection with the imaginary axis for $K = 4$, $\omega = \pm\sqrt{2}$.

c) Intersection with the imaginary axis for $K = 7.5$, $\omega = \pm\sqrt{1.5}$.

Conclusions about the step response of the corresponding systems:

a) Asymptotically stable all $K > 0$.
Small K: No oscillations, larger K gives faster system.
Larger K: Oscillations. Larger K gives more oscillations.

b) Asymptotically stable for $0 < K < 4$. Oscillating all $K > 0$.
Small K: larger K gives faster system.
Larger K: larger K gives more oscillating system. Unstable for large K (> 4).

c) Asymptotically stable for $K > 7.5$. Unstable for $K < 7.5$. Stable and oscillating for $K > 7.5$. Larger K gives faster system, until the real pole becomes dominating, then larger K gives a slower system.

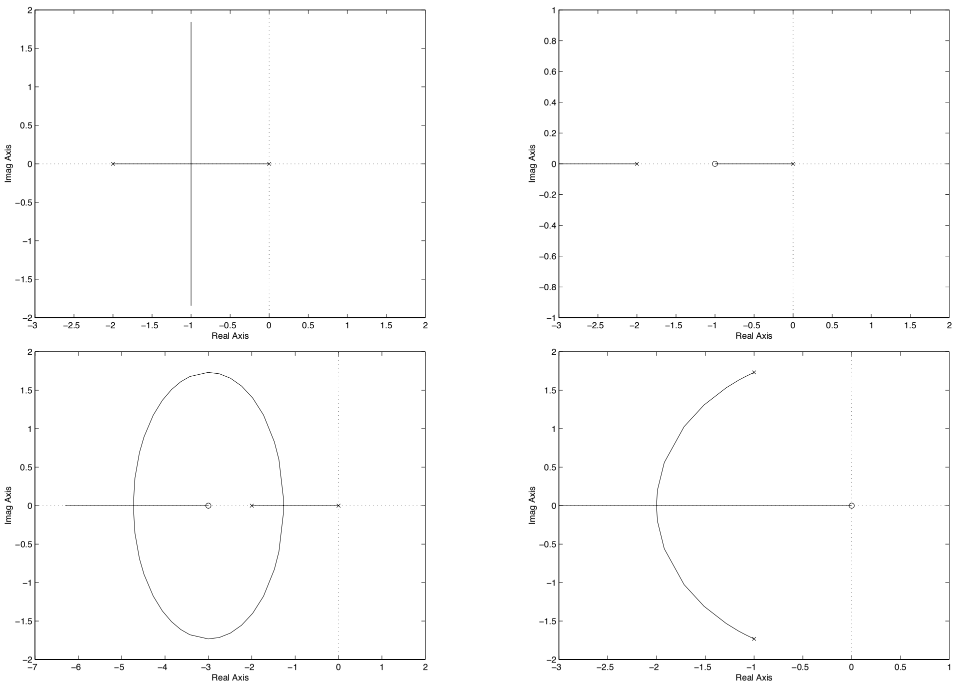

Figure 3.7a

3.7 General characteristic equation:

$$s(s+2) + 4K(1+\alpha s) = 0$$

The root loci are shown in Figure 3.7a.

a) Asymptotically stable for all $K > 0$, oscillatory for large K.

b) Asymptotically stable for all $K > 0$, not oscillatory for any K.

c) Asymptotically stable for all $K > 0$, no oscillations for small and large K, faster for large K.

d) Asymptotically stable for all $\alpha > 0$. Oscillatory for small α. Larger α gives more damped system.

With the tachometer feedback we can make the system both fast and well damped. The tachometer feedback is equivalent to the D-part in a PID controller.

3.8 a) The system is unstable so ω will grow to infinity.

b) The root locus is shown in Figure 3.8a. The system is asymptotically stable for $K > 12$.

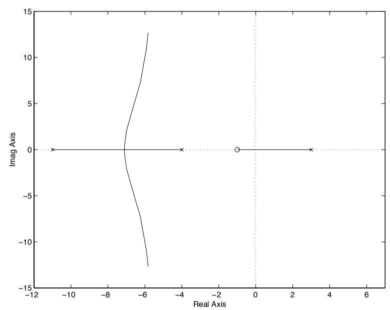

Figure 3.8a

c) No. When $K = 12$, $s = 0$ is one pole but the other two are complex.

3.9 a) Starting points: $s = -5.5 \pm \sqrt{5.5^2 - 10 - K_1}$. The starting points are all real for $K_1 \leq 20.25$, while we have complex starting points for $K_1 > 20.25$. The two principal root loci are shown in Figure 3.9a. The system is asymptotically stable for $0 < K_2 < 11K_1 + 110$.

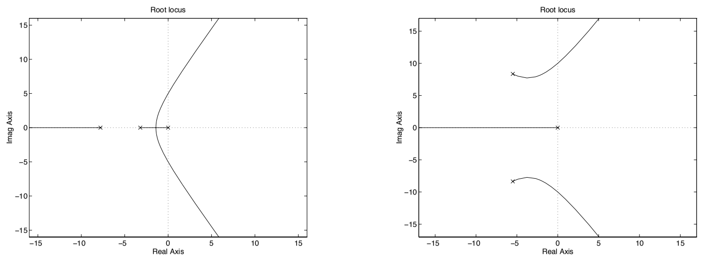

Figure 3.9a

b) A larger K_1 gives stability for larger K_2.

3.10 Root loci in Figure 3.10a.

a) The system is unstable for all K.

b) Asymptotically stable for $K > 5$.

3.11 a) The root locus is shown in Figure 3.11a. The system is asymptotically stable for $a > 1$.

b) The smallest amplitude is 0.1.

3.12

K	Step
4	C
10	D
18	B
50	A

3.13 The poles of the system all tends to points in the LHP or to $-\infty$ for large K.

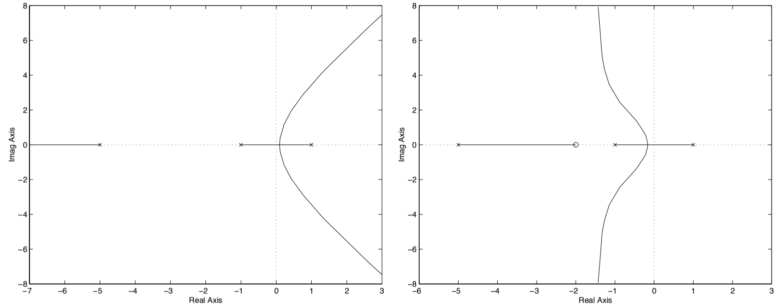

Figure 3.10a

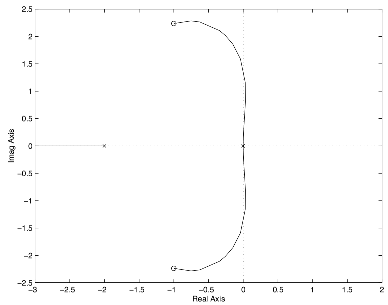

Figure 3.11a

3.14 The system is stable for $K/A < \pi$.

3.15 a) The closed loop system is stable in (i), (ii), and (iv).

b) Stable when: (i) $K < 2.5$, (ii) $K > 0$, (iii) $K < 1/2$, and (iv) $K < 1/4$ or $K > 1/2$.

3.16 The Nyquist curves are shown in Figure 3.16a.

3.17 a) $K < 2/3$

b) $\frac{1}{1+2K}$ when $K < 2/3$

c) $K < 2/3$

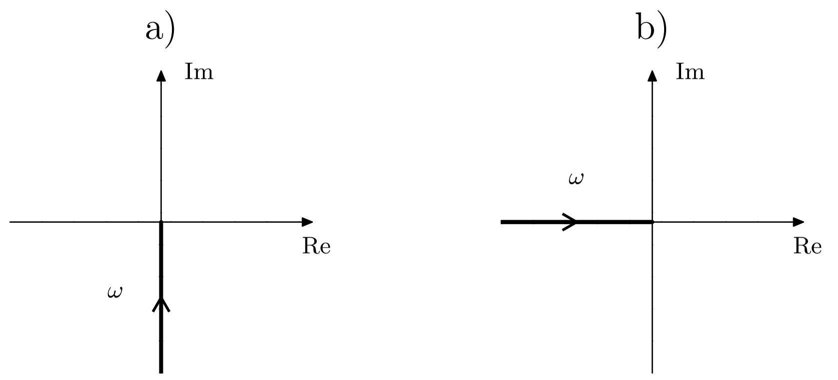

Figure 3.16a

3.18 $\tau = 1.69$
$T = \frac{\pi}{2} - \arctan \tau = 0.53$
$T_1 = \pi - 2 \arctan \frac{\tau}{2} = 1.74$

3.19 $K < 2$

3.20 Root locus 1.

3.21 P $\Rightarrow$ $b_0 = b_2 = 0$
I $\Rightarrow$ $b_0 = b_1 = 0$
D $\Rightarrow$ $b_1 = b_2 = 0$

3.22 a) The root locus with respect to K_{P} is shown in Figure **??**. When K_{P} increases the two complex poles move towards the imaginary axis, that is, the closed loop system becomes more oscillatory. Finally, for $K_{\mathrm{P}} \approx 6.2$, the poles cross the imaginary axis and the closed loop system becomes unstable. This result is in accordance with Problem **??**. For small values of K_{P} the properties of the step response are mainly determined by the real pole close to the origin. For larger values the complex poles start to dominate and when the complex poles cross the imaginary axis the amplitude of the oscillations in the step response increases and the system becomes unstable.

Note, however, that the root locus alone does not give sufficient information to tell how the steady state error changes with the parameter.

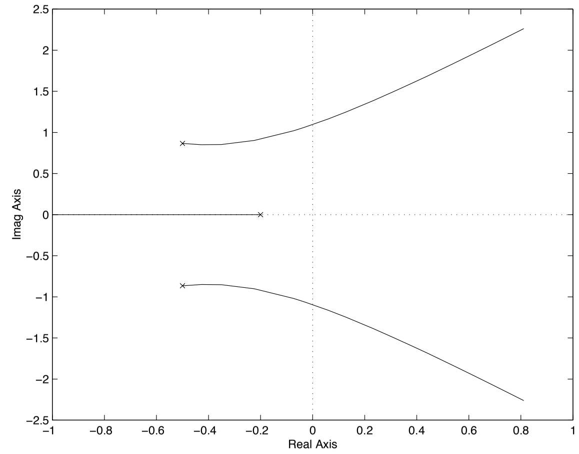

Figure 3.22a

b) The root locus with respect to K_{I} is shown in Figure **??**. For small K_{I} the response of the closed loop system is dominated by the poles on the real axis close to the origin. When K_{I} increases the poles become complex and move towards the imaginary axis, that is, the closed loop system becomes more oscillatory. Finally, for $K_{\mathrm{I}} \approx 1.5$, the poles cross the imaginary axis, that is, the closed loop system becomes unstable. As can be seen in Problem **??** a small value of K_{I}, that is, a pole close to the origin, gives a slow step response. When K_{I} increases the dominating poles become complex and the step response becomes oscillatory.

A large settling time will typically follow if the system is slow or has poor damping. Here, the large settling time for small K_{I} is due to the system being slow. That the steady state error is eliminated cannot easily be seen in the root locus.

c) The root locus with respect to K_{D} is shown in Figure **??**. When K_{D} increases the complex poles closest to the origin move towards the origin and and at the same time the damping of the poles is increased. When K_{D} increases even more the second pair of complex poles moves towards the imaginary axis giving a high frequency oscillation which finally gives instability.

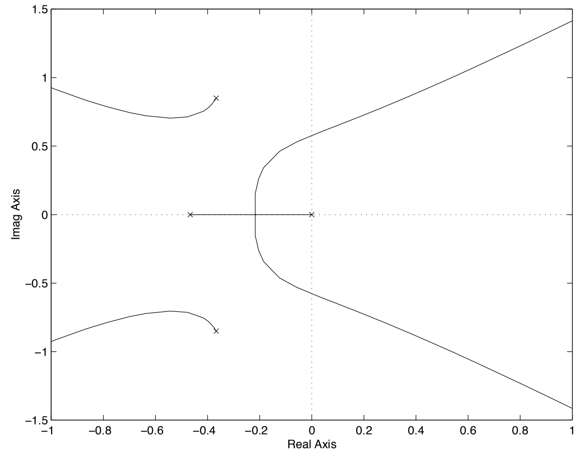

Figure 3.22b

3.23 a) The Nyquist curve is "far away" from the point −1 for all frequencies and the step response of the closed loop system is well damped. As K_P increases the Nyquist curve grows in size and for $K_P = 6.2$ the Nyquist curve reaches −1 and thus is the limit of stability.

b) For low frequencies the Nyquist curve is now far away from the origin since the integrating part makes $|G(i\omega)|$ large for low frequencies. The Nyquist curve now passes closer to −1 which results in a more oscillatory closed loop system. The system becomes unstable around $K_I = 1.44$.

c) The Nyquist curve is now further away from −1 which corresponds to an improved damping of the closed loop system. The system becomes unstable around $K_D = 66$.

3.24 a) $\omega_c = 0.38$, $\omega_p = 1.1$, $\varphi_m = 94°$ and $A_m = 3.1$.

b) The closed loop system is now much more oscillatory due to the reduced phase and gain margins.

c) $K_P = 3.1$.

3.25 A–*iii*, B–*i*, C–*iv*, D–*ii*.

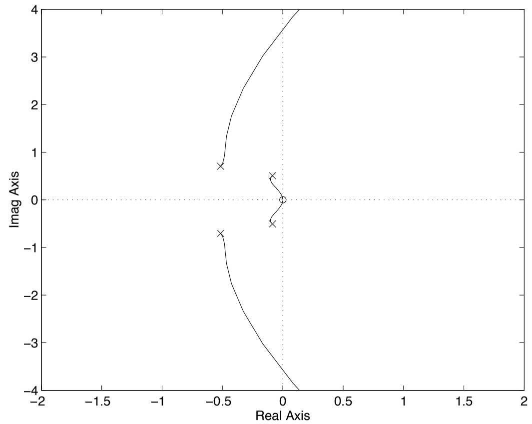

Figure 3.22c

3.26 a) K_P small ⇒ Both poles on the real axis, but one pole very close to the origin ⇒ Slow but not oscillatory system.

$K_P = 1/(4\tau^2 k_0)$ ⇒ Both poles in $-1/(2\tau)$, that is, faster than in (1) but still no oscillations.

K_P large ⇒ Complex poles with large imaginary part relative to the real part, that is, oscillatory system.

b) If the reference is a step,

$$\lim_{t\to\infty} e(t) = 0$$

If the reference is a ramp,

$$\lim_{t\to\infty} e(t) = \frac{A}{K_P k_0 \tau}$$

c) $\lim_{t\to\infty} e(t) = 0$

3.27 $G_c = \frac{G_o}{1+G_o}$

3.28 a) $G_o = FG$

b) $G_c = \frac{FG}{1+FG}$

c) $G_{ny} = -\frac{FG}{1+FG}$

d) $G_{re} = \frac{1}{1+FG}$

3.29 a) $\frac{3A}{3+K}$

b) $F(s) = \frac{1}{s}$ (for example)

c) Poles in -2, -2. No zeros.

3.30 A–4, B–2, C–3, D–1.

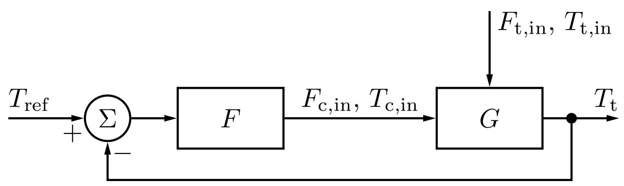

Figure 3.31a

3.31 a) See the block diagram in Figure 3.31a. There, the signals are classified as:

- ◇ Input $F_{c,in}$ and $T_{c,in}$
- ◇ Output T_t
- ◇ Disturbance $F_{t,in}$ and $T_{t,in}$

b) The model is given by

$$V_t \frac{dT_t}{dt} = F_t(T_{t,in} - T_t) + \frac{U}{c^p_{t,in}\rho_t}(T_c - T_t)$$
$$V_c \frac{dT_c}{dt} = F_c(T_{c,in} - T_c) - \frac{U}{c^p_c \rho_c}(T_c - T_t)$$

c) $T_{t\Delta}(s) = \frac{32}{(s+3.675)(s+0.185)} F_{c\Delta}(s)$

d,e) The root locus for a P controller is shown in Figure 3.31b.

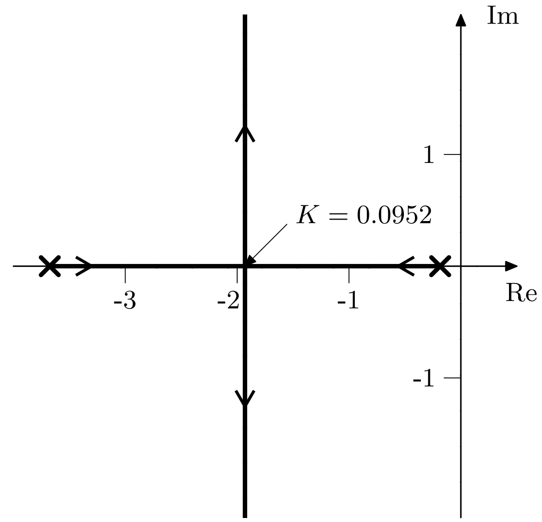

Figure 3.31b

3.32 a) The system has a pole in -3.

b) The system is stable for $K \leq -3$.

3.33 $K > \mu$

3.34

4 Frequency Description

4.1 $G(s) = \frac{0.16}{s+0.16}$

4.2 a) See figure in the solution. $\omega_c = 0.025$, $\varphi_m = 31°$, $A_m = 2.5$.

b) The period time will be 108 seconds, $K = 1.25$.

c) $B = 8°$, $\beta = 0.02$ rad/s and $\varphi = -42°$.

4.3 a) Figure 4.3a in Solutions.

b) Figure 4.3b in Solutions.

4.4 A–B, B–C, C–D, D–A.

4.5 a)

System	$G(0)$	ω_B	ω_r	M_p
G_A	1	0.64		
G_B	1	1.5	1	2.5
G_C	1	0.21		
G_D	1	1.27	0.7	1.15
G_E	1	2.54	1.4	1.15

b) The bandwidth of a system is (approximately) inversely proportional to the rise time. The damping is inversely proportional to the height of the resonance peak. A large peak implies low damping and large overshoot.

4.6 $y(t) = \frac{1}{\sqrt{5}} \sin(2t - 1/2 - 4 - \frac{\pi}{2} - \arctan 2)$.

4.7 a) $0.45 \sin(2t - 1.1)$

b) Unstable system.

c) $0.11 \sin(2t - 2.4)$

d) $0.45 \sin(2t - 2.1)$

4.8 a,b)

ω	$\lvert G(i\omega)\rvert$			$\arg G(i\omega)$		
1	1	=	0 dB_{20}	−0.2 rad	=	−11°
5	0.8	=	−1.9 dB_{20}	−0.9 rad	=	−52°
10	0.5	=	−6 dB_{20}	−1.6 rad	=	−92°
20	0.2	=	−14 dB_{20}	−2.2 rad	=	−126°

c) See Figure 4.8a in Solutions.

4.9 G_1–B, G_2–D, G_3–A, G_4–C, G_5–E.

4.10 Bode gain–step response pairs: A–D, B–C, C–A, D–B.

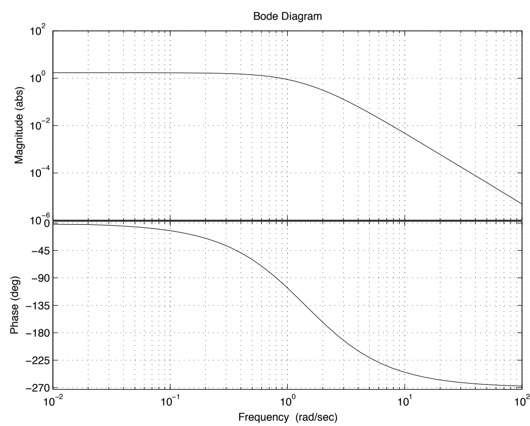

Figure 4.11a

4.11 a) The bode digram of the system is shown in Figure 4.11a.

b) $K = \frac{1}{0.1946} = 5.14$

4.12 a) $K \leq 5.04$

b) $F(s) = 1.58 \frac{s+0.01}{s}$

5 Compensation

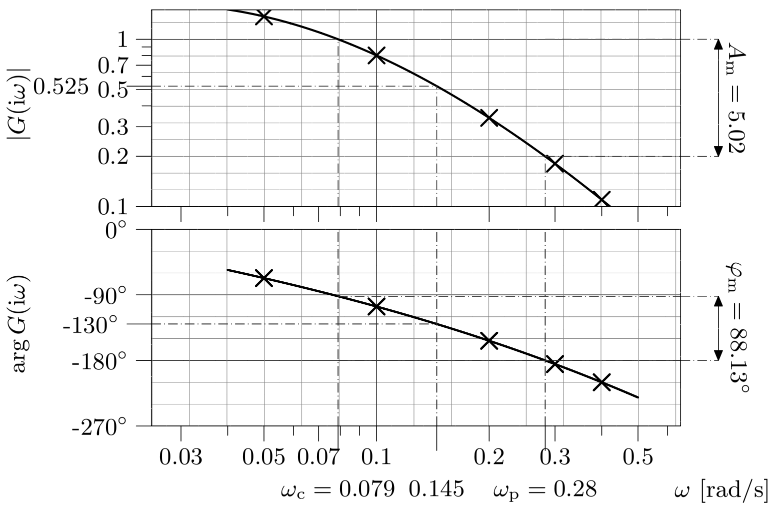

Figure 5.2a

5.1 For example, the following controller fulfills the requirements:

$$F(s) = 3.33 \cdot \frac{s + 0.185}{s + 0.555} \frac{s + 0.032}{s + 0.0036}$$

5.2 a) See Figure 5.2a.

b) Largest crossover frequency: 0.14 rad/s.

c) One controller that fulfills the requirements is the lead compensator (with gain adjustment)

$$F(s) = 1.9 \cdot 7 \frac{s + 0.106}{s + 0.106 \cdot 7}$$

5.3 a) See Figure 5.3a.

b) Smallest value of ramp error 0.067 and crossover frequency 150 rad/s.

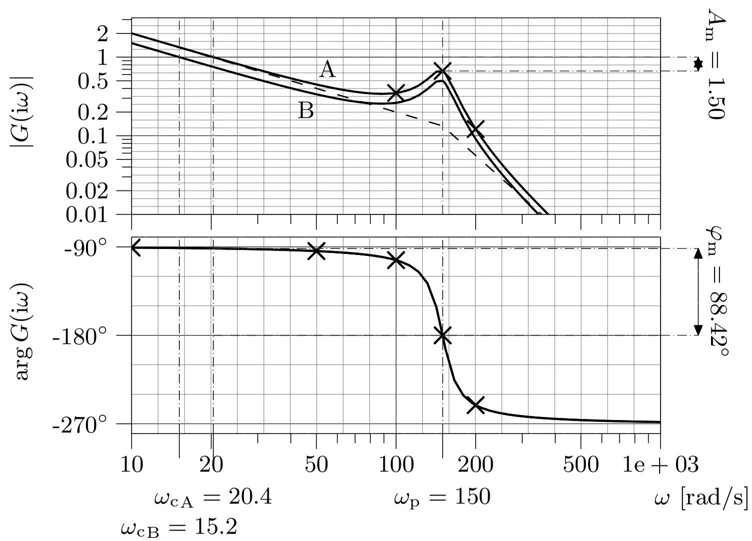

Figure 5.3a

c) One controller that fulfills the constraints is

$$F(s) = 0.75 \frac{s + 1.4}{s + 0.1}$$

5.4 One controller which fulfills the requirements is

$$F(s) = 1.02 \cdot 4 \cdot \frac{s + 6.3}{s + 25} \cdot \frac{s + 1.26}{s + 0.13}$$

5.5 A–E–C, B–C–E, C–A–B, D–D–D, E–B–A.

5.6 One controller which satisfies the demands is

$$F(s) = 1.2 \cdot 5 \frac{s + 8.0}{s + 5 \cdot 8.0} \cdot \frac{s + 1.8}{s + 1.8/84}$$

5.7 The system is stable when $0 < K < 0.2$ or $1.67 < K < 5$.

5.8 a) $T < 0.698$ s

b) $0.1 \text{ s} < T < 0.279$ s

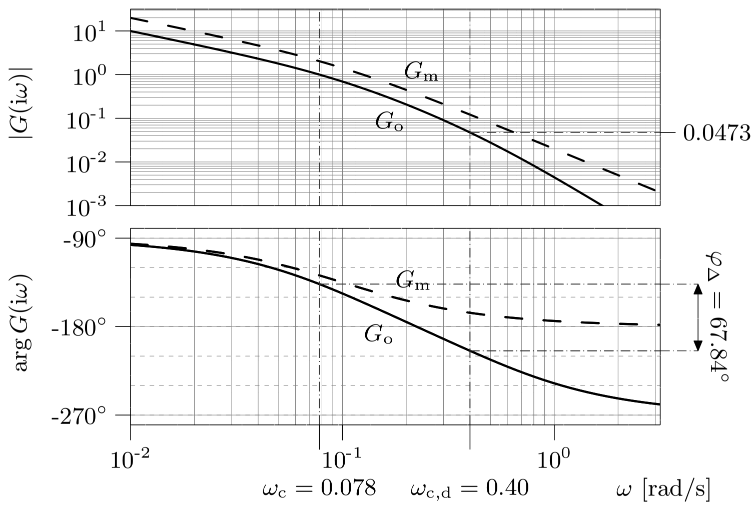

Figure 5.9a

5.9 a) $k_A = 0.25$ and $a = 0.5$. The Bode plot is given in Figure 5.9a.

b) One controller that does the job is

$$F(s) = 10.6 \cdot \left(4\frac{(s+0.2)}{(s+0.2\cdot 4)}\right)^2$$

5.10 The following compensator fulfills the requirements:

$$F(s) = 4.4 \cdot \left(4\frac{s+0.53}{s+0.53\cdot 4}\right)^2 \frac{s+0.105}{s+0.105/195}$$

5.11 a) See Figure 5.11a.

b) Asymptotically stable for $0 < K < 10$.

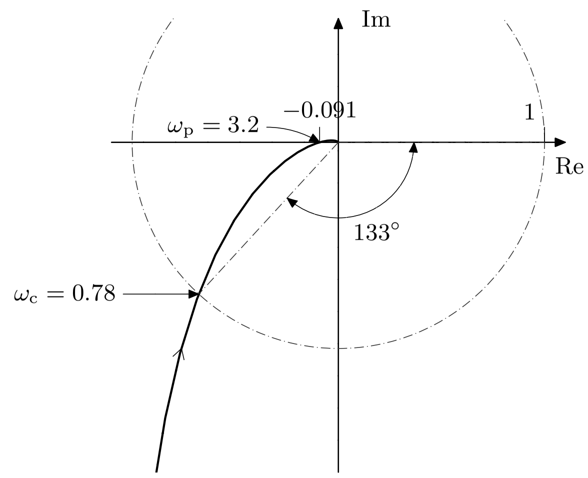

Figure 5.11a

c) $\lim_{t\to\infty} e(t) = 5$

d) $T < 0.4$

5.12 a) Impossible to determine.

b) It is stable.

5.13 a) $\omega_c = 5$ rad/s, $\omega_p = 9.5$ rad/s, $A_m = 3.5$ and $\varphi_m = 27°$.

b)-d) See solution.

5.14 The following controller will do:

$$F(s) = 35.7 \cdot 3\frac{s+0.12}{s+3\cdot 0.12}\frac{s+0.02}{s}$$

5.15 a) $e_0 = 0$, $e_1 = \frac{4}{K}$, provided $K < 4000$. Larger K results in an unstable system.

b) The following controller will do:

$$F(s) = 756 \cdot 7\,\frac{s+37.8}{s+264.6} \cdot \frac{s+10}{s+1.9}$$

6 Sensitivity and Robustness

6.1 The gain of the sensitivity is:

$$|S(1\mathrm{i})| = \frac{\sqrt{2}}{\sqrt{(K-1)^2+1}}$$

and the requirement on K becomes $K > 2$.

6.2 The maximum bandwidth is $\omega_\mathrm{B} = 1$.

6.3 See the solution, Figure 6.3a.

6.4 Yes.

6.5 a) See the solution, Figure 6.5a.

b)

$$\left|\frac{F(\mathrm{i}\omega)G(\mathrm{i}\omega)}{1+F(\mathrm{i}\omega)G(\mathrm{i}\omega)}\right| < \frac{1}{2}$$

6.6 a) No, stability cannot be guaranteed when $\tilde{G}(s) = 1$.

b) $\alpha > -2/25$. This is not contradictory since the robustness criterion is a *sufficient* but not *necessary* condition.

6.7 a) Asymptotically stable for $\alpha > 3$. See the solution, Figure 6.7a.

b) $\alpha > 4$

c) The robustness criterion gives a sufficient but not necessary condition.

6.8 $0 \leq \gamma < \frac{1}{35}$

6.9 $y(t) = \frac{1}{\sqrt{2}}\sin(t - \frac{\pi}{4}) - \sin(t)$

6.10 a) $\frac{1}{G_\Delta(s)} = -\frac{s+1}{s}$

b) Stability cannot be guaranteed for $F(s) = 1$, while it can be guaranteed for the regulator from Problem **??**.

6.11 The amplitude of the steady state error will be 0.2.

6.12 The controller also stabilizes the system for the stirring speed 400 r/min.

6.13 $K \geq \sqrt{396} \approx 19.9$

7 Special Controller Structures

7.1 a) See Figure 7.1a in the solution. ω_c and φ_m are undefined and $A_m = 43.5$. The stability requirement gives $K_1 = 21.75$ which implies

$$\lim_{t\to\infty} e(t) = 0.0487 \cdot a$$

where a is the size of the step.

b) See Figure 7.1b in the solution. ω_c and φ_m are undefined, and $A_m = 16$. The requirement gives $K_1 = 8$ which implies

$$\lim_{t\to\infty} e(t) = 0.111 \cdot a$$

7.2 a) $F_f(s) = 1$, and $h(t) = -\frac{0.1}{A\cdot 2}(1 - e^{-2t})$.

b) Zero steady state error.

7.3 a)

$$F_f = -\frac{G_v}{G_u} = -\frac{3(s+3)}{2(s+4)}$$

b) The amplitude of the control signal is 3.

c) $\lim_{t\to\infty} y(t) = \frac{9(1-b/2)}{12+4Kb}$

7.4 a) $F_f(s) = -\frac{4(s+1)}{3(s+2)(s+5)}$

b) $\lim_{t\to\infty} y(t) = -0.012$

c) $\lim_{t\to\infty} y(t) = -\frac{0.012}{3K+1}$

d) $y(t)$ doesn't have a final value.

8 State Space Description

8.1

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -1/\tau \end{pmatrix} x + \begin{pmatrix} 0 \\ K \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

8.2

$$\begin{aligned} \dot{x}_{1\Delta} &= x_{2\Delta} \\ \dot{x}_{2\Delta} &= \omega_0^2 x_{1\Delta} + u_\Delta \\ y_\Delta &= x_{1\Delta} \end{aligned}$$

8.3

$$\begin{aligned} \dot{x}_1(t) &= K_2 x_2(t) + M_l(t) \\ \dot{x}_2(t) &= -x_1(t) + x_3(t) \\ \dot{x}_3(t) &= -K_2 x_2(t) + K_1 i(t) \end{aligned}$$

8.4 a)

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= -6x_1(t) - 11x_2(t) - 6x_3(t) + 6u(t) \\ y(t) &= x_1(t) \end{aligned}$$

b)

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + x_3(t) + 4u(t) \\ \dot{x}_2(t) &= -3x_1(t) + 2u(t) \\ \dot{x}_3(t) &= -5x_1(t) + x_2(t) + u(t) \\ y(t) &= x_1(t) \end{aligned}$$

c)

$$\begin{aligned} \dot{x}_1(t) &= -2x_1(t) - u(t) \\ \dot{x}_2(t) &= -3x_2(t) + 3u(t) \\ y(t) &= x_1(t) + x_2(t) \end{aligned}$$

8.5

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + 2u(t) \\ \dot{x}_2(t) &= -4x_2(t) + 3u(t) \\ y(t) &= x_1(t) + x_2(t) \end{aligned}$$

8.6 $G(s) = \frac{s}{(s+2)(s+3)}$

8.7

$$x(t_0 + T) = e^{AT} x(t_0) + \left(\int_{t_0}^{t_0+T} e^{A(t_0+T-s)} \, \mathrm{d}s \right) B u_0$$

8.8 a) The state space description of the closed loop system

$$\dot{x}(t) = \begin{pmatrix} -3 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} h_{\text{ref}}(t)$$
$$h(t) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t)$$

b,c) The state space description of the closed loop system with noise

$$\dot{x}(t) = \begin{pmatrix} -3 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} n(t)$$
$$h(t) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t)$$

8.9 Yes, since the system is controllable.

8.10 a) Dimensions: 2 and 1. Subspaces: $\left\{ \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} -2 & 3 & -6 \end{pmatrix}^{\mathrm{T}} \right\}$ and $\left\{ \begin{pmatrix} 0 & -1 & 2 \end{pmatrix}^{\mathrm{T}} \right\}$.

b) Dimensions: 2 and 1. Subspaces: $\left\{ \begin{pmatrix} 0 & 4 & -2 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 0 & -8 & 8 \end{pmatrix}^{\mathrm{T}} \right\}$ and $\left\{ \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}} \right\}$.

8.11 a) $x_1 = 1 - e^{-t}$, $x_2 = 0.5(e^{2t} - 1)$

b) No. Yes.

c) Controllable, not observable.

d) Unobservable growing state ⇒ simulation collapses.

8.12 Poles: $1 \pm \mathrm{i}\sqrt{2}$. Zeros: -1.

8.13 a)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \frac{1}{\alpha}x_1 - \frac{u}{\alpha}\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= x_3 - u\end{aligned}$$

b) $\det \mathcal{S} = \frac{1}{\alpha^2}(1 - \frac{1}{\alpha})^2$. Thus, the system is controllable except for the case $\alpha = 1$, that is, when the two pendulums have the same lengths.

8.14 a)

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -1 & 0 \\ 1 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u\\ y &= \begin{pmatrix} 1 & 1 \end{pmatrix} x\end{aligned}$$

b) $u = -5x_1 + x_2 + 3.2r$

c) $Y(s) = \frac{3.2(2s+5)}{(s+4)^2} R(s)$

8.15 a) The model is given by

$$\begin{aligned}V\frac{\mathrm{d}c_\mathrm{A}}{\mathrm{d}t} &= -Vk_1c_\mathrm{A}^3 + qc_{\mathrm{A,in}} - qc_\mathrm{A}\\ V\frac{\mathrm{d}c_\mathrm{B}}{\mathrm{d}t} &= \frac{Vk_1c_\mathrm{A}^3}{3} - qc_\mathrm{B}\end{aligned}$$

b) The linearized model is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} c_{\mathrm{A},\Delta} \\ c_{\mathrm{B},\Delta} \end{pmatrix} &= \begin{pmatrix} \frac{-q-3k_1c_\mathrm{A}^*V}{V} & 0 \\ k_1c_\mathrm{A}^* & \frac{-q}{V} \end{pmatrix}\begin{pmatrix} c_{\mathrm{A},\Delta} \\ c_{\mathrm{B},\Delta} \end{pmatrix} + \begin{pmatrix} \frac{q}{V} \\ 0 \end{pmatrix} u\\ y &= \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} c_{\mathrm{A},\Delta} \\ c_{\mathrm{B},\Delta} \end{pmatrix}\end{aligned}$$

9 State Feedback

9.1 a) State feedback. Poles in $\{-3, -5\}$ gives the state feedback

$$u = -6x_1 - 14x_2 + y_{\text{ref}}$$

Poles in $\{-10, -15\}$ gives the state feedback

$$u = -23x_1 - 149x_2 + y_{\text{ref}}$$

b) Observer poles in -20 gives the observer gain

$$K = \begin{pmatrix} 38 \\ -399 \end{pmatrix}$$

9.2 a)

$$\dot{x} = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ K_1 \end{pmatrix} u$$

b) $u = -\frac{1}{8K_1K_2}x_1 - \frac{3}{4K_1}x_2 - \frac{3}{2K_1}x_3$

c) Observer gain $K^{\mathrm{T}} = \begin{pmatrix} 6 & 12/K_2 & 8/K_2 \end{pmatrix}$

9.3 a) $u = -\frac{2}{c_1\tau^2}\theta - \frac{1}{\tau c_1}\omega + \frac{2}{c_1\tau^2}\theta_{\text{ref}}$

b) $u = -\frac{2}{c_1\tau^2}\theta - \frac{1}{\tau c_1}\omega + \frac{2}{c_1\tau^2}\theta_{\text{ref}} - \frac{c_2}{c_1}\hat{x}_3$

9.4 State feedback gain $L = \begin{pmatrix} 6 & -2 \end{pmatrix}$. Observer gain $K^{\mathrm{T}} = \begin{pmatrix} 16 & 9 \end{pmatrix}$.

9.5 The system is observable and the poles of the observer may be placed arbitrarily.

9.6 a) Yes, since the system is controllable.

b) Closed loop poles in -3 gives

$$u = -3x_1 - 5x_2 - 4x_3 + y_{\text{ref}}$$

c) The system is observable with the sensor at x_1 or x_3. The sensor at x_1 and observer poles in -4 give $K^{\mathrm{T}} = \begin{pmatrix} 6 & 14 & 14 \end{pmatrix}$.

9.7 $\hat{X}_3(s) = \frac{K_1}{s+K}U(s) + \frac{K^2 s}{s+K}X_2(s)$

9.8 a) $L = \begin{pmatrix} 1 & 2 \end{pmatrix}$

b) In steady state: $h = -0.1$.

c) $F_{\text{f}}(s) = 2$ gives, in steady state, $h = 0$.

d) $h = \frac{2(k_1-1)}{k_1}v$

e) Introduce the integral of the height as a new state

$$z(t) = \int_0^t h(s)\,\mathrm{d}s \Rightarrow \dot{z} = h$$

9.9 $T < \frac{\arctan 2\omega_{\text{c}}}{\omega_{\text{c}}} = 0.65s$

9.10 a) $K^{\mathrm{T}} = \begin{pmatrix} -13 & 38 \end{pmatrix}$

b) The transfer function from v to $\tilde{x}_1$ is

$$-C_1(sI - A + KC)^{-1}K = \frac{13s - 12}{s^2 + 15s + 50}$$

where $C_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

9.11 a) The initial value theorem gives

$$\dot{y}(0) = -\frac{\beta^2}{\alpha}$$

and hence $\dot{y}(0)$ decreases as α decreases.

b) No, since the zero is not affected by the feedback.

9.12 A very fast closed loop system:

- implies that the poles are far into the LHP which implies a need for generating large input signals.
- easily becomes unstable in case of model uncertainties.
- becomes sensitive to measurement noise.
- has a sensitivity function with a large peak.

9.13 a) $L = \begin{pmatrix} 1 & 0 \end{pmatrix}$

b) $r(t) = r_0 e^{-5t}$

9.14 a) $x_2 = y$ (motor angle) and $x_1 = \dot{y}$ (angular velocity).

b) The pole locations give similar rise and settling times. With complex poles the maximum value of the input is lower.

c) Larger weight on the motor angle gives faster response.

d) Increasing weight on the input makes the system slower.

e) Increasing weight on the velocity makes the system slower.

9.15 a) Yes the system is controllable.

b) Poles in -0.1 gives the state feedback

$$u = -0.13x_1 - 0.128x_2$$

c) It is desirable that the estimation error converges to zero faster than the dynamics of the system. Thus, we should place the eigenvalues of the observer to the left of the poles of the closed loop system. To avoid large amplification of the measurement noise the poles of the observer should not be placed too far into the left hand plane.

d) Observer poles in -0.1 gives the observer gain

$$K = \begin{pmatrix} 0.45 \\ 0.33 \end{pmatrix}$$

9.16 Are the specifications 1–4 fulfilled?

??. The bandwidth requirement is not fulfilled.

??. The system is stable despite the model errors.

??. The gain is different from 1 when $\kappa \neq 1$

??. Both measurement and process noise are amplified for some frequency.

9.17 a) $u = 4x_1 + x_2$

b) Yes! It is essential that the input u is known since u is required in the observer design to get an asymptotically vanishing state estimation error.

c) Yes, by introducing a third state $x_3 = u$. This new system is observable hence a observer can be designed to estimate u.

9.18 a) The poles are pure complex and thus the system doesn't have a well defined stationary error or speed of response.

b) A linear combination of r and x_2 is given by

$$u = l_0 r - l_2 x_2$$

with this controller the poles can be placed with l_2 as

$$s = \frac{-l_2}{2} \pm \sqrt{\frac{l_2^2 - 4}{4}}$$

and by setting $l_0 = 1$ the stationary error will be zero when $w = 0$. If $w \neq 0$ and $l_0 = 1$ then there will be stationary error of size $l_2 w$.

c) Designing a observer with the following observer gains $k_1 = -11$, $k_2 = 6$, and $k_3 = -8$. Let the control law be $u = l_0 r - l_2 \hat{x}_2 - l_3 \hat{x}_3$. With $l_3 = l_2$ and $l_0 = 1$ there will be no error. Place the poles to the closed loop with l_2.

11 Implementation

11.1 $\beta_1 = 0.905$, $\alpha_1 = 19.14$, and $\alpha_2 = -18.95$.

11.2 a) $y_{k+1} - y_k = Tu_k$

b) $0 < K < \frac{2}{T}$

11.3 a)

$$A = \frac{1}{\sqrt{1 + (\omega_2 T_1)^2}}$$
$$\omega_1 = \frac{2\pi}{T} - \omega_2$$
$$\varphi = \pi + \arctan \omega_2 T_1$$

b) $T_1 = T/\pi$ gives $A = \frac{1}{\sqrt{1+(\omega_2 T/\pi)^2}}$.

Reglerteknik: Solutions

- **Solutions**

This version: August 2013

AUTOMATIC CONTROL
COMMUNICATION SYSTEMS
LINKÖPINGS UNIVERSITET

Solutions

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1 Mathematics

1.1 a) A step has Laplace transform $\frac{A}{s}$.

b) A ramp has Laplace transform $\frac{A}{s^2}$.

c) $\frac{1}{s+2}$

d) $\frac{s}{s^2+25}$

e) $sU(s) - u(0)$

f) $sU(s)$. ($u(0) = 0$ is a common assumption in the course.)

g) $s^2U(s) - su(0) - \dot{u}(0)$

h) $s^2U(s)$. ($u(0) = \dot{u}(0) = 0$ is a common assumption in the course.)

i) A time delayed signal has Laplace transform, $e^{-sT}U(s)$.

1.2 a) Insert $\dot{y}(t) = 0$ och $u(t) = 5$ directly into the differential equation $\Rightarrow$ $y(t) = 5/2$. It is also possible to solve the differential equation and let $t \to \infty$, or to use b) and the final value theorem.

b) Use Laplace transform on the differential equation $Y(s) = \frac{1}{s+2}U(s)$. The denominator coincides with the characteristic polynomial of the differential equation. Note that we also have assumed $y(0) = 0$.

1.3 The general solution is given by

$$y(t) = C_1e^{-2t} + (C_2 + C_3t)e^{-t} - \frac{3}{100}(\cos(2t) + 7\sin(2t))$$

1.4 a)

$$y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}, \quad t \geq 0$$

b) The Laplace transform of the input

$$u(t) = 1 + \sin t$$

yields

$$U(s) = \frac{1}{s} + \frac{1}{s^2+1}$$

The differential equation

$$\dot{y}(t) + y(t) = u(t)$$

may be represented by the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+1}$$

Hence, the Laplace transform of the system output is given by

$$Y(s) = \underbrace{\frac{1}{s} \cdot \frac{1}{s+1}}_{Y_1(s)} + \underbrace{\frac{1}{s+1} \cdot \frac{1}{s^2+1}}_{Y_2(s)}$$

Rewriting the first term using partial fractions leads to

$$Y_1(s) = \frac{1}{s} \cdot \frac{1}{s+1} = \frac{1}{s} - \frac{1}{s+1}$$

with inverse transform

$$y_1(t) = 1 - e^{-t}$$

Rewriting the second term using partial fractions leads to

$$Y_2(s) = \frac{1}{s+1} \cdot \frac{1}{s^2+1} = \frac{0.5}{s+1} - \frac{0.5s}{s^2+1} + \frac{0.5}{s^2+1}$$

with inverse transform

$$y_2(t) = 0.5e^{-t} - 0.5\cos t + 0.5\sin t$$

Hence, the system output is

$$y(t) = 1 - 0.5e^{-t} + 0.5\sin t - 0.5\cos t$$

1.5 a) The abolute value is $|1 + \mathrm{i}| = \sqrt{2}$, and the argument is $\arctan\frac{1}{1} = \frac{\pi}{4} = 45°$. Hence, the polar form is

$$\sqrt{2}e^{\mathrm{i}\frac{\pi}{4}}$$

b) The absolute value is

$$\frac{|1+\mathrm{i}|}{5\left|1+\sqrt{(3)}\mathrm{i}\right|} = \frac{\sqrt{2}}{5\cdot 2} \approx 0.14$$

The argument is

$$\begin{aligned}\arg\left(\frac{1+\mathrm{i}}{5\mathrm{i}(1+\sqrt{3}\mathrm{i})}\right) &= \arg(1+\mathrm{i}) - \arg 5\mathrm{i} - \arg(1+\sqrt{3}\mathrm{i})\\ &= \arctan 1 - 90^\circ - \arctan\sqrt{3} = 45^\circ - 90^\circ - 60^\circ\\ &= -105^\circ\end{aligned}$$

Hence, the polar form is

$$\frac{\sqrt{2}}{10}e^{-\mathrm{i}\frac{105}{180}\pi}$$

c) $2e^{\mathrm{i}\frac{\pi}{3}} = 2\cos\frac{\pi}{3} + 2\mathrm{i}\sin\frac{\pi}{3} = 1+\sqrt{3}\mathrm{i}$

d) $5e^{-\mathrm{i}\pi} = 5\cos(-\pi) + 5\mathrm{i}\sin(-\pi) = -5$

1.6 The amplification in deciBel is computed as $10\log|F|^2 = 20\log|F|$, where F is the absolute value of the amplification. The amplification $F = 100$ hence corresponds to $20\log 100 = 40$ dB_{20}.

deciBel (dB_{20})	Definition		Amplification F
20	$20\log F = 20$	$\Rightarrow$	$F = 10^1 = 10$
-3	$20\log F = -3$	$\Rightarrow$	$F = 10^{-3/20} \approx 0.708 \approx \frac{1}{\sqrt{2}}$
0	$20\log F = 0$	$\Rightarrow$	$F = 10^0 = 1$
10	$20\log F = 10$	$\Rightarrow$	$F = 10^{0.5} = \sqrt{10} \approx 3.16$
-10	$20\log F = -10$	$\Rightarrow$	$F = 10^{-0.5} = \frac{1}{\sqrt{10}} \approx 0.316$

1.7 Multiplication of the two matrices gives the unit matrix.

1.8 The eigenvalues (λ) of the matrix A are given by the equation $\det(\lambda I - A) = 0$, and the corresponding eigenvectors (v) are given by the equation $(\lambda I - A)v = 0$.

$$\lambda_1 = 3 \qquad v_1 = \begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$$\lambda_2 = -1 \qquad v_2 = \begin{pmatrix}1\\0\\3\end{pmatrix}$$

$$\lambda_3 = 4 \qquad v_3 = \begin{pmatrix}-1\\0\\2\end{pmatrix}$$

1.9

$$T = \begin{pmatrix}1 & 1 & -1\\ 1 & -1 & 0\\ 1 & 0 & 1\end{pmatrix}$$

1.10 A basis for the null space is for example

$$\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}$$

A basis for the range space is

$$\begin{pmatrix}2\\0\\3\\1\end{pmatrix} \quad \begin{pmatrix}1\\1\\1\\0\end{pmatrix} \quad \begin{pmatrix}2\\1\\3\\1\end{pmatrix}$$

The rank of the matrix is hence 3.

1.11 a) Writing the function with partial fractions yields

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

The inverse transform is then computed by use of a Laplace transform table:

$$f(t) = 1 - e^{-t}$$

This means that $f(t) \to 1$ as $t \to \infty$. The same result can also be obtained by use of the final value theorem, that is, by computing $\lim_{s\to 0} sF(s)$.

b) Writing the function with partial fractions yields

$$F(s) = -\frac{0.5}{s+1} + \frac{0.5}{s-1}$$

The inverse transform is then computed by use of a Laplace transform table:

$$f(t) = -0.5e^{-t} + 0.5e^{t}$$

This means that $f(t)$ will grow without bound as $t \to \infty$. Here, the final value theorem cannot be used since $f(t)$ lacks a final value.

c) The inverse transform can be computed by use of the relation

$$\mathcal{L}^{-1}\{G(s+a)\} = e^{-at} \cdot g(t)$$

Here, $G(s) = \frac{1}{s^2}$ and $a = 1$. The inverse transform of G is $g(t) = t$, so

$$f(t) = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\} = e^{-t} \cdot t$$

which tends to 0 as $t \to \infty$. This result can also be obtained by use of the final value theorem.

1.12 The relation between inflow and water level is given by the transfer function

$$Y(s) = \frac{1}{s+1} Z(s)$$

and the relation between control signal and inflow may be written as

$$Z(s) = \frac{1}{s^2+s+1} U(s)$$

This means that the Laplace transforms of the control signal and water level are related by

$$Y(s) = \frac{1}{(s+1)}\frac{1}{(s^2+s+1)} U(s) = \frac{1}{s^3+2s^2+2s+1} U(s)$$

which corresponds to the differential equation

$$y^{(3)} + 2\ddot{y} + 2\dot{y} + y = u$$

2 Dynamic Systems

2.1 a) We start from the equations

$$J\ddot{\theta} = -f\dot{\theta} + M \tag{2.1}$$
$$M = k_{\mathrm{a}} i \tag{2.2}$$
$$v = k_{\mathrm{v}}\dot{\theta} \tag{2.3}$$

Voltage equilibrium gives

$$u - R_{\mathrm{a}} i - L_{\mathrm{a}}\frac{\mathrm{d}i}{\mathrm{d}t} - v = 0 \tag{2.4}$$

where $L_{\mathrm{a}} = 0$. Equation (2.2) in (2.1) gives

$$J\ddot{\theta} + f\dot{\theta} = k_{\mathrm{a}} i \tag{2.5}$$

From (2.4) and (2.3) we get

$$i = (u - k_{\mathrm{v}}\dot{\theta})/R_{\mathrm{a}}$$

which in (2.5) gives

$$J\ddot{\theta} + f\dot{\theta} = k_a(u - k_{\mathrm{v}}\dot{\theta})/R_{\mathrm{a}}$$

that is

$$\ddot{\theta} + \frac{R_{\mathrm{a}} f + k_{\mathrm{a}} k_{\mathrm{v}}}{J R_{\mathrm{a}}}\dot{\theta} = \frac{k_{\mathrm{a}}}{J R_{\mathrm{a}}} u$$

Let

$$\frac{1}{\tau} = \frac{R_{\mathrm{a}} f + k_{\mathrm{a}} k_{\mathrm{v}}}{J R_{\mathrm{a}}} \qquad k_0 = \frac{k_{\mathrm{a}}}{J R_{\mathrm{a}}}$$

which gives

$$\ddot{\theta} + \frac{1}{\tau}\cdot\dot{\theta} = k_0 u \tag{2.6}$$

b) Laplace transformation of (2.6) gives

$$(s^2 + \frac{1}{\tau}\cdot s)\theta(s) = k_0 U(s)$$

and this gives the transfer function

$$G(s) = \frac{\theta(s)}{U(s)} = \frac{k_0}{s(s+1/\tau)}$$

c) Suppose that u is a unit step, that is,

$$u = \begin{cases} 0, & t < 0 \\ 1 & t \geq 0 \end{cases}$$

that is

$$U(s) = \frac{1}{s}$$

This gives

$$\theta(s) = G(s)U(s) = \frac{k_0}{s(s+1/\tau)}\cdot\frac{1}{s} = \left(\frac{k_0\tau}{s} - \frac{k_0\tau}{s+1/\tau}\right)\cdot\frac{1}{s}$$

Inverse Laplace transformation gives

$$\theta(t) = k_0\tau t - k_0\tau^2(1 - e^{-t/\tau})$$

that is, θ will grow unlimited when t increases.

2.2 (1) Asymptotically stable system. Monotonic step response, that is, real poles: $K = 0.1$.

(2) Very oscillative system. Poles close to the imaginary axis: $K = 2.5$.

(3) Unstable system. Poles in the right half plane: $K = 3$.

(4) Asymptotically stable system. Oscillative step response, that is, complex poles in the left half plane: $K = 0.5$.

2.3 The inverse Laplace transform gives the step response

$$d_1(t) = \mathcal{L}^{-1}\left\{\frac{\beta}{1+sT}\cdot\frac{1}{s}\right\} = \beta(1 - e^{-t/T})$$

For the final value, we have

$$d_1(t) \to \beta,\ t \to \infty$$

The figure gives $\beta = 10$. At the time $t = T$, the system time constant, the step response has reached 63% of the final value, that is,

$$d_1(T) = 0.63 \cdot 10$$

The figure gives $T = 3$, which gives the total transfer function

$$G(s) = \frac{10}{1+3s}$$

If we measure the signal $d_2(t)$ we introduce an additional time delay of $\frac{L}{V}$ time units. The total transfer function then becomes

$$G(s) = \frac{10e^{-\frac{L}{V}s}}{1+3s}$$

Answer:

$$G(s) = \frac{10e^{-\frac{L}{V}s}}{1+3s}$$

2.4 Use the system description

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

In the first figure $\omega_0 = 1$ and $\zeta = 0.5$.

a) For the system

$$G(s) = \frac{1}{s^2 + as + 1}$$

we have $\omega_0 = 1$ and $\zeta = 0.5a$. The step response is more oscillative than in the case $\zeta = 0.5$, that is, $\zeta < 0.5$. This gives $a < 1$.

b) For the system

$$G(s) = \frac{b^2}{s^2 + bs + b^2}$$

we have $\omega_0 = b$ and $\zeta = 0.5$. The step response is in this case pure time scaling compared to the case $\omega_0 = 1$. The figures show that the step response is twice as fast as in the case $\omega_0 = 1$. This gives $b = \omega_0 = 2$.

2.5 The pairs of plots that belong to the same system will be written in the form *pole-zero-letter–step-response-letter.*

Pole-zero diagram B has a single pole in the origin which gives a ramp as step response, that is, B–F. Pole-zero diagram D also has a pole in the origin which gives an infinitely growing step response, D–C. Pole-zero diagram F has complex poles which gives an oscillative step response, F–D. Pole-zero diagram A has a zero in the origin which gives final value zero, A–B. Pole-zero diagram C cannot be step response E, since two real poles and no zeros give no overshoot. Hence C–A, and step response E is the only alternative left for pole-zero diagram E.

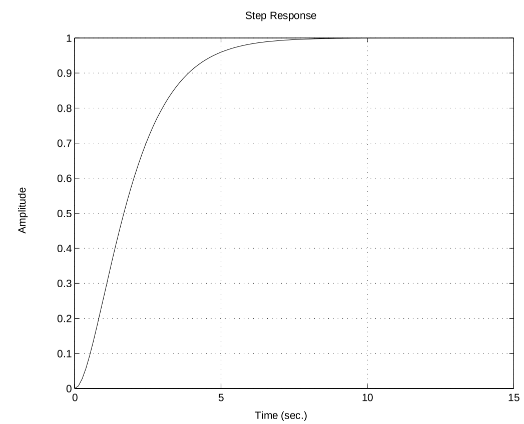
Answer: *A–B, B–F, C–A, D–C, E–E, F–D.*

2.6 a) Enter the systems.

```
>> s = tf( 's' );
>> GA = 1 / ( s^2 + 2*s + 1 );
>> GB = 1 / ( s^2 + 0.4*s + 1 );
>> GC = 1 / ( s^2 + 5*s + 1 );
>> GD = 1 / ( s^2 + s + 1 );
>> GE = 4 / ( s^2 + 2*s + 4 );
```

Compute and plot the step response.

```
>> step( GA ); grid
```



The systems $G_B(s)$, $G_C(s)$, $G_D(s)$, and $G_E(s)$ can be simulated in a similar way. The values of T_r, T_s, and M for the different step responses can be found by a right click in the figure and selecting "Characteristics" and then selecting the desired property. Use the "Properties…" menu item (of the right click menu) to change the interval for the settling time. (The default interval is 2%, while we use 5% in the course.)

b) Compute the poles.

```
>> pole( GA )
ans =
    -1
    -1
```

The other systems are handled in the same way.

c) The results from a) and b) can be summarized in the following table.

System	T_r	T_s	M	poles
G_A	3.37	4.74	0%	$-1, -1$
G_B	1.21	13.7	52.7%	$-0.2 \pm i0.98$
G_C	10.5	14.6	0%	$-4.8, -0.2$
G_D	1.65	5.29	16.3%	$-0.5 \pm i0.87$
G_E	0.824	2.64	16.3%	$-1 \pm i1.73$

Using this table we can draw the following conclusions. (i): The speed of the step response (mainly) depends on the distance between the poles and the origin. Poles further away from the origin give a faster step response and shorter rise time. (ii): The damping of the system depends on the relationship between the imaginary part and the real part of the poles. Poles with large imaginary part relative to the real part give a poorly damped (oscillatory) step response.

Remark: We see that even though the distance to the origin is nearly the same in system G_A and G_B the rise time is almost 3 times faster in system B. Note that speed is not only rise time, also the settling time should be considered! Look at the following system

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

The poles of this system are given by $s = \omega_0(-\zeta \pm i\sqrt{1-\zeta^2}) = \omega_0(-\cos\phi \pm i\sin\phi)$ where $\cos\phi = \zeta$. The parameter ζ is called relative damping and $0 \le \zeta \le 1$. We see that ω_0 is the distance from the origin to the poles and in Figure 2.6a the step responses for different ζ are shown when ω_0 is constant. We see clearly that the rise time is faster when ζ is small but when ζ is small the settling time is big!

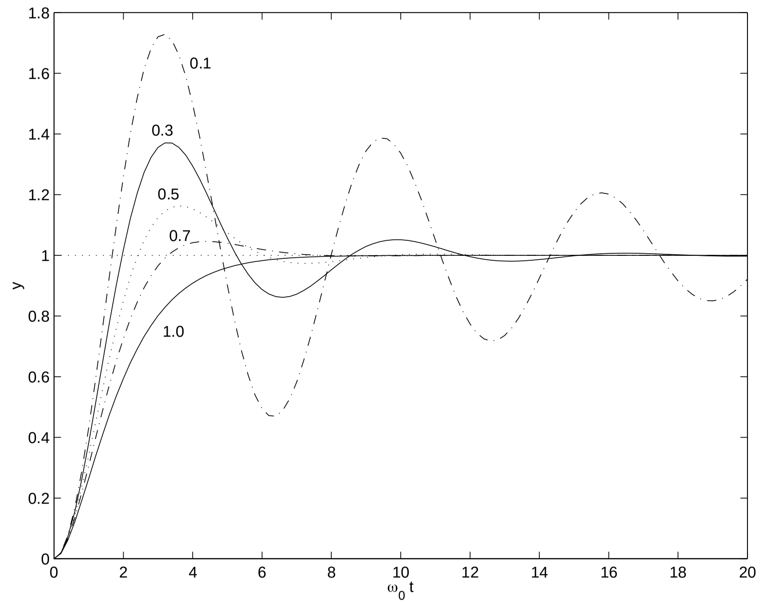


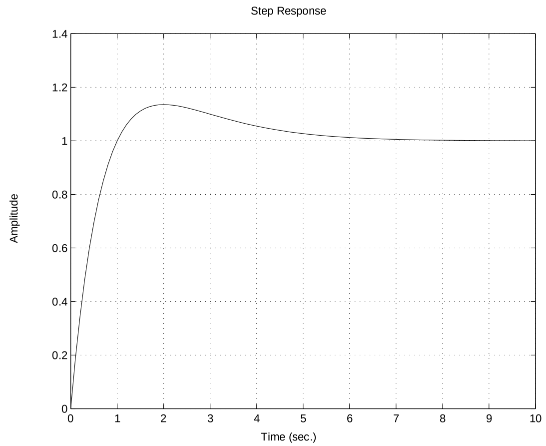
Figure 2.6a

2.7 Enter the system. Here we consider the case $\alpha = 2$, that is the system has a zero in -0.5.

```
>> s = tf( 's' );
>> G1 = ( 2*s + 1 ) / ( s^2 + 2*s + 1 );
```

Plot the step response.

```
>> step( G1, 10 ); grid
```



A zero located close to the origin on the negative real axis causes an overshoot in the step response. A zero on the positive real axis causes the step response to initially move in the negative direction. This means that in some cases the zeros of the system can have significant influence in the system properties. Systems with zeros in the right half plane normally imply extra difficulties for the design of control systems.

2.8 The Laplace transform of a step is $U(s) = \frac{1}{s}$. The step response is hence given by

$$y(t) = \mathcal{L}^{-1}(G(s)\frac{1}{s}).$$

If $G(s)$ is a rational function the inverse Laplace transform can be computed by first doing a partial fraction expansion and then using a transform table. When the system is available one can let the input $u(t)$ be a step and measure $y(t)$.

2.9 a) The steady state value is 1.5.

b) The output signal almost reaches 1.9, which is slightly less than 0.4 over the final value. The overshoot is hence $\frac{0.4}{1.5} \approx 26\%$.

c) Find the time points where the output is 10% (0.15) and 90% (1.35) of the steady state value. The rise time is the difference between these values, here approximately $T_\mathrm{r} \approx 1.5$ s.

d) Find the earliest time such that the output then lies within ±5% of the steady state value. Here, the interval is $[\,1.425,\, 1.575\,]$, and the settling time is $T_\mathrm{s} \approx 7.8$.

2.10 G_1–C: G_1 is poorly damped, which gives an oscillatory behavior.

G_2: Can be excluded since it is the only system having static gain $\frac{1}{2}$, and among the step responses there is always more than one match for each of the present final values.

G_3–B: This case has the shortest rise time, and some overshoot due to the pair of complex poles. The static gain is 2.

G_4–A: The pole in -2 dominates, which gives slower step response than systems G_3 and G_5. The static gain is 1.

G_5–D: The dominating pole is in -3, which is slower than for G_3 but faster than for G_4. The static gain is 2.

G_6: Can be excluded due to instability.

2.11 a) The signals can be classified as

- ⋄ Disturbances signal: Acid process flow (unknown pH and flow)
- ⋄ Control signal: NaOH solution
- ⋄ Measured and controlled signal: The pH of the outflow

b) A block diagram where the control strategy is based on feedback could look like Figure 2.11a

2.12 a) At steady state the inflow is equal to the outflow (constant volume). From mass balance

$$\rho^* q^* = \rho_1^* q_1^* + \rho_2^* q_2^*$$

Assuming the densities are equal ($\rho = \rho_1 = \rho_2$) gives $q^* = q_1^* + q_2^* = 1 + 0.5 = 1.5\ \mathrm{m}^3/\mathrm{min}$. From component balance for component A

$$q^* c_\mathrm{A}^* = q_1^* c_{\mathrm{A},1}^* + q_2^* c_{\mathrm{A},2}^*$$

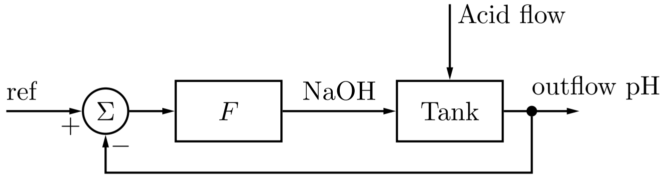


Figure 2.11a

which gives $c_A^* = 2.0\ \text{kmol/m}^3$.

b) The amount of mass in the tank is given by ρV (assuming ρ is constant). The change in mass is given by the mass coming in subtracted by the mass going out of the tank

$$\frac{\mathrm{d}(\rho V)}{\mathrm{d}t} = \rho(q_{in} - q_{out}) \tag{2.1}$$

where $q_{\text{in}} = q_1 + q_2$ and $q_{\text{out}} = q$. Assuming that the volume is constant gives $\frac{\mathrm{d}(\rho V)}{\mathrm{d}t}$ which means

$$q_1 + q_2 = q \tag{2.2}$$

The amount of component A contained in the tank is given by Vc_A. The change is then given by

$$\frac{\mathrm{d}(Vc_A)}{\mathrm{d}t} = q_1 c_{A,1} + q_2 c_{A,2} - q c_A \tag{2.3}$$

Constant V and (2.2) gives

$$V\frac{\mathrm{d}c_A}{\mathrm{d}t} = q_1(c_{A,1} - c_A) + q_2(c_{A,2} - c_A) \tag{2.4}$$

The model (2.1), (2.3) is nonlinear since it contains products between variables. Assuming volumes and flows to be constant gives a linear model.

c) Assume that all the other independent variables (q_1, q_2, $c_{A,2}$) are constant. Take their values from a). Equation (2.4) then gives

$$V\frac{\mathrm{d}c_A}{\mathrm{d}t} = q_1^*(c_{A,1}(t) - c_A(t)) + q_2^*\left(c_{A,2}^* - c_A(t)\right) = -1.5c_A(t) + c_{A,1}(t) + 2$$

The equation can be written

$$\frac{\mathrm{d}c_A}{\mathrm{d}t} = -1.5c_A(t) + 3.2$$

for $t \geq 0$. The corresponding Laplace transform equation is

$$s(\mathcal{L}c_A)(s) - (\mathcal{L}c_A)(0) = -1.5(\mathcal{L}c_A)(s) + 3.2\frac{1}{s}$$

or

$$(\mathcal{L}c_A)(s) = \frac{1}{s+1.5}\left(2 + \frac{3.2}{s}\right) = 2\frac{1}{s+1.5} + 3.2\frac{1}{s+1.5}\frac{1}{s}$$

which transforms back to

$$c_A(t) = 2e^{-1.5t} + \frac{3.2}{1.5}\left(1 - e^{-1.5t}\right)$$

Rearranging yields

$$\frac{3.2}{1.5} + \left(2 - \frac{3.2}{1.5}\right)e^{-1.5t} = 2 - \left(2 - \frac{3.2}{1.5}\right)\left(1 - e^{-\frac{t}{\frac{1}{1.5}}}\right)$$

where the sought constants can be identified: $k_0 = 2.0\ \text{kmol/m}^3$, $k_1 = 0.13\ \text{kmol/m}^3$, and $\tau = \frac{1}{1.5} = 0.67$ min.

2.13 a) The equilibrium equation is

$$y_i = \frac{\alpha x_i}{1 + (\alpha - 1)x_i} \tag{2.1}$$

Mass balance gives

$$\frac{\mathrm{d}M_i}{\mathrm{d}t} = L_{i-1} + V_{i+1} - L_i - V_i \tag{2.2}$$

Component balance gives

$$\frac{\mathrm{d}(M_i x_i)}{\mathrm{d}t} = M_i\frac{\mathrm{d}x_i}{\mathrm{d}t} + x_i\frac{\mathrm{d}M_i}{\mathrm{d}t} = L_{i-1}x_{i-1} + V_{i+1}y_{i+1} - L_i x_i - V_i y_i \tag{2.3}$$

Combining (2.2)–(2.3) gives

$$\begin{aligned} M_i\frac{\mathrm{d}x_i}{\mathrm{d}t} &= -x_i(L_{i-1} + V_{i+1} - L_i - V_i) + L_{i-1}x_{i-1} + V_{i+1}y_{i+1} - L_i x_i - V_i y_i \\ &= L_{i-1}(x_{i-1} - x_i) + V_{i+1}(y_{i+1} - x_i) + V_i(x_i - y_i) \end{aligned} \tag{2.4}$$

The dynamic model for $M_i(t)$ and $x_i(t)$ is described by (2.1), (2.2), and (2.4).

b) The stationary point for (2.2) gives $L_{i-1}^* + V_{i+1}^* - L_i^* - V_i^* = 0$. Introduce the difference variables

$$\begin{aligned} x_{i\Delta} &= x_i - x_i^* & x_{i+1,\Delta} &= x_{i+1} - x_{i+1}^* & y_{i\Delta} &= y_i - y_i^* \\ y_{i-1,\Delta} &= y_{i-1} - y_{i-1}^* & L_{i+1,\Delta} &= L_{i+1} - L_{i+1}^* & V_{i-1,\Delta} &= V_{i-1} - V_{i-1}^* \\ L_{i\Delta} &= L_i - L_i^* & V_{i\Delta} &= V_i - V_i^* \end{aligned}$$

The assumption that the change of mass on the plate is zero gives

$$\frac{\mathrm{d}M_{i\Delta}}{\mathrm{d}t} = 0$$

which means that

$$L_{i-1} + V_{i+1} - L_i - V_i = 0$$

this will simplify (2.4) to

$$M_i \frac{\mathrm{d}x_i}{\mathrm{d}t} = L_{i-1}x_{i-1} + V_{i+1}y_{i+1} - L_i x_i - V_i y_i \tag{2.5}$$

Linearization of (2.5) gives

$$\begin{aligned} M_i^* \frac{\mathrm{d}x_{i\Delta}}{\mathrm{d}t} &= L_{i-1}^* x_{i-1,\Delta} + V_{i+1}^* y_{i+1,\Delta} - L_i^* x_{i\Delta} - V_i^* y_{i\Delta} \\ &\quad + x_{i-1}^* L_{i-1,\Delta} + y_{i+1}^* V_{i+1,\Delta} - x_1^* L_{i\Delta} - y_i^* V_{i\Delta} \end{aligned} \tag{2.6}$$

Linearization of (2.1) gives

$$y_{i\Delta} = \frac{\alpha}{\left(1 + (\alpha - 1)x_i^*\right)^2} x_{i\Delta} \tag{2.7}$$

The linearized model is described by (2.6)–(2.7).

2.14 Från blockdiagramet fås $Y(s) = G_2(s)[F_2(s)Y(s) + G_1(s)U(s) + F_1(s)U(s)]$, vilket ger överföringsfunktionen $\dfrac{Y(s)}{U(s)} = \dfrac{G_2(s)(G_1(s) + F_1(s))}{1 - F_2(s)G_2(s)}$.

2.15 D: En integrator vars stegsvar är en ramp. Ger 1. B: Nollställe i högra halvplanet vilket ger ett stegsvar som initialt går åt fel håll. Ger 5. A: Polerna till A och B är samma, vilket ger samma relativa dämpning. Ger 2. C: Polerna har relativ dämpning $\zeta = 0.15$ vilket är mindre än alla andra. Ger 4. F: Polerna har relativ dämpning $\zeta = 1$ och snabbhet $\omega_0 = 3$. Inget annat system är så snabbt. Ger 3. E: Enda systemet kvar. Ger 6.

Svar: A-2, B-5, C-4, D-1, E-6 and F-3

3 Feedback Systems

3.1 a) To begin with, the transfer function for the tank system is derived. The mass balance equation is, assuming that the bottom area of the tank is 1 m^2

$$\dot{h}(t) = x(t) - v(t)$$

that is (note that all initial conditions are zero when deriving transfer functions)

$$sH(s) = X(s) - V(s)$$

Hence

$$H(s) = G_t(s)(X(s) - V(s))$$

where

$$G_t(s) = \frac{1}{s}$$

The block diagram becomes like in Figure 3.1a.

b) The transfer function for the valve is

$$G_v(s) = \frac{k_v}{1 + Ts}$$

With the input taken as a unit step signal, that is,

$$U(s) = \frac{1}{s}$$

it follows that

$$X(s) = \frac{k_v}{1 + Ts} \cdot \frac{1}{s}$$

The final value theorem gives

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = k_v$$

The time constant T is the time it takes for the step response to reach 63% of its final value. From the plot it follows that $T = 5$ and $k_v = 2$, that is

$$G_v(s) = \frac{2}{1 + 5s}$$

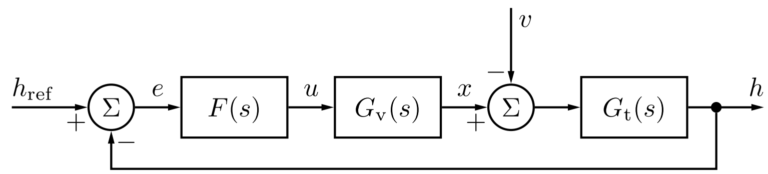


Figure 3.1a

c) By using the controller $F(s)$, the closed loop system shown in Figure 3.1a is obtained. From the block diagram, the following equations are obtained:

$$E(s) = H_{ref}(s) - H(s)$$
$$H(s) = G_t(s)\big(F(s)G_v(s)E(s) - V(s)\big)$$

This leads to

$$H(s) = G_t(s)\Big(G_v(s)F(s)\,[H_{ref}(s) - H(s)] - V(s)\Big)$$
$$\Longleftrightarrow$$
$$H(s)\Big(1 + G_t(s)G_v(s)F(s)\Big) = G_t(s)\Big(G_v(s)F(s)H_{ref}(s) - V(s)\Big)$$
$$\Longleftrightarrow$$
$$H(s) = \underbrace{\frac{G_t(s)G_v(s)F(s)}{1 + G_t(s)G_v(s)F(s)}}_{G_c(s)} H_{ref}(s) - \underbrace{\frac{G_t(s)}{1 + G_t(s)G_v(s)F(s)}}_{-G_{v,h}(s)} V(s)$$

That the expression for the output is a sum over all inputs, with each term given by a rational transfer function multiplied by the input, is no coincidence; this will always be true of any transfer function between points in a block diagram with rational transfer functions and summation points. In particular, the output is a linear (dynamic) function of the inputs. This leads to a conclusion that will be used frequently hereafter: *When computing the transfer function from one input to the output, all other inputs may be set to zero.* The reader is encouraged to try this by taking $H_{ref}(s) = 0$ in the first equation above.

Inserting the expressions for $G_{\mathrm{t}}(s)$ and $G_{\mathrm{v}}(s)$ in the equation above, it follows that $G_{\mathrm{v,h}}$ is given by

$$\frac{H(s)}{V(s)} = -\frac{1+5s}{s(1+5s)+2F(s)}$$

and G_{c} by

$$\frac{H(s)}{H_{\mathrm{ref}}(s)} = \frac{2F(s)}{s(1+5s)+2F(s)}$$

Assume $F(s) = \frac{F_{\mathrm{b}}(s)}{F_{\mathrm{a}}(s)}$ with $F_{\mathrm{b}}(s)$ and $F_{\mathrm{a}}(s)$ polynomials, then the characteristic polynomial becomes $p(s) = s(1+5s)F_{\mathrm{a}}(s) + 2F_{\mathrm{b}}(s)$ in both cases.

d) Proportional feedback

$$F(s) = K$$

gives

$$\frac{H(s)}{H_{\mathrm{ref}}(s)} = \frac{0.4K}{s^2+0.2s+0.4K}$$

The closed loop poles are given by

$$s^2 + 0.2s + 0.4K = 0$$

That is

$$s = -0.1 \pm \mathrm{i}\sqrt{0.4K - 0.01} \quad \text{if } K > 0.025$$

The closed loop poles belong to the pre specified region provided that $|\mathrm{Re}\,| > |\mathrm{Im}\,|$ or

$$0.01 > 0.4K - 0.01$$

Hence $K < 0.05$.

e) When v is a unit step signal we have

$$V(s) = \frac{1}{s}$$

The control error $e = h_{\mathrm{ref}} - h = -h \quad (h_{\mathrm{ref}} = 0)$ is given by

$$E(s) = -H(s) = \frac{s+0.2}{s^2+0.2s+0.4K}\cdot\frac{1}{s}$$

The final value theorem gives (the system is stable for $K > 0$)

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{1}{2K}$$

f) A PI controller, that is,

$$F(s) = \frac{K_{\mathrm{P}}s + K_{\mathrm{I}}}{s}$$

gives

$$E(s) = \frac{s(s+0.2)}{s^2(s+0.2)+0.4(K_{\mathrm{P}}s+K_{\mathrm{I}})}\cdot\frac{1}{s}$$

when v is a unit step signal. The final value theorem gives (provided that the closed loop system is asymptotically stable)

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s(-H(s)) = 0$$

3.2 a) The closed loop poles (from Solution 3.1) are given by

$$s = -0.1 \pm \mathrm{i}\sqrt{0.4K - 0.01}$$

$K = 1$ gives

$$s = -0.1 \pm \mathrm{i}\sqrt{0.39}$$

b) PD control

$$F(s) = K_{\mathrm{P}} + K_{\mathrm{D}}s$$

and using the expressions derived in Solution 3.1, this results in

$$H(s) = \frac{2F(s)}{s(1+5s)+2F(s)}H_{\mathrm{ref}}(s) = \frac{0.4(sK_{\mathrm{D}}+K_{\mathrm{P}})}{s^2+(0.2+0.4K_{\mathrm{D}})s+0.4K_{\mathrm{P}}}H_{\mathrm{ref}}(s)$$

The characteristic polynomial is

$$s^2 + (0.2+0.4K_{\mathrm{D}})s + 0.4K_{\mathrm{P}} = 0$$

Compare with the standard form

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

where ω_0 denotes the fundamental frequency and ζ denotes the relative damping. Assume $K_{\mathrm{P}} = 1$ and determine K_{D} so that $\zeta > 1/\sqrt{2}$. A comparison with the standard form then gives

$$\omega_0 = \sqrt{0.4}$$

$$\zeta = \frac{0.2+0.4K_{\mathrm{D}}}{2\sqrt{0.4}} > \frac{1}{\sqrt{2}}$$

which gives $K_{\mathrm{D}} > 1.7$.

3.3 The dynamics of the astronaut is given by

$$F = ma$$

where $m = 100$, F is the control signal u and $a = \ddot{y}$. This gives the model

$$100\ddot{y} = u$$

and

$$Y(s) = \frac{1}{100s^2}U(s)$$

The control law is given by

$$u = K_1(r - y) - K_1K_2\dot{y} = K_1((r - y) - K_2\dot{y})$$

or

$$U(s) = K_1(R(s) - Y(s) - K_2sY(s))$$

The transfer function from r to e is given by

$$E(s) = \frac{s^2 + 0.01K_1K_2s}{s^2 + 0.01K_1K_2s + 0.01K_1}R(s)$$

When $r(t) = t$ we have

$$R(s) = \frac{1}{s^2}$$

The final value theorem then gives (provided that K_1 and K_2 are chosen such that the closed loop is asymptotically stable) (also note that the transfer function from r to e must have at least one zero at the origin for the final value to exist, but this is satisfied regardless of the choice of K_1 and K_2)

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = K_2 < 1$$

The transfer function from r to y is given by

$$G_c(s) = \frac{Y(s)}{R(s)} = \frac{0.01K_1}{s^2 + 0.01K_1K_2s + 0.01K_1}$$

The standard form for the characteristic equation

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

gives with $\zeta = 1/\sqrt{2} \approx 0.7$

$$s^2 + \sqrt{2}\omega_0 + \omega_0^2 = 0$$

A comparison with

$$s^2 + 0.01K_1K_2s + 0.01K_1 = 0$$

gives $\omega_0 = 0.1\sqrt{K_1}$. We hence obtain

$$K_1 = \frac{200}{K_2^2}$$

Answer: Choose $K_2 < 1$ and $K_1 = 200/K_2^2$.

3.4 We shall determine how the control error $e(t) = y_{\text{ref}}(t) - y(t)$ depends on the disturbance signal f_c. We can assume that $y_{\text{ref}}(t) = 0$, since the size of the error as a function of f_c is sought for.

$$E(s) = Y_{\text{ref}}(s) - G(s)\cdot(F_c(s) + F(s)E(s))$$

where

$$G(s) = \frac{1}{ms^2 + ds}$$

gives

$$E(s) = -\frac{G(s)}{1 + G(s)F(s)}\cdot F_c(s)$$

$f_c(t)$ is a step disturbance, that is

$$F_c(s) = \frac{a}{s}$$

a) Proportional control, $F(s) = K$, gives

$$E(s) = -\frac{1}{ms^2 + ds + K}\cdot\frac{a}{s}$$

Using the final value theorem it follows that (provided that K is chosen such that the closed loop is asymptotically stable)

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = -a/K$$

b) Proportional-Integral control

$$F(s) = \frac{K_1 s + K_2}{s}$$

gives

$$E(s) = -\frac{s}{ms^3 + ds^2 + K_1 s + K_2} \cdot \frac{a}{s}$$

The final value theorem in this case gives (provided that K_1 and K_2 are chosen such that the closed loop is asymptotically stable)

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = 0$$

3.5 a) Enter the system.

```
>> s = tf( 's' );
>> G = 0.2 / ( ( s^2 + s + 1 ) * ( s + 0.2 ) );
```

Generate a proportional regulator.

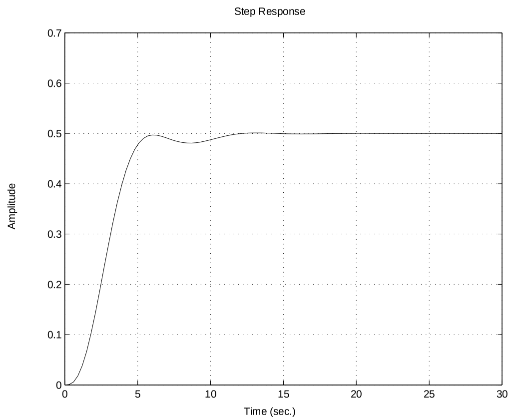
```
>> F = 1;
```

Generate the closed loop system.

```
>> Gc = feedback( F * G, 1 );
```

Compute and plot the step response.

```
>> step( Gc, 30 ); grid
```



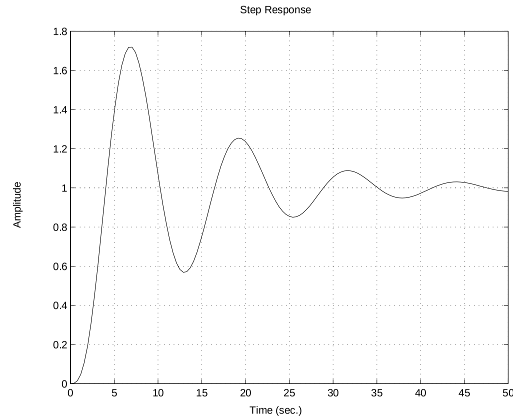
By trying some different values of K_P the following behavior can be seen: For small values of K_P the step response is slow, well damped and the steady state error is large. For increasing K_P the step response becomes faster but more oscillatory, while the error is reduced. For large K_P the amplitude of the oscillations increases over time, that is, the closed loop system becomes unstable.

b) Generate a PI controller with $K_P = 1$ and $K_I = 1$.

```
>> KP = 1; KI = 1;
>> F = KP + KI / s;
```

Plot the result.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 ); grid
```



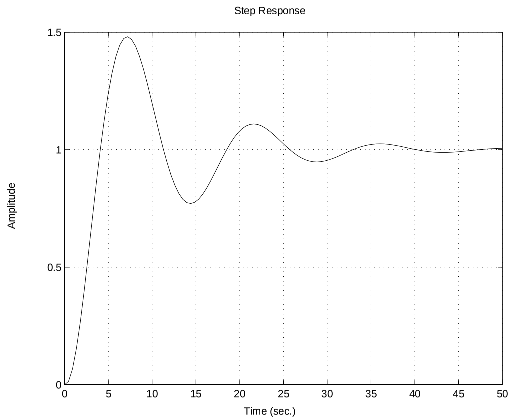
The following effects of the integrator can be found by trying some different values of K_I. (i): The integrator in the regulator eliminates the steady state error. (ii): A too small value of K_I gives a large settling time while a too large value gives an oscillatory (finally unstable) closed loop system.

c) Generate a PID controller with $K_P = 1, K_I = 1, K_D = 2$ and $T = 0.1$.

```
>> KP = 1; KI = 1; T = 0.1; KD = 1;
>> FP = KP;
>> FI = KI / s;
>> FD = KD * s / ( s*T + 1 );
>> F = FP + FI + FD;
```

Plot the result.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 ); grid
```



Using the (approximate) derivative of the error in the regulator increases the damping of the closed loop system. Increasing K_D too much, however, gives that an oscillation with higher frequency appears in the step response and finally (approximately when $K_D > 65$) the closed loop system becomes unstable.

3.6 a) The transfer function for the closed loop system is

$$G_c(s) = \frac{G_o(s)}{1+G_o(s)} = \frac{K(s+2)}{s(s+1)(s+3)+K(s+2)}$$

The characteristic equation is

$$s(s+1)(s+3)+K(s+2) = P(s)+KQ(s) = 0$$

that is

$$P(s) = s(s+1)(s+3) \qquad Q(s) = s+2$$

- ◇ Starting points: ⇔ zeros of $P(s): 0, -1, -3$
  End points: ⇔ zeros of $Q(s): -2$
- ◇ Number of asymptotes: 2
  Directions: $\frac{1}{2}[\pi + 2k\pi] = \pm\pi/2$
  Intersection with the real axis: $\frac{1}{2}[0+(-1)+(-3)-(-2)] = -1$

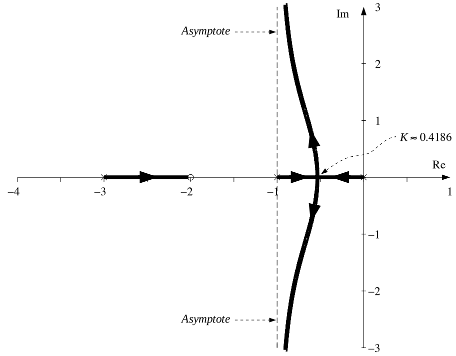


Figure 3.6a

- ◇ Real axis: $[-3, -2)$ and $[-1, 0]$ belongs to the root locus
- ◇ Intersection with the imaginary axis: Set $s = i\omega$ and solve the characteristic equation

$$i\omega(i\omega+1)(i\omega+3)+K(i\omega+2) = -i\omega^3 - 4\omega^2 + (3+K)i\omega + 2K = 0$$

$$\Rightarrow \left.\begin{array}{r}(-\omega^2+3+K)\omega = 0\\ -4\omega^2+2K = 0\end{array}\right\} \Rightarrow \begin{array}{c}\omega = K = 0\\ \text{(starting point)}\end{array}$$

This gives the root locus in Figure 3.6a.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for all $K > 0$. For small values of K there are no oscillations and the speed is increasing with increasing K. For a certain value of K the system becomes oscillating. The damping is decreasing with increasing K.

b) The transfer function for the closed loop system is

$$G_c(s) = \frac{G_o(s)}{1+G_o(s)} = \frac{K}{s(s^2+2s+2)+K}$$

The characteristic equation reads

$$s(s^2+2s+2)+K = 0$$

that is

$$P(s) = s(s^2+2s+2) \qquad Q(s) = 1$$

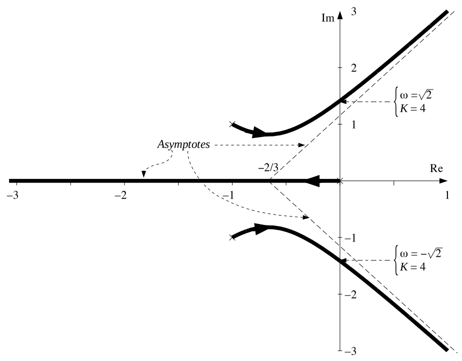


Figure 3.6b

◇ Starting points: ⇔ zeros of $P(s): 0, -1 \pm \mathrm{i}$
End points: ⇔ There are no zeros of $Q(s)$

◇ Number of asymptotes: 3
Directions: $\frac{1}{3}[\pi + 2k\pi] = \pi, \pm\pi/3$
Intersection of asymptotes: $\frac{1}{3}[0 + (-1+\mathrm{i}) + (-1-\mathrm{i})] = -2/3$

◇ Part of the real axis that belongs to the root locus: $(-\infty, 0]$

◇ Intersection with the imaginary axis: Set $s = \mathrm{i}\omega$ and solve the characteristic equation

$$\mathrm{i}\omega((\mathrm{i}\omega)^2 + 2\mathrm{i}\omega + 2) + K = -\mathrm{i}\omega^3 - 2\omega^2 + 2\mathrm{i}\omega + K = 0$$

$$\Rightarrow \left.\begin{array}{r}(-\omega^2+2)\omega = 0\\ -2\omega^2 + K = 0\end{array}\right\} \Rightarrow \left.\begin{array}{c}\omega = K = 0 \\ \text{(start point)}\end{array} \quad \text{or} \quad \begin{array}{r}\omega = \pm\sqrt{2}\\ K = 4\end{array}\right\}$$

This gives the root locus in Figure 3.6b.

Answer: All poles are in the left half plane. That is, the system is asymptotically stable for $0 < K < 4$. The step response is oscillating for all K. To begin with the system will be faster with increasing K. However, for K sufficiently large the oscillating part is dominating. The damping will decrease with increasing K and for $(K \geq 4)$ the closed loop system is unstable.

c) The transfer function for the closed loop system is

$$G_c(s) = \frac{G_o(s)}{1 + G_o(s)} = \frac{K(s+1)}{s(s-1)(s+6) + K(s+1)}$$

The characteristic equation is

$$s(s-1)(s+6) + K(s+1) = P(s) + KQ(s) = 0$$

$$P(s) = s(s-1)(s+6) \qquad Q(s) = s+1$$

◇ Starting points: ⇔ zeros of $P(s): 0, 1, -6$
End points: ⇔ zeros of $Q(s): -1$

◇ Number of asymptotes: $3 - 1 = 2$
Directions: $\frac{1}{2}[\pi + 2k\pi] = \pm\pi/2$
Intersection of the asymptotes: $\frac{1}{2}[0 + 1 + (-6) - (-1)] = -2$

◇ Part of the real axis that belongs to the root locus: $[-6, -1)$ and $[0, 1]$

◇ Intersection with the imaginary axis: Set $s = \mathrm{i}\omega$ and solve the characteristic equation:

$$\mathrm{i}\omega(\mathrm{i}\omega - 1)(\mathrm{i}\omega + 6) + K(\mathrm{i}\omega + 1) = -\mathrm{i}\omega^3 - 5\omega^2 + (K-6)\mathrm{i}\omega + K = 0$$

$$\Rightarrow \left.\begin{array}{r}(-\omega^2 + K - 6)\omega = 0\\ -5\omega^2 + K = 0\end{array}\right\} \Rightarrow \left.\begin{array}{c}\omega = K = 0\\ \text{(start point)}\end{array}\right\} \text{or} \begin{array}{c}\omega = \sqrt{\frac{3}{2}}\\ K = 7.5\end{array}$$

This gives the root locus in Figure 3.6c.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for $K > 7.5$. For small values on K the closed loop system is (as the open loop system) unstable. For $K > 7.5$ the closed loop system is stable and oscillating. As K is increasing from the critical value both the damping and the response speed are increasing (the time constant is always $\geq 1/2$s), until they both are beginning to decrease. The damping is decreasing with increasing K.

3.7 The transfer function for the closed loop system is obtained from

$$\theta(s) = \frac{1}{s}\dot{\theta}(s) = \frac{1}{s} \cdot \frac{k}{1 + s\tau} \cdot K \cdot (\theta_{\text{ref}}(s) - \alpha s\theta(s) - \theta(s))$$

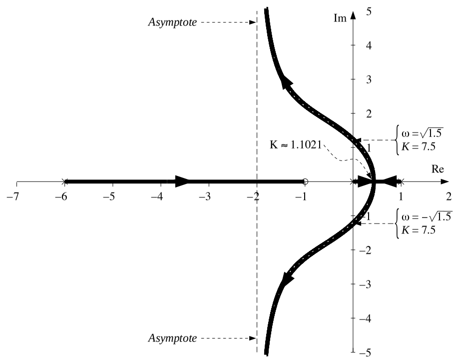


Figure 3.6c

$\Rightarrow$

$$G(s) = \frac{\theta(s)}{\theta_{\text{ref}}(s)} = \frac{k \cdot K}{s(1+s\tau) + k \cdot K(1+\alpha s)} = \frac{4K}{s(s+2) + 4K(1+\alpha s)}$$

The characteristic equation is:

$$s(s+2) + 4K(1+\alpha s) = 0$$

a) $\alpha = 0$. The characteristic equation is then

$$s(s+2) + 4K = s^2 + 2s + 4K = 0$$

with the solution

$$s = -1 \pm \sqrt{1-4K}$$

This gives the root locus in Figure 3.7a.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for all $K > 0$.

b) $\alpha = 1$. The characteristic equation is then

$$s(s+2) + 4K(1+s) = 0$$

that is

$$P(s) = s(s+2) \qquad Q(s) = 4(1+s)$$

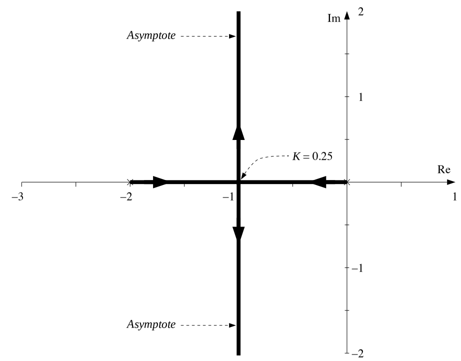


Figure 3.7a

- ◇ Starting points: $\Leftrightarrow$ zeros of $P(s)$: 0, −2
  End points: $\Leftrightarrow$ zeros of $Q(s)$: −1
- ◇ Number of asymptotes: $2 - 1 = 1$.
  Direction of asymptotes: $\frac{1}{1} \cdot \pi$, that is, the negative real axis.
- ◇ Part of the real axis that belongs to the root locus: $(-\infty, -2]$ and $(-1, 0]$
- ◇ Intersection with the imaginary axis: Set $s = \mathrm{i}\omega$ and solve the characteristic equation:

$$\mathrm{i}\omega(\mathrm{i}\omega + 2) + 4K(1 + \mathrm{i}\omega) = -\omega^2 + (2+4K)\mathrm{i}\omega + 4K = 0$$

$$\Rightarrow \left.\begin{array}{r}(2+4K)\omega = 0\\ -\omega^2 + 4K = 0\end{array}\right\} \Rightarrow \begin{array}{c}\omega = K = 0\\ \text{(start point)}\end{array}$$

This gives the root locus in Figure 3.7b.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for all $K > 0$.

c) $\alpha = 1/3$. The characteristic equation is then

$$s(s+2) + 4K(1+s/3) = P(s) + KQ(s) = 0$$

which gives

$$P(s) = s(s+2) \qquad Q(s) = 4(1+s/3)$$

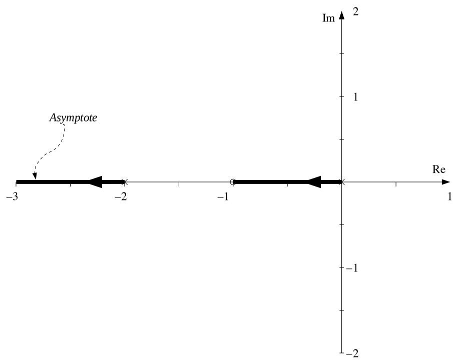


Figure 3.7b

- ◇ Starting points ⇔ zeros of $P(s)$: $0, -2$
  End points ⇔ zeros of $Q(s)$: -3
- ◇ Number of asymptotes: $2 - 1 = 1$ Direction: $\frac{1}{1} \cdot \pi$, that is, the negative real axis
- ◇ Part of the real axis that belongs to the real axis $(-\infty, 3)$ and $[-2, 0]$
- ◇ Intersection with the imaginary axis. Set $s = \mathrm{i}\omega$ and solve the characteristic equation:

$$\mathrm{i}\omega(\mathrm{i}\omega + 2) + 4K(1 + \mathrm{i}\omega/3) = -\omega^2 + (2 + \frac{4}{3}K)\mathrm{i}\omega + 4K = 0$$

$$\Rightarrow \left.\begin{array}{r} (2 + \frac{4}{3}K)\omega = 0 \\ -\omega^2 + 4K = 0 \end{array}\right\} \Rightarrow \begin{array}{c} \omega = K = 0 \\ \text{(start point)} \end{array}$$

This gives the root locus in Figure 3.7c.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for all $K > 0$.

d) $K = 1$. The characteristic equation becomes

$$s(s + 2) + 4(1 + \alpha s) = s^2 + 2s + 4 + 4\alpha s = 0$$

that is

$$P(s) = s^2 + 2s + 4 \qquad Q(s) = 4s$$

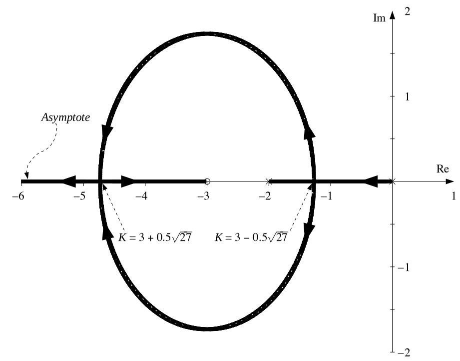


Figure 3.7c

- ◇ Starting points ⇔ zeros of $P(s)$: $-1 \pm \mathrm{i}\sqrt{3}$
  End points ⇔ zeros of $Q(s)$: 0
- ◇ Number of asymptotes: $2 - 1 = 1$
  Direction: $\frac{1}{1} \cdot \pi$, that is, the negative real axis
- ◇ Part of the real axis that belongs to the root locus: $(-\infty, 0)$
- ◇ Intersection with the imaginary axis: $s = \mathrm{i}\omega$ solves the characteristic equation

$$-\omega^2 + 2\mathrm{i}\omega + 4 + 4\mathrm{i}\omega\alpha = 0$$

$$\Rightarrow \left.\begin{array}{r} \omega(2 + 4\alpha) = 0 \\ -\omega^2 + 4 = 0 \end{array}\right\} \begin{array}{r} \text{has no solution} \\ (\alpha < 0 \text{ is of no interest}) \end{array}$$

To get further insights into the behavior of the closed loop system the intersection with the real axis is determined. That is, a real valued double root to the characteristic equation has to be determined

$$s^2 + 2s + 4 + 4\alpha s = (s + a)^2 = s^2 + 2as + a^2$$

$$\Rightarrow \begin{array}{l} 2a = 2 + 4\alpha \\ a^2 = 4 \end{array} \Rightarrow \left.\begin{array}{l} a = 2 \\ \alpha = 1/2 \end{array}\right\}$$

This gives the root locus in Figure 3.7d.

Answer: All poles are in the left half plane, that is, the closed loop system is stable for all $\alpha \geq 0$. From d) it follows that the system will be

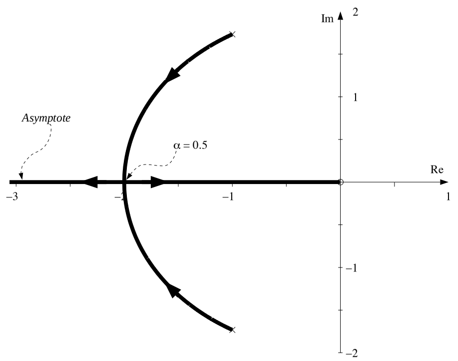


Figure 3.7d

more damped for larger values on α (compare b, c: in b) the system is not oscillating for any value on K). For α sufficiently large, the time constant can be arbitrary large. This is natural since the term $-\alpha\dot{\theta}\cdot K$ (D-term) that appears in the input voltage of the motor reduces the velocity of the axis. The effect is as if the motor has been drained with thick oil. With a suitable viscosity α the system can be made fast and stable as in c). With $\alpha = 0$ as in a) and K large enough, the system is not becoming faster just less damped.

3.8 a)

$$\frac{\omega(s)}{\delta_{\text{ref}}(s)} = G_1(s)\cdot G_2(s) = \frac{10(s+1)}{(s+10)(s+4)(s-3)}$$

The open loop system is unstable (a pole in $s = 3$). Hence $\omega(t)$ is increasing when $\delta_{\text{ref}}(t)$ is a step signal. Observe that the model is valid for small changes with respect to a large reference input θ_0 for the pitch, and for predetermined values on the static and the dynamic pressure.

b)

$$\omega(s) = G_1(s)\cdot G_2(s)\cdot K\cdot(\omega_{\text{ref}}(s) - \omega(s))$$

gives

$$\omega(s) = \frac{K\cdot G_1(s)\cdot G_2(s)}{1+K\cdot G_1(s)\cdot G_2(s)}\omega_{\text{ref}}(s) = \frac{10K(s+1)}{(s+10)(s+4)(s-3)+10K(s+1)}\omega_{\text{ref}}(s)$$

The characteristic equation is

$$(s+10)(s+4)(s-3)+10K(s+1) = 0$$

which gives

$$P(s) = (s+10)(s+4)(s-3) \qquad Q(s) = 10(s+1)$$

- ◇ Starting points: ⇔ zeros of $P(s)$: -10, -4, 3
  End points: ⇔ zeros of $Q(s)$: -1
- ◇ Number of asymptotes: $3-1 = 2$
  Directions: $\frac{1}{2}(\pi + 2k\pi) = \pm\pi/2$
  Intersection with the real axis: $\frac{1}{3-1}[(-10)+(-4)+3-(-1)] = -5$
- ◇ Part of the real axis that belong to the root locus: $[-10, -4]$ and $(-1, 3]$
- ◇ Intersection with the imaginary axis: $s = \mathrm{i}\omega$ solves the characteristic equation

$$(\mathrm{i}\omega+10)(\mathrm{i}\omega+4)(\mathrm{i}\omega-3)+10K(\mathrm{i}\omega+1) =$$
$$= -\mathrm{i}\omega^3 - 11\omega^2 + (10K-2)\mathrm{i}\omega + 10K - 120 = 0$$
$$\Rightarrow \left.\begin{array}{r}(-\omega^2+10K-2)\omega = 0\\ -11\omega^2+10K-120 = 0\end{array}\right\} \Rightarrow \left.\begin{array}{r}\omega = 0\\ K = 12\end{array}\right\}$$

This gives the root locus in Figure 3.8a.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for all $K > 12$.

c) The question is: Is there any $K > 12$ for which all poles are real valued? For $K = 12$ it is known that $s = 0$ is a solution to the characteristic equation. The other roots are given by

$$(s+10)(s+4)(s-3)+10\cdot 12(s+1) = s(s^2+11s+118)$$

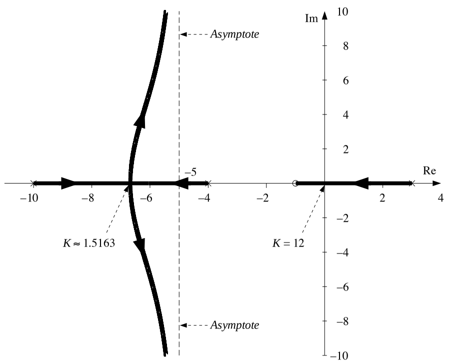


Figure 3.8a

That is, the two remaining poles are

$$-\frac{11}{2} \pm \sqrt{(\frac{11}{2})^2 - 118} = -5.5 \pm 9.4\mathrm{i}$$

which shows that they are complex for $K > 12$. The answer is hence no.

3.9 This text serves as a workaround for an obscure bug in LaTeX.

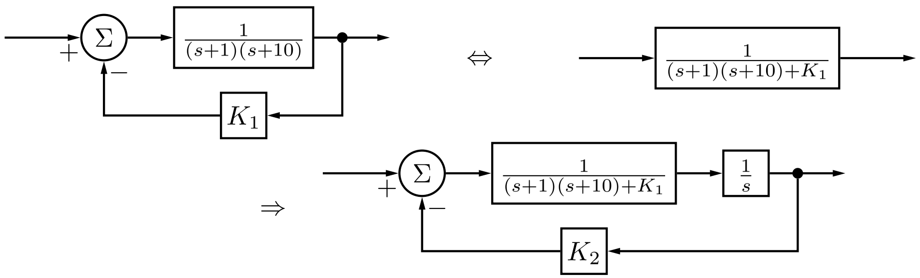


Figure 3.9a

a) The block diagram is given in Figure 3.9a. Hence, the characteristic equation is $\underbrace{s\left((s+1)(s+10)+K_1\right)}_{P(s)} + K_2 = 0$

Start points: 0 and the roots to $s^2 + 11s + 10 + K_1 = 0$, that is, $s = -5.5 \pm \sqrt{5.5^2 - 10 - K_1}$

The roots are real when $K_1 \leq 20.25$. (I)

The roots are complex when $K_1 > 20.25$. (II)

(I)
- ◇ start points: 0, $-5.5 \pm \alpha$
- ◇ end points: missing
- ◇ asymptotes: 3: $\frac{\pi}{3}$, π, $\frac{5\pi}{3}$
  intersection of the asymptotes: $\frac{1}{3}(-11) = -\frac{11}{3}$
- ◇ parts of real axis: $(-\infty, -5.5-\alpha], [-5.5+\alpha, 0]$

The root locus is given in Figure 3.9b.

(II)
- ◇ start points: 0, $-5.5 \pm \mathrm{i}\beta$
- ◇ end points: missing
- ◇ asymptotes: as in (I)
- ◇ parts of real axis: $(-\infty, 0]$

The root locus is given in Figure 3.9c

Imaginary axis crossings: put $s = \mathrm{i}\omega$ in the characteristic equation $\Rightarrow$

$$\mathrm{i}\omega(-\omega^2 + 11\mathrm{i}\omega + 10 + K_1) + K_2 = 0 \quad \Leftrightarrow$$
$$-\mathrm{i}\omega^3 - 11\omega^2 + 10\mathrm{i}\omega + K_1\mathrm{i}\omega + K_2 = 0 \quad \Rightarrow$$
$$\begin{cases} -\omega^3 + 10\omega + K_1\omega = 0 & (1) \\ -11\omega^2 + K_2 = 0 & (2) \end{cases}$$

$\omega = 0$ solution to (1) $\Rightarrow$ in (2) $K_2 = 0$.

$\omega^2 = 10 + K_1$ solution to (1) $\Rightarrow -110 - 11K_1 + K_2 = 0 \Rightarrow K_2 = 11K_1 + 110$

Answer: The closed loop system is asymptotically stable when $0 < K_2 < 11K_1 + 110$

b) By using the inner feedback ($K_1 > 0$) a larger value of K_2 is allowed.

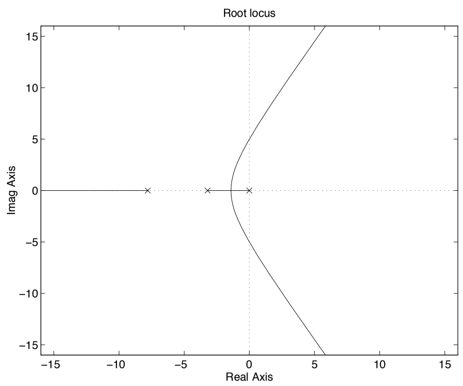


Figure 3.9b

3.10 Set

$$G_o(s) = \frac{1}{(s+1)(s-1)(s+5)}$$

With $U(s) = F(s)E(s)$, the transfer function of the closed loop system becomes

$$G_c(s) = \frac{G_o(s)F(s)}{1 + G_o(s)F(s)}$$

a) Here, $F(s) = K$, so

$$G_c(s) = \frac{K}{(s+1)(s-1)(s+5) + K}$$

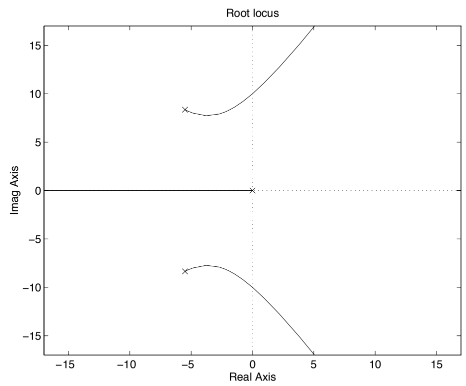


Figure 3.9c

The characteristic equation is

$$(s+1)(s-1)(s+5) + K = 0$$

which gives

$$P(s) = (s+1)(s-1)(s+5) \qquad Q(s) = 1$$

◇ Starting points: $\Leftrightarrow$ Zeros of $P(s)$: -1, 1, -5
End points: $\Leftrightarrow$ Zeros of $Q(s)$: none

◇ Number of asymptotes: $3 - 0 = 3$
Directions: $\frac{1}{3}[\pi + 2k\pi] = \pi,\ \pm\pi/3$
Intersection point: $\frac{1}{3}[-1 + 1 + (-5)] = -5/3$

- ◇ Real axis: $(-\infty, -5]$ and $[-1, 1]$ belongs to the root locus
- ◇ Intersection with the imaginary axis, set $s = \mathrm{i}\omega$:

$$(\mathrm{i}\omega+1)(\mathrm{i}\omega-1)(\mathrm{i}\omega+5)+K = -\mathrm{i}\omega^3-5\omega^2-\mathrm{i}\omega+K-5=0$$

$$\Longleftrightarrow \begin{cases} (\omega^2+1)\omega = 0 \\ -5\omega^2+K-5=0 \end{cases} \Longleftrightarrow \begin{cases} \omega = 0 \\ K = 5 \end{cases}$$

(A simple root!)

This gives the root locus in Figure 3.10a.

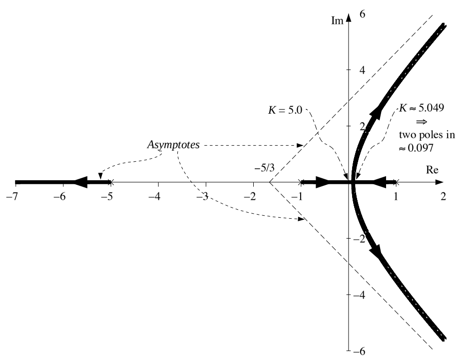


Figure 3.10a

Answer: There exists at least one pole in the RHP. Hence, the system is not asymptotically stable for any value of K.

b) Here, $F(s) = K(1+0.5s)$. Hence

$$G_c(s) = \frac{K(1+0.5s)}{(s+1)(s-1)(s+5)+K(1+0.5s)}$$

The characteristic equation is

$$(s+1)(s-1)(s+5)+K(1+0.5s)=0$$

which gives

$$P(s) = (s+1)(s-1)(s+5) \qquad Q(s) = 1+0.5s$$

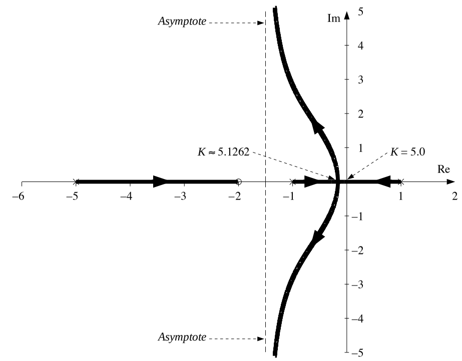


Figure 3.10b

- ◇ Starting points ⇔ Zeros of $P(s)$: -1, 1, -5
  End points ⇔ Zeros of $Q(s)$: -2
- ◇ Number of asymptotes: $3-1=2$
  Directions: $\frac{1}{2}[\pi+2k\pi] = \pm\pi/2$
  Intersection point: $\frac{1}{2}[-1+1-5-(-2)] = -\frac{3}{2}$
- ◇ Real axis: $[-5, -2)$ and $[-1, 1]$ belongs to the root locus
- ◇ Intersection with the imaginary axis, set $s = \mathrm{i}\omega$:

$$(\mathrm{i}\omega+1)(\mathrm{i}\omega-1)(\mathrm{i}\omega+5)+K(1+0.5\mathrm{i}\omega)=0$$

$$\Longleftrightarrow \begin{cases} -\omega^3+\omega(0.5K-1)=0 \\ -5\omega^2-5+K=0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \omega = 0 \\ K = 5 \end{cases} \text{ or } \begin{cases} \omega^2 = -1 \text{ , not real!} \\ K = 0 \end{cases}$$

This gives the root locus in Figure 3.10b.

Answer: The system is asymptotically stable (all poles in the LHP) if $K > 5$.

3.11 a) The closed loop system

$$G_c(s) = \frac{\frac{k}{s(s+2)}}{1+\frac{ka}{s(s+2)(s+a)}} = \frac{k(s+a)}{s(s+2)(s+a)+ka}$$

has the characteristic equation

$$s(s+2)(s+a)+ka=0$$

Choose $k=6$, and draw a root locus with respect to a. The characteristic equation can be written

$$s^3+2s^2+a(s^2+2s+6)=0$$

that is,

$$P(s)=s^2(s+2) \qquad Q(s)=s^2+2s+6$$

- ◇ Starting points: 0, 0, −2
  End points: $-1 \pm \mathrm{i}\sqrt{5}$
- ◇ Number of asymptotes: $3-2=1$, direction: π.
- ◇ Parts of the real axis: $(-\infty, -2]$
- ◇ Intersection with the imaginary axis: $s=\mathrm{i}\omega$

$$6a-\omega^2(2+a)+\mathrm{i}\omega(2a-\omega^2)=0$$

Im: $\omega(2a-\omega^2)=0 \quad \Longleftarrow \quad \omega=0$ or $\omega^2=2a$

Re: $6a-\omega^2(2+a)=0$

$$\omega=0 \quad \Longleftarrow \quad a=0$$

$$\omega^2=2a \quad \Longleftarrow \quad 2a-2a^2=0 \quad \Longleftarrow \quad a=0 \text{ or } a=1$$

Intersection points: $s=0$, $s=\pm\sqrt{2}$

This gives the root locus in Figure 3.11a.

Answer: The system is asymptotically stable for $a>1$

b) For y to have a stationary value of 1 the system must first of all be stable. When the system is stable, the stationary value will be 1 when r is a unit step since the system contains an integrator.

Next, consider

$$y_{\mathrm{m}}(t)=\sin(10t) \Rightarrow y_{\mathrm{f}}(t)=\left|\frac{a}{a+10\mathrm{i}}\right|\sin(10t+\varphi)$$

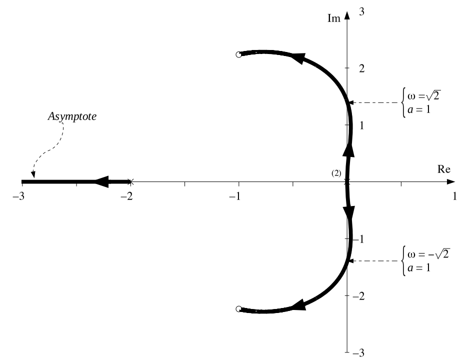


Figure 3.11a

(the expression for $y_{\mathrm{f}}(t)$ is valid after a long time, that is, when the transient has vanished). The amplitude is given by

$$A=\left|\frac{a}{a+10\mathrm{i}}\right|=\frac{1}{\sqrt{1+\frac{100}{a^2}}}$$

Now, choose a as small as possible but $a>1$ to maintain stability. The lowest amplitude is $A \approx 0.1$.

Answer: $A=0.1$

3.12 When K is small the system has a real unstable pole, that is, the magnitude of the step response grows without bound and the step response has no oscillations $\Rightarrow K=4$ corresponds to step response C.

When K is larger we have an unstable complex-conjugated pole pair, that is, the magnitude of the step response grows without bound and the step response is oscillative. $\Rightarrow K=10$ corresponds to step response D.

For even larger values of K all poles end up in the LHP. As K grows the step response becomes faster since the dominating poles move away from the origin. $K=18$ corresponds to step response B and $K=50$ to step response A.

Answer:

K	Step
4	C
10	D
18	B
50	A

3.13

$$G(s) = \frac{s^{n-1} + b_1 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{T_{n-1}(s)}{N_n(s)}$$

With a proportional feedback the closed loop system becomes

$$G_c(s) = \frac{G(s)}{1 + KG(s)} = \frac{T_{n-1}(s)}{N_n(s) + KT_{n-1}(s)}$$

with the characteristic equation

$$N_n(s) + KT_{n-1}(s) = 0$$

that is,

$$P(s) = N_n(s) \qquad Q(s) = T_{n-1}(s)$$

- Starting points: The zeros of $N_n(s)$
  End points: The zeros of $T_{n-1}(s)$
- Number of asymptotes: 1 since $\deg N_n(s) - \deg T_{n-1}(s) = 1$
  Direction: π

When K tends to infinity, one root approaches $-\infty$, the remaining roots approaches the zeros of $T_{n-1}(s)$. The zeros of $T_{n-1}(s)$ are in the LHP according to the problem formulation. Hence, if K is large enough, the system is asymptotically stable.

3.14 Since $q_{\text{out},\Delta}(t) = 0$ we get

$$\frac{d}{dt} h_\Delta(t) = \frac{1}{A}(q_{\text{in},\Delta}(t) - q_{\text{out},\Delta}(t)) = \frac{1}{A} q_{\text{dam},\Delta}(t - T)$$
$$= \frac{K}{A}(h_{\text{ref},\Delta}(t - T) - h_\Delta(t - T))$$

which gives

$$sH_\Delta(s) = \frac{K}{A} e^{-sT}(H_{\text{ref},\Delta}(s) - H_\Delta(s))$$

The transfer function of the open loop system is hence

$$G_o(s) = \frac{K}{A} \cdot \frac{e^{-sT}}{s}$$

Now draw the Nyquist curve:

- Big semi-circle in the RHP:

  $$s = Re^{i\theta} \qquad -\pi/2 < \theta < \pi/2$$

  Since $\operatorname{Re} s > 0$ we have $\left|e^{-sT}\right| < 1$, that is,

  $$|G_o(s)| < \frac{K}{A} \cdot \frac{1}{R}$$

  The large half circle is hence mapped onto the origin.

- Imaginary axis:

  $$|G_o(i\omega)| = \frac{K}{A} \cdot \frac{1}{\omega} \qquad \arg G_o(i\omega) = -\frac{\pi}{2} - \omega T$$

  As ω goes from r to R, the gain monotonically decreases towards zero and the argument goes from $-\pi/2$ to $-\infty$. The resulting Nyquist curve makes a spiral motion towards the origin. The first time the curve crosses the real axis is for $\omega T = \pi/2$, that is, $\omega = \pi$. The absolute value is then $\frac{K/A}{\pi}$.

- Small semi-circle to the right of the origin:

  $$G_o(re^{i\omega}) \approx \frac{K}{A} \cdot \frac{1}{r} \cdot e^{-i\omega}$$

  The small half circle is hence mapped into a large half circle in the RHP.

This gives the Nyquist path in Figure 3.14a. The system $G_o(s)$ has no poles in the RHP. According to the Nyquist criterion, the closed loop system is asymptotically stable if the Nyquist curve does not enclose the point -1. In this case the condition reads

$$\frac{K/A}{\pi} < 1$$

Answer: $K/A < \pi$

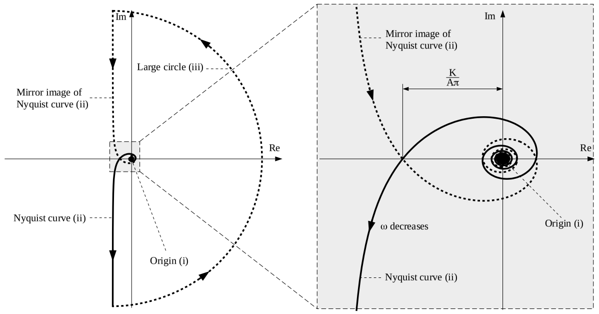


Figure 3.14a

3.15 The system $G(s)$ has no poles in the RHP. The closed loop system is asymptotically stable if the Nyquist curve of $KG_o(s)$ does not enclose the point -1. In the problem, Nyquist diagrams for $G(s)$ are given. The axes must hence be rescaled with a factor K.

a) (i) Yes. (ii) Yes. (iii) No. (iv) Yes.

b) (i) Stable if $0.4K < 1$, that is, $K < 2.5$.

(ii) Stable for $K > 0$.

(iii) Stable if $2K < 1$, that is, $K < 1/2$.

(iv) Stable if $4K < 1$ or $2K > 1$, that is, $K < 1/4$ or $K > 1/2$.

3.16 a) $G(i\omega) = \frac{1}{i\omega}$ gives

$$|G(i\omega)| = \frac{1}{\omega} \qquad \arg G(i\omega) = -90^\circ$$

b) $G(i\omega) = \frac{1}{-\omega^2}$ gives

$$\arg G(i\omega) = \frac{1}{\omega^2} \qquad \arg G(i\omega) = -180^\circ$$

This gives the Nyquist curves in Figure 3.16a.

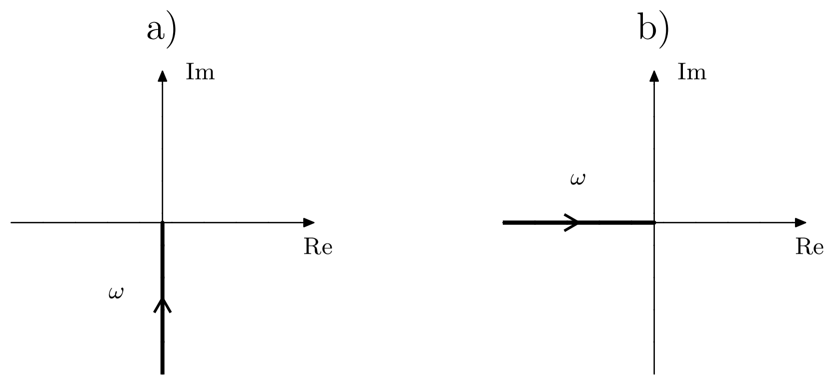


Figure 3.16a

3.17 a) Since $G(i\omega) \to 0,\ \omega \to \infty$, we assume that the large half circle is mapped onto the origin. The small half circle is mapped onto the point 2. The point -1 is not encircled by the curve. This means that the closed loop system is stable if $1.5 \cdot K < 1$. Hence $K < 2/3$.

b)

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \cdot \frac{1}{1+KG(s)} \cdot \frac{1}{s} = \frac{1}{1+2K}$$

for $K < 2/3$ according to a.

c) The Nyquist criterion can also be applied to

$$\frac{K}{s} \cdot G(s)$$

as the open loop system. On the large half circle $\frac{1}{s} \approx 0$ which means that it is mapped onto the origin even for $\frac{1}{s} \cdot G(s)$. On the small half circle

$$s = r \cdot e^{i\theta} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

we have $G(s) \approx 2$ and

$$\frac{1}{s} = \frac{1}{r} e^{-i\theta}$$

Hence, it is transformed by $\frac{1}{s} \cdot G(s)$ to a large half circle in the RHP. Setting $s = i\omega$ in $\frac{1}{s}$ gives the absolute value $\frac{1}{\omega}$ and the argument $-\pi/2$.

The Nyquist curve is turned 90° and "increased" by a factor $\frac{1}{\omega}$. This gives the Nyquist path in Figure 3.17a.

Answer: The closed loop system is asymptotically stable if $\frac{3}{2}K < 1$. This means that also in this case we have $K < 2/3$.

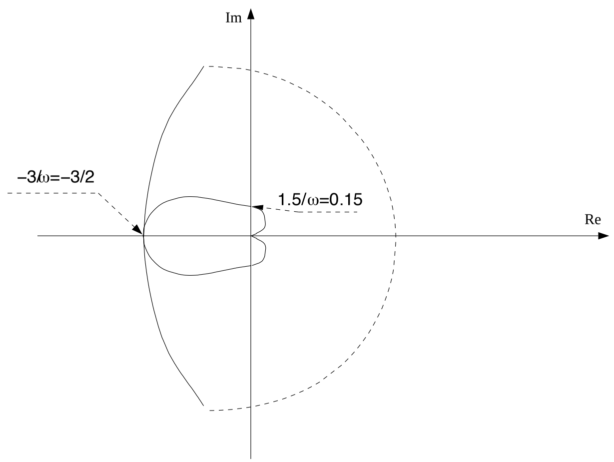


Figure 3.17a

3.18 The system

$$G(s) = \frac{1}{s(1+\tau s)}$$

is controlled by

$$u(t) = -Ky(t-T)$$

which gives the open loop system

$$G_o(s) = Ke^{-sT}G(s)$$

During self oscillations the open-loop gain is equal to -1:

$$Ke^{-i\omega T}G(i\omega) = -1$$

that is,

$$Ke^{-i\omega T} \cdot \frac{1}{\omega}e^{-i\frac{\pi}{2}} \frac{1}{\sqrt{\omega^2\tau^2+1}}e^{-i\arctan\omega\tau} = e^{-i\pi}$$

$\omega = 1$ gives

$$\begin{cases} -T - \frac{\pi}{2} - \arctan\tau = -\pi & (1) \\ \frac{K}{\sqrt{\tau^2+1}} = 1 & (2) \end{cases}$$

$K_1 = \frac{1}{3}K$ gives a self oscillations with $\omega = 0.5$. This gives

$$\begin{cases} -\frac{T_1}{2} - \frac{\pi}{2} - \arctan\frac{\tau}{2} = -\pi & (3) \\ \frac{K_1}{0.5\cdot\sqrt{\frac{\tau^2}{4}+1}} = 1 & (4) \end{cases}$$

The equations (1) - (4) give $\tau = 1.69$ and hence

$$T = \frac{\pi}{2} \cdot \arctan\tau = 0.53$$

$$T_1 = \pi - 2\arctan\frac{\tau}{2} = 1.74$$

3.19 From the Nyquist curve it is seen that for $\omega = 1$

$$\arg G(1i) = -135^\circ \qquad |G(1i)| = 1/\sqrt{2}$$

and

$$\arg F(1i) = -45^\circ \qquad |F(1i)| = K/\sqrt{2}$$

This gives $\arg F(1i)G(1i)) = -180^\circ$. According to the Nyquist criterion, asymptotic stability is achieved if

$$|F(1i)G(1i)| = K/2 < 1 \quad \Rightarrow \quad K < 2$$

3.20 Since $|G(i\omega)|$ does not tend to ∞ as $\omega \to 0$ the system does not have integrating factor for $K = 0$. Thus reject root locus no 2. Furthermore, since the gain can be increased arbitrarily without causing the Nyquist curve to encircle -1, that is, without making the closed loop system unstable, we reject root loci 3 and 4.

Answer: Root locus no 1.

3.21 P $\Rightarrow$ $b_0 = b_2 = 0$
I $\Rightarrow$ $b_0 = b_1 = 0$
D $\Rightarrow$ $b_1 = b_2 = 0$

3.22 a) The characteristic equation of the closed loop system is given by

$$(s^2 + s + 1)(s + 0.2) + K_P \cdot 0.2 = 0$$

that is,

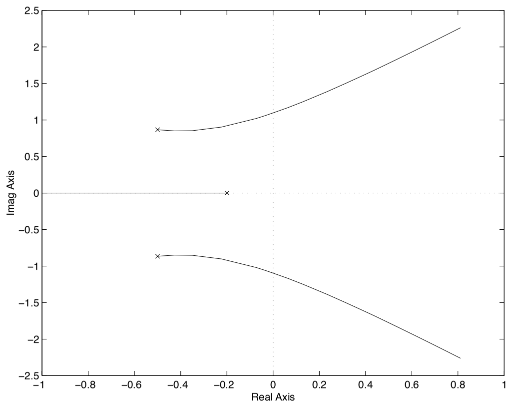
$$P(s) = (s^2 + s + 1)(s + 0.2) \qquad Q(s) = 0.2$$

Enter $P(s)$ and $Q(s)$.

```
>> s = tf( 's' );
>> P = ( s^2 + s + 1) * ( s + 0.2 );
>> Q = 0.2;
```

Draw the root locus. Click in the figure to determine the imaginary axis crossings.

```
>> rlocus( Q / P )
```



When K_P increases the two complex poles move towards the imaginary axis, that is, the closed loop system becomes more oscillatory. Finally, for $K_P \approx 6.2$, the poles cross the imaginary axis and the closed loop system becomes unstable. This result is in accordance with Problem **??**. For small values of K_P the properties of the step response are mainly determined by the real pole close to the origin. For larger values the complex poles start to dominate and when the complex poles cross the imaginary axis the amplitude of the oscillations in the step response increases and the system becomes unstable.

Note, however, that the root locus alone does not give sufficient information to tell how the stationary error changes with the parameter.

b) The characteristic equation of the closed loop system using the PI controller with $K_P = 1$ is given by

$$s((s^2 + s + 1)(s + 0.2) + 0.2) + K_I \cdot 0.2 = 0$$

that is,

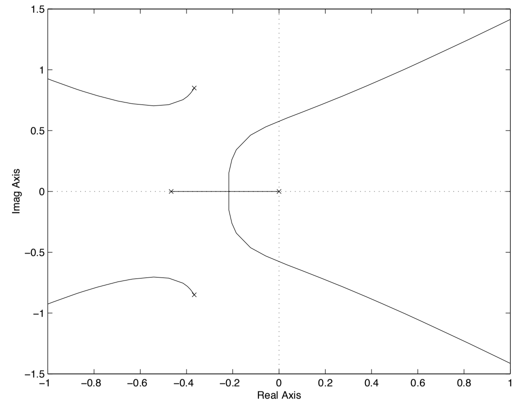
$$P(s) = s(s^3 + 1.2s^2 + 1.2s + 0.4) \qquad Q(s) = 0.2$$

Enter $P(s)$ and $Q(s)$.

```
>> P = s * ( s^3 + 1.2*s^2 + 1.2*s + 0.4 );
>> Q = 0.2;
```

Draw the root locus. Click in the figure to determine the imaginary axis crossings.

```
>> rlocus( Q / P )
```



For small K_I the response of the closed loop system is dominated by the poles on the real axis close to the origin. When K_I increases the poles become complex and move towards the imaginary axis, that is, the closed loop system becomes more oscillatory. Finally, for $K_I \approx 1.5$, the poles cross the imaginary axis, that is, the closed loop system becomes unstable. As can be seen in Problem **??** a small value of K_I, that is, a pole close to the origin, gives a slow step response. When K_I increases the dominating poles become complex and the step response becomes oscillatory.

A large settling time will typically follow if the system is slow or have poor damping. Here, the large settling time for small K_I is due to the system being slow. That the steady state error is eliminated cannot easily be seen in the root locus.

c) Using PID control with $K_P = 1, K_I = 1$ and $T = 0.1$ the characteristic equation of the closed loop system is given by

$$(0.1s + 1)(s(s^2 + s + 1)(s + 0.2) + 0.2(s + 1)) + K_D \cdot 0.2s^2 = 0$$

that is,

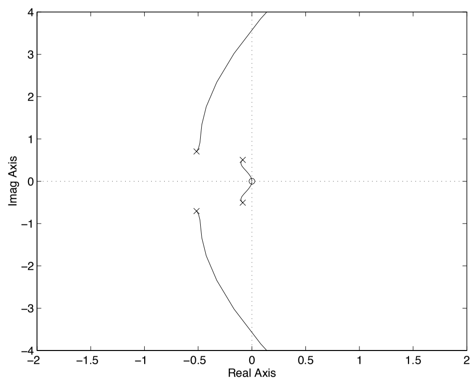
$$P(s) = (0.1s + 1)(s^4 + 1.2s^3 + 1.2s^2 + 0.4s + 0.2) \qquad Q(s) = 0.2s^2$$

Enter $P(s)$ and $Q(s)$.

```
>> P = ( 0.1*s + 1 ) * ...
       ( s^4 + 1.2*s^3 + 1.2*s^2 + 0.4*s + 0.2 );
>> Q = 0.2*s^2;
```

Draw the root locus. By changing the axes or using the function `zoom` the region of interest can be seen more clearly.

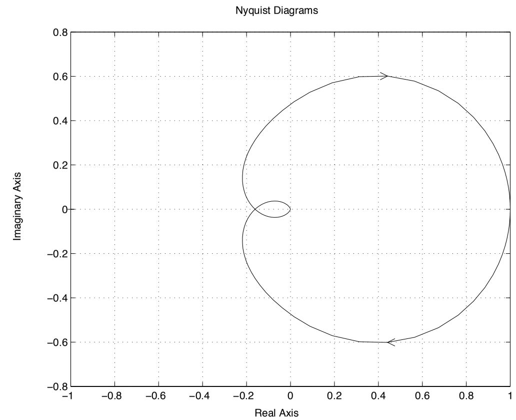
```
>> rlocus( Q / P )
>> axis([ -2 2 -4 4 ])
```



When K_D increases the complex poles closest to the origin move towards the origin and and at the same time the damping of the system is increased. When K_D increases even more the second pair of complex poles moves towards the imaginary axis giving a high frequency oscillation which finally gives instability.

3.23 a) Enter the system and the regulator. Plot the Nyquist curve of the open loop system.

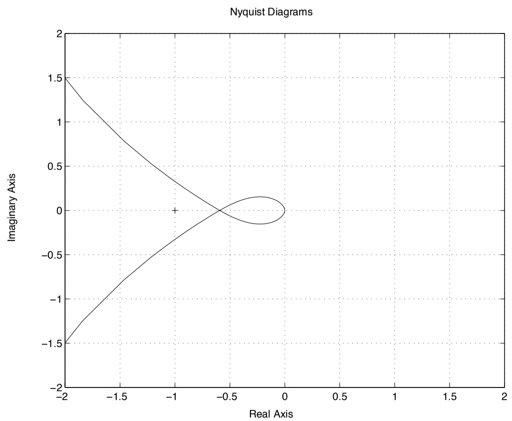
```
>> s = tf( 's' );
>> G = 0.2 / ( ( s^2 + s + 1 ) * ( s + 0.2) );
>> F = 1;
>> nyquist( F * G )
```



The Nyquist curve is “far away” from the point -1 for all frequencies and the step response of the closed loop system is well damped. As K_P increases the Nyquist curve grows in size and for $K_P = 6.2$ the Nyquist curve reaches -1 and thus is the limit of stability.

b) Generate a PI controller. Plot the Nyquist curve of the open loop system.

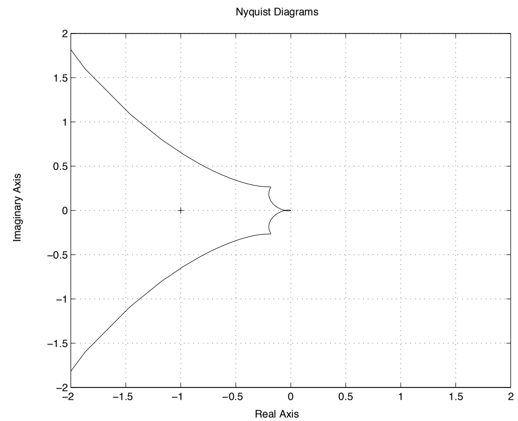
```
>> F = 1 + 1/s;
>> nyquist( F * G )
>> axis([ -2 2 -2 2 ])
```



For low frequencies the Nyquist curve is now far away from the origin since the integrating part makes $|G(i\omega)|$ large for low frequencies. The Nyquist curve now passes closer to -1 which results in a more oscillatory closed loop system. The system becomes unstable around $K_I = 1.44$.

c) Generate a PID controller. Plot the Nyquist curve of the open loop system. Here with the parameters $K_P = 1$, $K_I = 1$, $K_D = 2$, and $T = 0.1$

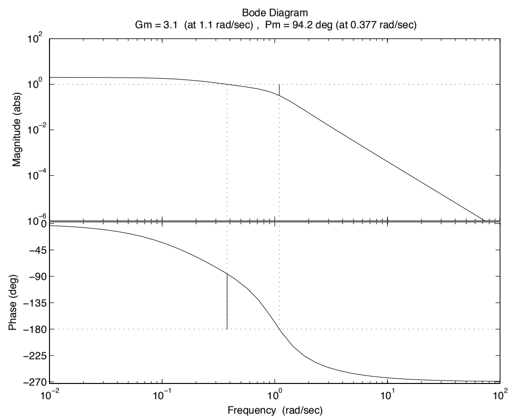
```
>> F = 1 + 1/s + 2*s / ( 0.1*s + 1 );
>> nyquist( F * G )
>> axis([ -2 2 -2 2 ])
```



The Nyquist curve is now further away from -1 which corresponds to an improved damping of the closed loop system. The system becomes unstable around $K_D = 66$.

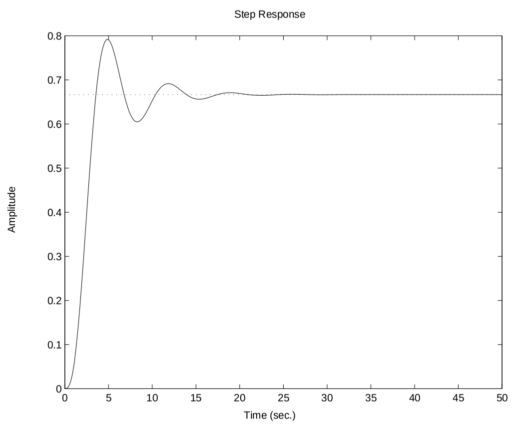
3.24 a) Enter the systems and the regulator. Make a Bode plot of the open loop system when the regulator and the system are put in series. This gives $\omega_c = 0.38$, $\omega_p = 1.1$, $\varphi_m = 94°$ and $A_m = 3.1$.

```
>> s = tf( 's' );
>> G = 0.4 / ( ( s^2 + s + 1 ) * ( s + 0.2 ) );
>> F = 1;
>> margin( F * G )
```



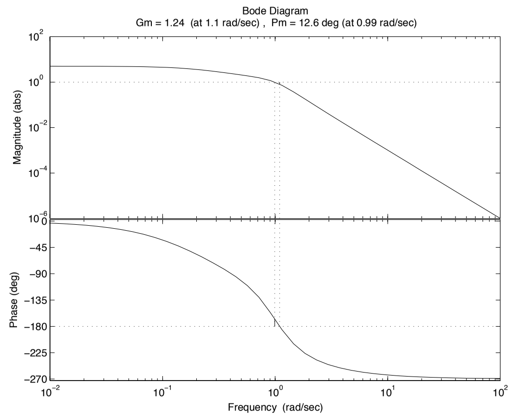
Plot the step response.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 )
```



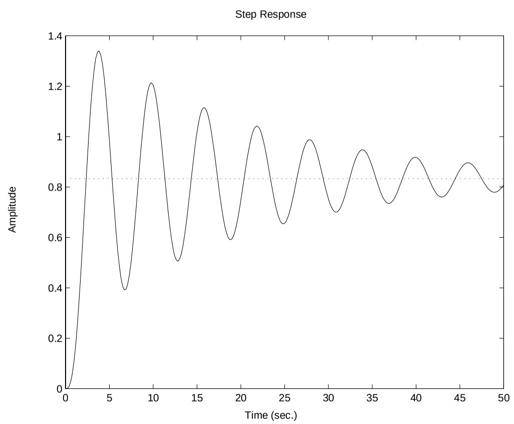
b) Increase the gain in the regulator. Make a Bode plot. The crossover frequency ω_c has increased while ω_p is the same, since only the amplitude curve is changed when the gain is changed. Both the gain and phase margins have decreased.

```
>> F = 2.5;
>> margin( F * G )
```



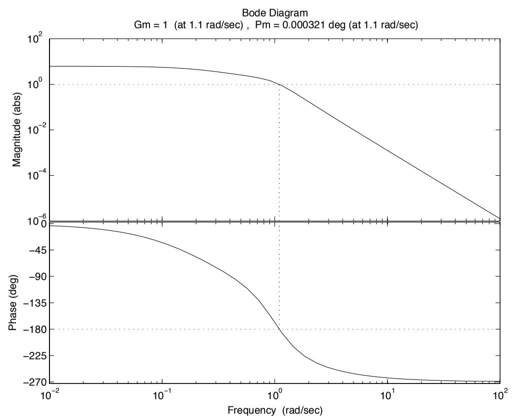
Plot the step response. The closed loop system is now much more oscillatory due to the reduced phase and gain margins.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 )
```



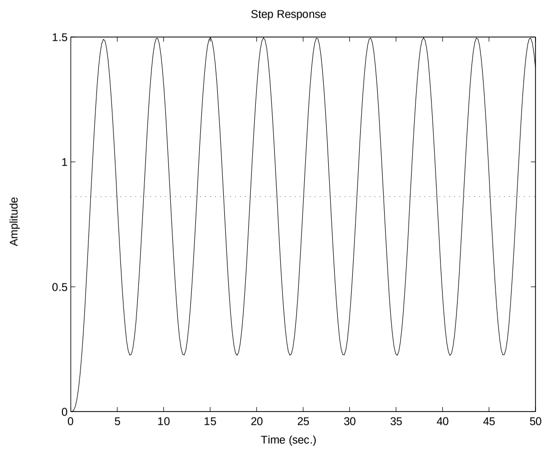
c) Increase the gain to 3.1, that is, the value of A_m in a). Both the gain and phase margin are at the limit between what would give an stable or unstable closed loop system. Any further increase of the gain will give an unstable closed loop system.

```
>> F = 3.1;
>> margin( F * G )
```



Plot the step response. The output now oscillates with constant amplitude.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 )
```



3.25 The top row gives a steady state error $\Rightarrow K_I = 0$. Left column less oscillative than the right one $\Rightarrow K_D \neq 0$.

Answer: A–*iii*, B–*i*, C–*iv*, D–*ii*.

3.26 a) The motor transfer function is (from Solution 2.1))

$$\frac{\theta(s)}{U(s)} = G(s) = \frac{k_0}{s(s+1/\tau)}$$

Feedback control

$$U(s) = F(s)(\theta_{\text{ref}}(s) - \theta(s))$$

where $F(s)$ is the control law transfer function and θ_{ref} is the reference signal. The closed loop transfer function is given by

$$G_c(s) = \frac{\theta(s)}{\theta_{\text{ref}}(s)} = \frac{F(s)G(s)}{1+F(s)G(s)}$$

Proportional feedback $F(s) = K_P$ and $G(s)$ according to above give

$$G_c(s) = \frac{K_P k_0}{s^2 + s/\tau + K_P k_0}$$

The poles of the closed loop system are given by

$$s^2 + s/\tau + K_P k_0 = 0$$

that is,

$$s = \frac{-1 \pm \sqrt{1 - 4\tau^2 K_P k_0}}{2\tau}$$

(1) K_P small $\Rightarrow$ Both poles on the real axis, but one pole very close to the origin $\Rightarrow$ Slow but not oscillatory system.

(2) $K_P = 1/(4\tau^2 k_0)$ $\Rightarrow$ Both poles in $-1/(2\tau)$, that is, faster than in (1) but still no oscillations.

(3) K_P large $\Rightarrow$ Complex poles with large imaginary part relative to the real part, that is oscillative system.

b) The transfer function from the reference signal to the tracking error $e = \theta_{\text{ref}} - \theta$ is given by

$$E(s) = \frac{1}{1+F(s)G(s)}\theta_{\text{ref}}(s) = \frac{s(s+1/\tau)}{s(s+1/\tau)+K_{\text{P}}k_0}\theta_{\text{ref}}(s)$$

The reference signal is a step

$$\theta_{\text{ref}}(t) = \begin{cases} 0, & t < 0 \\ A, & t \geq 0 \end{cases}$$

which gives

$$\theta_{\text{ref}}(s) = \frac{A}{s}$$

The final value theorem gives

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s \cdot \frac{s(s+1/\tau)}{s(s+1/\tau)+K_{\text{P}}k_0} \cdot \frac{A}{s} = 0$$

The reference signal is a ramp

$$\theta_{\text{ref}}(t) = \begin{cases} 0, & t < 0 \\ At, & t \geq 0 \end{cases}$$

which gives

$$\theta_{\text{ref}}(s) = \frac{A}{s^2}$$

The final value theorem gives (the closed loop is asymptotically stable for all K_{P} according to a))

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s \cdot \frac{s(s+1/\tau)}{s(s+1/\tau)+K_{\text{P}}k_0} \cdot \frac{A}{s^2} = \frac{A}{K_{\text{P}}k_0\tau}$$

The error can be decreased by selecting K_{P} large, but according to a) the system becomes very oscillative for large K_{P}.

c) PI controller

$$u(t) = K_{\text{P}}e(t) + K_{\text{I}}\int_0^t e(\tau)\,\mathrm{d}\tau$$

that is

$$F(s) = K_{\text{P}} + K_{\text{I}}\frac{1}{s}$$

gives

$$E(s) = \frac{1}{1+F(s)G(s)}\theta_{\text{ref}}(s) = \frac{s^2(s+1/\tau)}{s^2(s+1/\tau)+k_0(K_{\text{P}}s+K_{\text{I}})}\theta_{\text{ref}}(s)$$

When θ_{ref} is a ramp according to b) we get

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = 0$$

Comment: The final value theorem can only be used when the denominator of $G(s)U(s)$ has all zeros in the left half plane or at the origin. $G(s)$ is the system transfer function and $U(s)$ is the input signal.

3.27 The transfer function for the loop gain is G_{o}.

The transfer function from the reference signal R to the output Y is obtained by using the block diagram and observing that

$$Y = G_{\text{o}}(R - Y)$$

Solving this equation for Y gives

$$Y = \frac{G_{\text{o}}}{1+G_{\text{o}}}R$$

that is, the transfer function for the closed loop system is $G_{\text{c}} = \frac{G_{\text{o}}}{1+G_{\text{o}}}$.

3.28 a) The loop gain, G_{o}, is FG.

b) The influence of the disturbance ($N = 0$) can be neglected. Use the solution to problem Solution 3.27. The transfer function from R to Y is $G_{\text{c}} = \frac{FG}{1+FG}$, that is, $Y = G_{\text{c}}R$.

c) The influence of the reference signal can be neglected. ($R = 0$). The block diagram gives

$$Y = FGE = -FG(Y + N)$$

which implies that the transfer function from N to Y is $G_{ny} = -\frac{FG}{1+FG}$.

d) The influence of the disturbance can be neglected ($N = 0$). The block diagram gives

$$E = R - Y = R - FGE$$

Solving for E gives

$$E = \frac{1}{1+FG}R$$

that is, the transfer function from R to E is $G_{re} = \frac{1}{1+FG}$.

3.29 a) The transfer function from reference signal to error signal is (see Solution 3.28d)

$$E = \frac{1}{1+FG}R(s) = \frac{1}{1+\frac{K}{(s+1)(s+3)}}R(s) = \frac{(s+1)(s+3)}{(s+1)(s+3)+K}R(s)$$

$r(t)$ step $\Rightarrow$ $R(s) = \frac{A}{s}$. The steady state value of the error is given by the final value theorem

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{(s+1)(s+3)}{(s+1)(s+3)+K}A = \frac{3A}{3+K}$$

b) In order to make the steady state error equal to zero the regulator has to contain an integrator. Using, for example, $F(s) = \frac{1}{s}$ one gets

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{A}{1+F(s)G(s)} = \lim_{s\to 0} \frac{A}{1+\frac{1}{s}\frac{1}{(s+1)(s+3)}} = 0$$

Notice though, that the integrating feedback normally has to be combined with proportional feedback.

c) The transfer function from R to Y using $F(s) = 1$ is

$$G_c(s) = \frac{FG}{1+FG} = \frac{1}{(s+1)(s+3)+1} = \frac{1}{s^2+4s+4} = \frac{1}{(s+2)^2}$$

The system has two poles in -2 and no zeros.

3.30 • The four step responses are characterized by, for example, that A and D have a steady state error, while C and B do not. Further, A shows better damping than D, and C shows better damping than B. It can also be noticed (although it is not as apparent as the other characteristics) that the error decays more slowly in C than in B.

• The four regulators are characterized by, for example, that regulators 1 and 4 don't have any integral action. Regulator 2 has more integral action than 3, and regulator 4 gives better damping than 1.

• The derivative part in the regulator improves the damping, while integral action eliminates the steady state error and reduces the damping. Besides, for small values of K_I, the error will decay slowly to zero.

Answer: A–4, B–2, C–3, D–1.

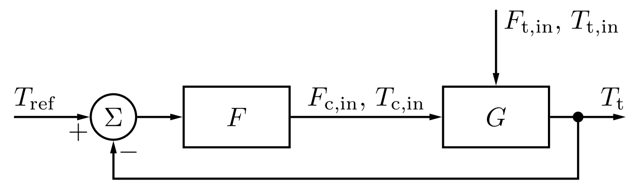


Figure 3.31a

3.31 a) See the block diagram in Figure 3.31a. There, the signals are classified as:

◇ Input $F_{c,in}$ and $T_{c,in}$

◇ Output T_t

◇ Disturbance $F_{t,in}$ and $T_{t,in}$

b) Assume prefect mixing in the tank. Mass balance for the tank

$$\frac{d(\rho_t V_t)}{dt} = \rho_{t,in}F_{t,in} - \rho_t F_t$$

Assume $\rho_{t,in} = \rho_t$ and that ρ_t is constant which gives

$$\frac{d(\rho_t V_t)}{dt} = 0 = F_{t,in} - F_t \quad \Rightarrow \quad F_t = F_{t,in}$$

Assume that there are no heat losses to the surroundings. The energy balance for the tank is

$$\frac{d\left(\rho_t V_t c_t^p (T_t - T_{ref})\right)}{dt} = \rho_t F_{t,in} c_{t,in}^p (T_{t,in} - T_{ref}) - \rho_t F_t c_t^p (T_t - T_{ref}) + U(T_c - T_t) \quad (3.1)$$

where U is a heat transfer constant. Assume that $c^{\mathrm{p}}_{\mathrm{t,in}} = c^{\mathrm{p}}_{\mathrm{t}}$ is constant and that T_{ref} is constant. This means that (3.1) can be simplified to

$$V_{\mathrm{t}}\frac{\mathrm{d}T_{\mathrm{t}}}{\mathrm{d}t} = F_{\mathrm{t}}(T_{\mathrm{t,in}} - T_{\mathrm{t}}) + \frac{U}{c^{\mathrm{p}}_{\mathrm{t}}\rho_{\mathrm{t}}}(T_{\mathrm{c}} - T_{\mathrm{t}}) \tag{3.2}$$

Mass balance for the heating system

$$\frac{\mathrm{d}(\rho_{\mathrm{c}}V_{\mathrm{c}})}{\mathrm{d}t} = \rho_{\mathrm{c,in}}F_{\mathrm{c,in}} - \rho_{\mathrm{c}}F_{\mathrm{c}}$$

Assume $\rho_{\mathrm{c,in}} = \rho_{\mathrm{c}}$ and that ρ_{c} is constant which gives $F_{\mathrm{c}} = F_{\mathrm{c,in}}$. Assume that there are no heat losses to the surroundings. The energy balance for the heating system is

$$\begin{aligned}&\frac{\mathrm{d}\left(\rho_{\mathrm{c}}V_{\mathrm{c}}c^{\mathrm{p}}_{\mathrm{c}}(T_{\mathrm{c}} - T_{\mathrm{ref}})\right)}{\mathrm{d}t}\\ &\quad= \rho_{\mathrm{c}}F_{\mathrm{c,in}}c^{\mathrm{p}}_{\mathrm{t,in}}(T_{\mathrm{c,in}} - T_{\mathrm{ref}}) - \rho_{\mathrm{c}}F_{\mathrm{c}}c^{\mathrm{p}}_{\mathrm{c}}(T_{\mathrm{c}} - T_{\mathrm{ref}}) - U(T_{\mathrm{c}} - T_{\mathrm{t}})\end{aligned} \tag{3.3}$$

Assume that $c^{\mathrm{p}}_{\mathrm{c}} = c^{\mathrm{p}}_{\mathrm{t,in}}$ is constant. This means that (3.3) can be simplified to

$$V_{\mathrm{c}}\frac{\mathrm{d}T_{\mathrm{c}}}{\mathrm{d}t} = F_{\mathrm{c}}(T_{\mathrm{c,in}} - T_{\mathrm{c}}) - \frac{U}{c^{\mathrm{p}}_{\mathrm{c}}\rho_{\mathrm{c}}}(T_{\mathrm{c}} - T_{\mathrm{t}}) \tag{3.4}$$

The dynamical model is described by (3.2) and (3.4).

c) Linearization of (3.2) and (3.4) (assuming $\rho_{\mathrm{c}} = \rho_{\mathrm{t}}$ and $c^{\mathrm{p}}_{\mathrm{c}} = c^{\mathrm{p}}_{\mathrm{t}}$) gives

$$\begin{aligned}V^*_{\mathrm{t}}\frac{\mathrm{d}T_{\mathrm{t}\Delta}}{\mathrm{d}t} &= -\left(F^*_{\mathrm{t}} + \frac{U}{c^{\mathrm{p}}_{\mathrm{t}}\rho_{\mathrm{t}}}\right)T_{\mathrm{t}\Delta} + F^*_{\mathrm{t}}T_{\mathrm{t,in}\Delta}\\ &\quad+ \frac{U}{c^{\mathrm{p}}_{\mathrm{t}}\rho_{\mathrm{t}}}T_{\mathrm{c}\Delta} + \left(T^*_{\mathrm{t,in}} - T^*_{\mathrm{t}}\right)F_{\mathrm{t}\Delta}\\ V^*_{\mathrm{c}}\frac{\mathrm{d}T_{\mathrm{c}\Delta}}{\mathrm{d}t} &= -\left(F^*_{\mathrm{t}} + \frac{U}{c^{\mathrm{p}}_{\mathrm{t}}\rho_{\mathrm{t}}}\right)T_{\mathrm{c}\Delta} + F^*_{\mathrm{c}}T_{\mathrm{c,in}\Delta}\\ &\quad+ \frac{U}{c^{\mathrm{p}}_{\mathrm{t}}\rho_{\mathrm{t}}}T_{\mathrm{c}\Delta} + \left(T^*_{\mathrm{c,in}} - T^*_{\mathrm{c}}\right)F_{\mathrm{c}\Delta}\end{aligned}$$

d) With numerical values for the stationary points and assuming that F_{t}, $T_{\mathrm{t,in}}$, and $T_{\mathrm{c,in}}$ is constant, the linearized model is

$$\frac{\mathrm{d}T_{\mathrm{t}\Delta}}{\mathrm{d}t} = -0.26T_{\mathrm{t}\Delta} + 0.16T_{\mathrm{c}\Delta} \tag{3.5}$$

$$\frac{\mathrm{d}T_{\mathrm{c}\Delta}}{\mathrm{d}t} = -3.6T_{\mathrm{c}\Delta} + 1.6T_{\mathrm{t}\Delta} + 200F_{\mathrm{c}\Delta} \tag{3.6}$$

Taking the Laplace transform of (3.5) and (3.6) gives

$$sT_{\mathrm{t}\Delta}(s) = -0.26T_{\mathrm{t}\Delta}(s) + 0.16T_{\mathrm{c}\Delta}(s) \tag{3.7}$$

$$sT_{\mathrm{c}\Delta}(s) = -3.6T_{\mathrm{c}\Delta}(s) + 1.6T_{\mathrm{t}\Delta}(s) + 200F_{\mathrm{c}\Delta}(s)$$

$$\Rightarrow T_{\mathrm{c}\Delta}(s) = \frac{1.6}{s+3.6}T_{\mathrm{t}\Delta}(s) + \frac{200}{s+3.6}F_{\mathrm{c}\Delta}(s) \tag{3.8}$$

Combining (3.7) and (3.8) gives

$$sT_{\mathrm{t}\Delta}(s) = -0.26T_{\mathrm{t}\Delta}(s) + \frac{0.256}{s+3.6}T_{\mathrm{t}\Delta}(s) + \frac{32}{s+3.6}F_{\mathrm{c}\Delta}(s)$$

$$\Rightarrow T_{\mathrm{t}\Delta}(s) = \frac{32}{(s+3.675)(s+0.185)}F_{\mathrm{c}\Delta}(s)$$

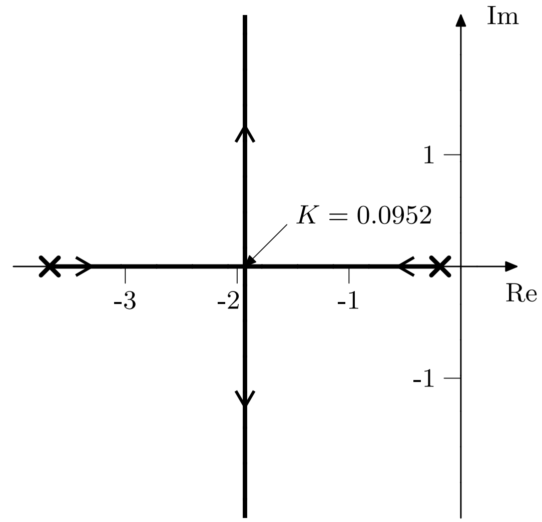


Figure 3.31b

e) The transfer function for the closed loop is

$$G_{\mathrm{c}}(s) = \frac{32K}{s^2 + 3.86s + 0.68 + 32K}$$

The characteristic equation is

$$s^2 + 3.86s + 0.68 + 32K = 0$$

with the solution

$$s = -1.93 \pm \sqrt{1.93^2 - 0.68 - 32K}$$

This gives the root locus in Figure 3.31b.

3.32 a) The system $G_o(s) = \frac{-1}{s^2+2s-3}$ has one pole in -3 and one pole in 1, hence the system is unstable.

b) The closed loop is given by

$$G_c(s) = \frac{-K}{s^2 + 2s - 3 - K}$$

The poles of the closed loop are given by

$$s = -1 \pm \sqrt{1 + 3 + K}$$

For $K \leq -3$ the closed loop will have all its pole in the LHP.

3.33 Given $\dot{y} = \mu y + u$ and $u = K(r - y)$ we have $\dot{y} = (\mu - K)y + Kr$. This system converges when the eigenvalues of $(\mu - K)$ are in the LHP, that is, when $K > \mu$.

3.34

a) The closed loop system

$$G_c(s) = \frac{G(s)K}{1 + G(s)K} = \frac{K(s+2)}{(s+1)^2 + K(s+2)}$$

has the characteristic equation

$$(s+1)^2 + K(s+2) = 0$$

which gives

$$P(s) = (s+1)^2 \qquad Q(s) = s + 2$$

- Starting points $\Longleftrightarrow$ Zeros of $P(s)$: -1,-1
  End points $\Longleftrightarrow$ Zeros of $Q(s)$: -2

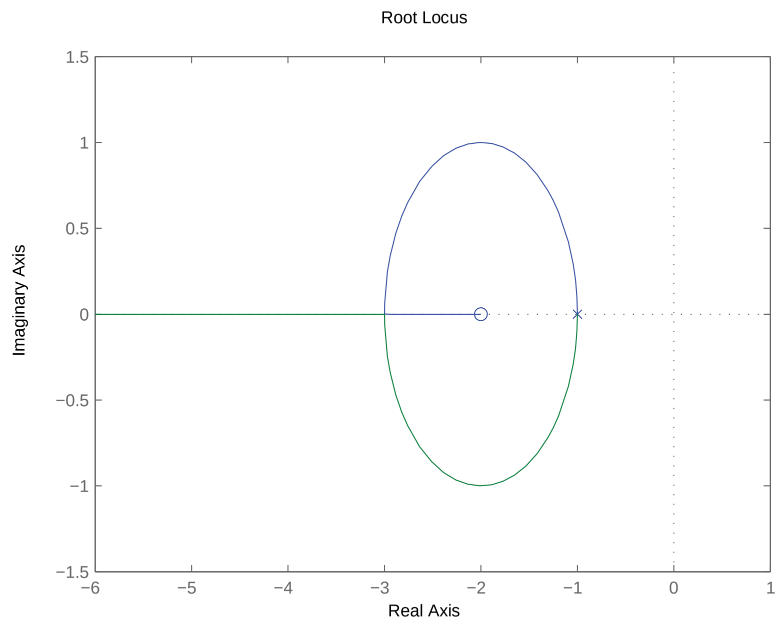


Figure 3.34a

- Number of asymptotes: $2 - 1 = 1$
  Direction: π
  Intersection point: $-1 - 1 + 2 = 0$
- Real axis: $(-\infty, -2]$ belongs to the root locus
- Intersection with the imaginary axis, set $s = j\omega$:

$$(j\omega + 1)^2 + K(j\omega + 2) = 0$$
$$\text{Im} : \omega(2 + K) = 0$$
$$\text{Re} : -\omega^2 + 1 + 2K = 0$$
$$\Longrightarrow \omega = 0, K = -\frac{1}{2}$$

which does not meet $K > 0$.

Intersection with the real axis, set $s = j\omega$:

$$(j\omega + a)^2 = (j\omega + 1)^2 + K(j\omega + 2)$$
$$\Longrightarrow (K = 0, a = 1), (K = 4, a = 3)$$

This gives the root locus in Figure 3.34a. The system is asymptotically stable. $K = 4$ (pole position -3) gives the fastest step response without fluctuations since it does not have any imaginary parts.

b) With a similar approach as in a), the closed loop system is

$$G_c(s) = \frac{G(s)F(s)}{1 + G(s)F(s)}$$

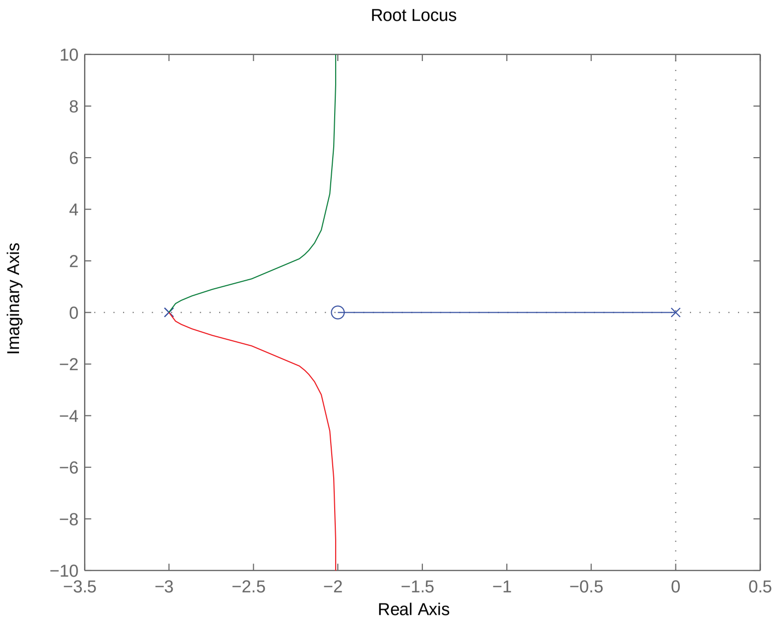


Figure 3.34b

where $F(s) = 4 + \frac{K_I}{s}$, $G(s) = \frac{s+2}{(s+1)^2}$. The characteristic equation is

$$1 + F(s)G(s) = s(s+1)^2 + (4s + K_I)(s+2) = 0$$

which gives

$$P(s) = s(s+1)^2 + 4s(s+2) = s(s+3)^2 \qquad Q(s) = K_I(s+2)$$

- Starting points $\Longleftrightarrow$ Zeros of $P(s)$: 0,-3,-3
  End points $\Longleftrightarrow$ Zeros of $Q(s)$: -2
- Number of asymptotes: $3 - 1 = 2$
  Direction: $\frac{\pi}{2}, \frac{3\pi}{2}$
  Intersection point: $\frac{0-3-3+2}{2} = -2$
- Intersection with the imaginary axis, set $s = j\omega$:

$$j\omega(j\omega + 3)^2 + K_I(j\omega + 2) = 0$$
$$\text{Im} : \omega(-\omega^2 + K_I + 9) = 0$$
$$\text{Re} : -6\omega^2 + 2K_I = 0$$
$$\Longrightarrow (\omega = 0, K = 0), (\omega^2 = K_I + 9, K_I = -\frac{54}{4} < -9) : \text{not real}$$

  which does not meet $K_I > 0$.

This gives the root locus in Figure 3.34b. The system is asymptotically stable.

c) The P-controller of a) gives a faster step response than the PI-controller of b) since the dominant pole $[-2, 0]$ is slower than -3. However, there is the stationary error of P-controller, see Figure 3.34c

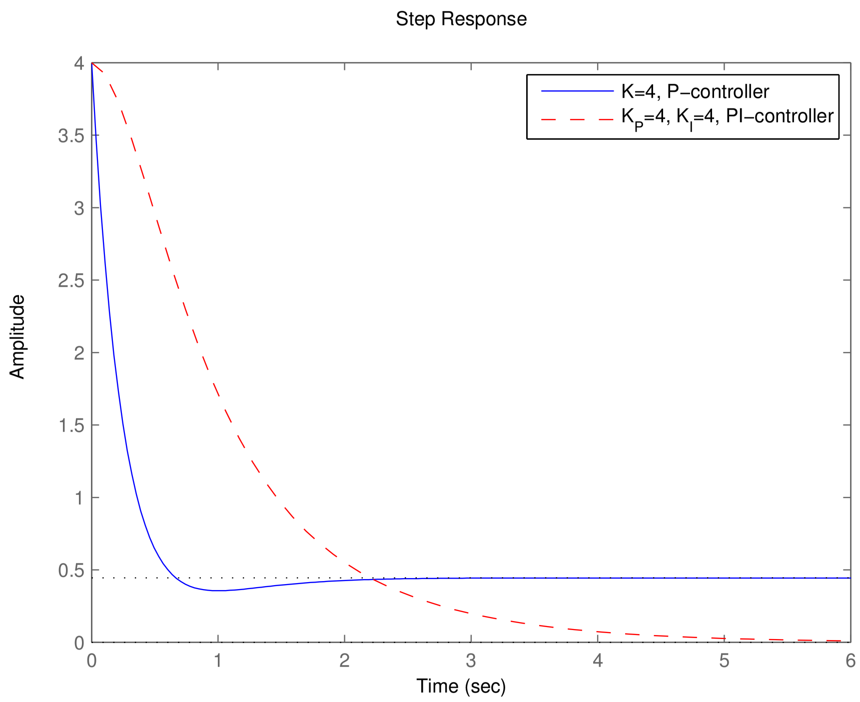

Step Response
Amplitude
Time (sec)
K=4, P−controller
K_P=4, K_I=4, PI−controller
0
0.5
1
1.5
2
2.5
3
3.5
4
0
1
2
3
4
5
6

Figure 3.34c

4 Frequency Description

4.1 If we let $\bar{u}(t)$ and $\bar{y}(t)$ denote the actual temperature and the measured temperature, respectively, we can divide the temperatures into their mean values and variations as follows:

$$\bar{u}(t) = u_0 + u(t)$$

and

$$\bar{y}(t) = y_0 + y(t)$$

where $u_0 = y_0 = 30$ °C.

The thermometer is modeled as the following first order linear time invariant dynamic system with

$$\frac{Y(s)}{U(s)} = G(s) = \frac{a}{s+b}$$

Since

$$u(t) = A\sin(\omega t)$$

it follows that after the transients have vanished (that is, in steady state)

$$y(t) = |G(\mathrm{i}\omega)|\, A\sin(\omega t + \phi)$$

where

$$\phi = \arg(G(\mathrm{i}\omega)) = -\arctan(\omega/b)$$

From the relationship $\omega = 2\pi/T$ and from the figure the following is obtained:

1. $\omega = \frac{2\pi}{0.314\cdot 60}$ rad/s $= 0.33$ rad/s
2. $\phi = \frac{-0.056}{0.314}\cdot 2\pi$ rad $= -1.12$ rad
3. $|G(\mathrm{i}\omega)| = \frac{0.9}{2.0} = 0.45$

Hence

$$\tan(\phi) = -\frac{\omega}{b} \quad \Rightarrow \quad b = \frac{0.33}{2.066} = 0.16$$

and

$$|G(\mathrm{i}\omega)| = \frac{a}{\sqrt{\omega^2 + b^2}} \quad \Rightarrow \quad a = 0.16$$

Answer:

$$G(s) = \frac{0.16}{s + 0.16}$$

4.2 The equation

$$\omega = \dot{\psi}$$

and

$$T_1 \cdot \dot{\omega} = -\omega + K_1 \cdot \delta$$

give the transfer function

$$G_{\mathrm{s}}(s) = \frac{K_1}{s(1+T_1 s)} = \frac{0.1}{s(1+s/0.01)}$$

The transfer function of the rudder machine is

$$G_{\mathrm{r}}(s) = \frac{1}{1+sT_2} = \frac{1}{1+s/0.1}$$

and the controller has the transfer function

$$F(s) = K\frac{1+s/a}{1+s/b} = K\frac{1+s/0.02}{1+s/0.05}$$

a) $K = 0.5$ gives

$$G_{\mathrm{o}}(s) = F(s)G_{\mathrm{r}}(s)G_{\mathrm{s}}(s) = \frac{0.05(1+s/0.02)}{s(1+s/0.01)(1+s/0.05)(1+s/0.1)}$$

It thus follows that

$$|G_{\mathrm{o}}(\mathrm{i}\omega)| = \frac{0.05\sqrt{1+(\frac{\omega}{0.02})^2}}{\omega\sqrt{1+(\frac{\omega}{0.01})^2}\sqrt{1+(\frac{\omega}{0.05})^2}\sqrt{1+(\frac{\omega}{0.1})^2}}$$

with low frequency asymptote

$$|G_{\mathrm{o}}(\mathrm{i}\omega)| \to \frac{0.05}{\omega},\ \omega \to 0$$

and

$$\arg G_{\mathrm{o}}(\mathrm{i}\omega) = \arctan\frac{\omega}{0.02} - 90^\circ - \arctan\frac{\omega}{0.01} - \arctan\frac{\omega}{0.05} - \arctan\frac{\omega}{0.1}$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote, $\left|\frac{0.05}{0.005}\right| = 10$, and the breakpoints and slopes of the asymptotes:

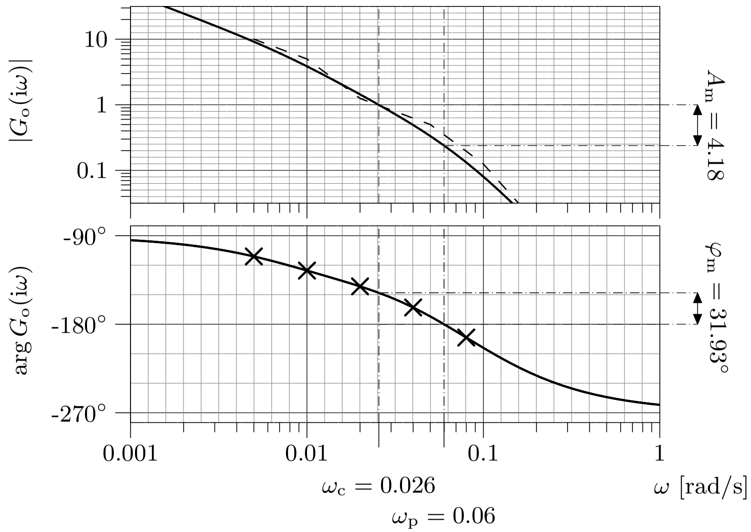


Figure 4.2a

Frequency [rad/s]		0.01		0.02		0.05		0.1	
Slope	−1		−2		−1		−2		−3

The phase shift is drawn based on a couple of samples:

Frequency [rad/s]	0.005	0.01	0.02	0.04	0.08
Phase	−111°	−125°	−142°	−163°	−194°

The Bode plot in Figure 4.2a gives: $\omega_c = 0.026$ rad/s, $\varphi_m = 32°$, $A_m = 4.2$.

b) The system starts to oscillate if K is chosen so that $\arg(G_o(i\omega_c)) = -180°$. This gives the crossover frequency $\omega_c = \omega_p = 0.06$ rad/s. This implies that the gain should be amplified 4.2 times. Therefore, choose $K = 0.5 \cdot 4.2 = 2.1$.

$$\omega = 2\pi/T \quad \Rightarrow \quad T = \frac{2\pi}{\omega_c} = \frac{2\pi}{0.06} = 105 \text{ s}$$

Answer: The period time will be 105 seconds, and $K = 2.1$.

c)
$$\Psi_{ref}(t) = A\sin(\alpha t)$$

gives

$$\Psi(t) = B\sin(\beta t + \varphi)$$

where $A = 5°$, $\alpha = 0.02$, $\beta = \alpha$, $B = A\,|G_c(i\alpha)|$ and $\varphi = \arg G_c(i\alpha)$. The transfer function for the closed loop system when $K = 0.5$ is

$$G_c(s) = \frac{G_o(s)}{1 + G_o(s)}$$

where

$$|G_o(i0.02)| = 1.44 \qquad \arg G_o(i0.02) = -142°$$

That is

$$G_o(i0.02) = -1.135 - i0.886$$

which gives

$$|G_c(i0.02)| = \frac{1.44}{\sqrt{0.135^2 + 0.886^2}} = 1.61 \quad \Rightarrow \quad B = 8°$$

and

$$\arg G_c(i0.02) = -142° + 180° - \arctan(\frac{0.886}{0.135}) = -0.76 \text{ rad}$$

Answer: $B = 8°$, $\beta = 0.02$ rad/s and $\varphi = -0.76$ rad.

4.3 a) As $\omega \to 0$, $|G(i\omega)| \to \infty$ and $\arg G(i\omega) \to -90°$. The gain is first decreasing (low frequencies). It then increases, and finally decreases again (approaching zero for high frequencies). The phase shift is increasing at low frequencies. As the frequency becomes higher the phase shift is positive in an interval until it decreases towards $-90°$. This gives the plot in Figure 4.3a.

b) A system with a Bode plot as the one shown above must have one pole in the origin since $\arg G(i\omega) \to -90°$ as $\omega \to 0$. Then two break points appear (up), since there is a positive phase shift. After that, there must be two break points (down), since the phase shift should approach $-90°$. Hence, the plot in Figure 4.3b is possible.

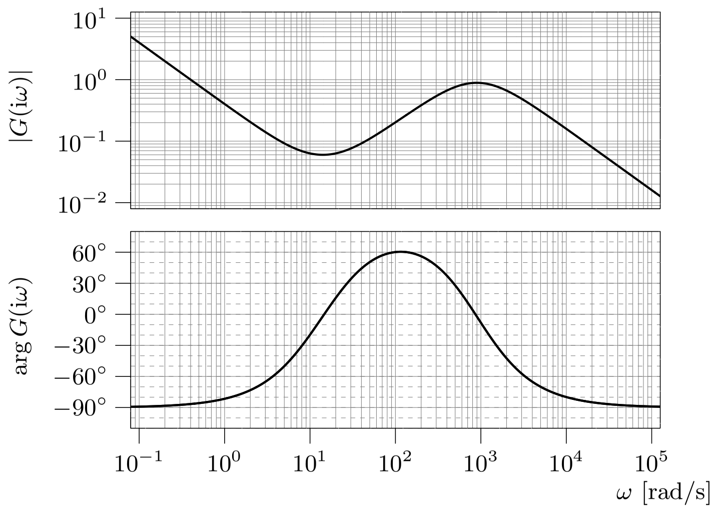


Figure 4.3a

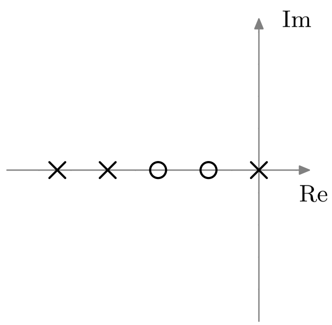
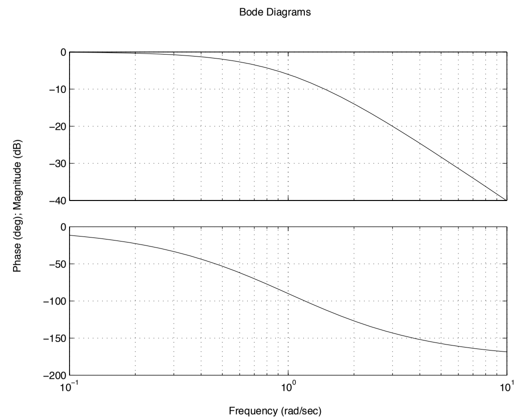


Figure 4.3b. Pole-zero diagram. Not accurate in scale; the diagram shall only be interpreted as a right to left ordering of poles and zeros, with the first pole at the origin.

4.4 From the final value of step response B (the only one greater than 1) and static gain in Bode gain C (the only one greater than 1), the step response–Bode gain pair B–C follows. Step responses C and A have approximately the same overshoots but different fundamental frequencies. Bode gains B and D have equal resonance peaks but D has a lower resonance frequency. This gives the combinations C–D and A–B. The remaining combination is D–A, which is a good match with small overshoot (resonance peak) and final value (static gain) 1.

4.5 a) Enter the system and make a Bode plot.

```
>> s = tf( 's' );
>> GA = 1 / ( s^2 + 2*s + 1 );
>> bode( GA )
```



Use, for example, curve handles and "Characteristics" in the right click menu to find static gain, bandwidth, resonance frequency, and resonance peak. The other systems are treated in the same way. The results can be summarized in the following table. (Note that gain values may be presented in dB_{20} in MATLAB.)

System	$G(0)$	ω_B	ω_r	M_p
G_A	1	0.64		
G_B	1	1.5	1	2.5
G_C	1	0.21		
G_D	1	1.27	0.7	1.15
G_E	1	2.54	1.4	1.15

b) Using the results in a) and in Problem ??, the following observations can be made. (i): The bandwidth of a system is (approximately) inversely proportional to the rise time. High bandwidth implies a short rise time and hence a fast system. (ii): The damping is inversely proportional to the height of the resonance peak. A large peak implies low damping and large overshoot.

4.6 From the frequency responce interpretation of the transfer function ("a sinusoid in gives a sinusoid out") and the input being

$$u(t) = 2\sin(2t - 1/2)$$

it follows that the output is

$$y(t) = 2\,|G(\mathrm{i}2)|\sin(2t - 1/2 + \arg G(\mathrm{i}2))$$

Here $G(s) = \frac{e^{-2s}}{s(s+1)}$, and hence

$$|G(\mathrm{i}2)| = \frac{1}{2\sqrt{2^2+1}} = \frac{1}{2\sqrt{5}}$$
$$\arg G(\mathrm{i}2) = -4 - \frac{\pi}{2} - \arctan 2$$

4.7 The input is a sinusoid with amplitude 1 and angular frequency $\omega = 2$ rad/s.

a) $0.45\sin(2t - 1.1)$.
(Gain: $\left|\frac{1}{\mathrm{i}2+1}\right| = \frac{1}{\sqrt{5}} \approx 0.45$, phase: $-\arg(\mathrm{i}2+1) \approx -1.1$ rad $= -63°$.)

b) The system is unstable. Hence, the system output will tend to infinity, and the system will not reach a steady state. To be more precise, the general form of the solution to the differential equation describing the system output is $y(t) = C_0e^t + \frac{1}{\sqrt{5}}\sin(2t - \pi + \arctan 2)$, and any initial state $y(0) \neq \frac{1}{\sqrt{5}}\sin(-\pi + \arctan 2)$ will lead to a solution that tends to infinity. This will almost always be the case in practice.

c) $0.11\sin(2t - 2.4)$
(Gain: $\left|\frac{1}{(\mathrm{i}2+1)(\mathrm{i}4+1)}\right| = \frac{1}{\sqrt{5}\sqrt{17}} \approx 0.11$, phase: $-\arg(\mathrm{i}2+1) - \arg(4\mathrm{i}+1) \approx -2.4$ rad $= -139°$.)

d) $0.45\sin(2(t-0.5) - 1.1) = 0.45\sin(2t - 2.1)$.
Similar to problem a), with an extra time delay of 0.5 s.

4.8 a) To determine the phase difference, ϕ, given a diagram with two sinusoids, $\sin(\omega t)$ and $K\sin(\omega t + \phi)$, one possibility is to consider the time points when the two curves pass 0. Determine t_1 and t_2 such that

$$\sin(\omega t_1) = 0$$
$$K\sin(\omega t_2 + \phi) = 0$$

This gives that $\omega t_1 = \omega t_2 + \phi$, that is,

$$\phi = -\omega t_\Delta = -\frac{2\pi \text{ rad}}{T} t_\Delta = -\frac{t_\Delta}{T} 2\pi \text{ rad}$$

where $t_\Delta = t_2 - t_1$ and T is the common period time. Here, the last expression may be interpreted as the delay expressed in parts ($\frac{t_\Delta}{T}$) of a whole revolution (2π). For example, consider the second graph where $t_\Delta \approx 0.18$ s and $T \approx 1.25$ s (which can either be read from the figure, or, in this problem, computed using $\omega = 5$ rad/s). Hence, $\phi = -\frac{0.18\text{ s}}{1.25\text{ s}} 2\pi$ rad $= -0.9$ rad. This results in the table below, where the answer to part b is also included.

ω	$\lvert G(\mathrm{i}\omega)\rvert$			$\arg G(\mathrm{i}\omega)$		
1	1	=	0 dB$_{20}$	−0.2 rad	=	−11°
5	0.8	=	−1.9 dB$_{20}$	−0.9 rad	=	−52°
10	0.5	=	−6 dB$_{20}$	−1.6 rad	=	−92°
20	0.2	=	−14 dB$_{20}$	−2.2 rad	=	−126°

b) Just evaluate the decibel formula to obtain the values in the table above.

c) A Bode plot of the system is given in Figure 4.8a

4.9 **Answer:** G_1–B, G_2–D, G_3–A, G_4–C, G_5–E.

- The Bode plot B has static gain 1 and no resonance peak, and hence G_1–B. It can also be seen that the Bode plot B decays by one decade (20 dB$_{20}$) when the frequency increases by a factor of ten ("the slope is −1") and that G_1 has one pole.

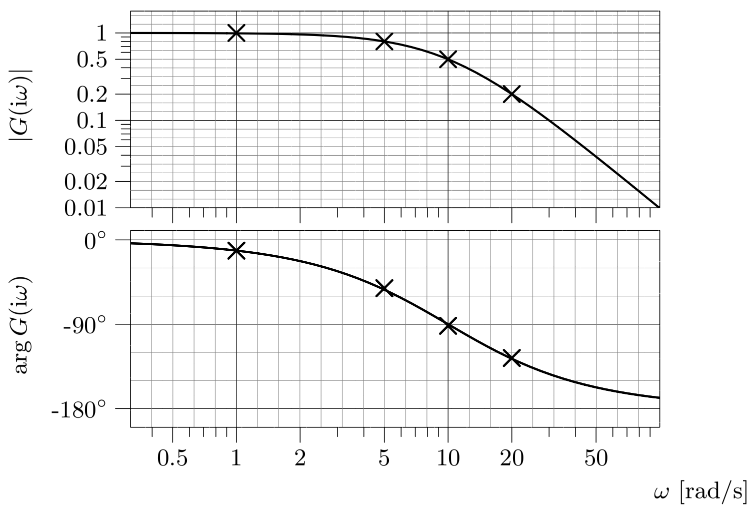


Figure 4.8a

- The Bode plots A and C have both infinite gain for when the frequency tends to zero, that is, they correspond to systems containing an integrator $\Rightarrow$ systems G_3 and G_4. The Bode plot C decays more rapidly for high frequencies $\Rightarrow$ the relative degree (number of poles − number of zeros) is higher. Hence G_3–A, G_4–C.

- The Bode plots D and E have peaks $\Rightarrow$ systems G_2 and G_5. (For G_2 the peak is caused by the zero where the curve "turns up" at $\omega = 1$.) The Bode plot E has larger slope than D for high frequencies, that is, E corresponds to a system with higher relative degree. G_2 has one pole more than zeros, G_5 has 2 poles, and hence G_2–D, G_5–E.

4.10
- In step response A and D the step responses tend to one, that is, they correspond to Bode gain A and C. Step response D has larger overshoot, that is, it corresponds to Bode gain A, and consequently step response A corresponds to Bode gain C. This gives the Bode gain–step response pairs A–D and C–A.

- Step response B has no overshoot, which implies that it corresponds to Bode gain D, which has no peak. This gives the combination D–B.

- The remaining combination is B–C. Step response C has an overshoot which can be related to the peak in the Bode gain plot. It can also be seen that this pair belongs to the fastest system.

4.11 a) The system can be rewritten as

$$G(s) = \frac{1.7}{(s+1)(\frac{s}{1.43}+1)(\frac{s}{2}+1)}$$

It thus follows that

$$|G(i\omega)| = \frac{1.7}{\sqrt{1+\omega^2}\sqrt{1+\left(\frac{\omega}{1.43}\right)^2}\sqrt{1+\left(\frac{\omega}{2}\right)^2}}$$

and

$$\arg G(i\omega) = -\arctan\omega - \arctan\frac{\omega}{1.43} - \arctan\frac{\omega}{2}$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote, $|G(i0)| = 1.7$, and the breakpoints and slopes of the asymptotes:

Frequency [rad/s]		1		1.43		2	
Slope	0		−1		−2		−3

The phase shift is drawn based on a couple of samples:

Frequency [rad/s]	0.1	0.5	1
Phase	−12.7°	−59.9°	−106.6°
Frequency [rad/s]	2	3	10
Phase	−162.9°	−192.4°	−244°

The bode plot in Figure 4.11a gives: $\omega_c = 0.874$ rad/s, $\varphi_m = 83.8°$, $A_m = 5.14$, and $\omega_p = 2.51$ rad/s.

b) The phase is $-180°$ at $\omega_p = 2.51$ where the amplitude is 0.1946. To make the pH oscillate with constant amplitude one has to choose $K = \frac{1}{0.1946} = 5.14$.

4.12 a) The phase is $-180°$ at $\omega_p = 0.334$ where the amplitude is 0.1984. To keep the reactor stabile one has to choose $K \leq \frac{1}{0.1984} = 5.04$.

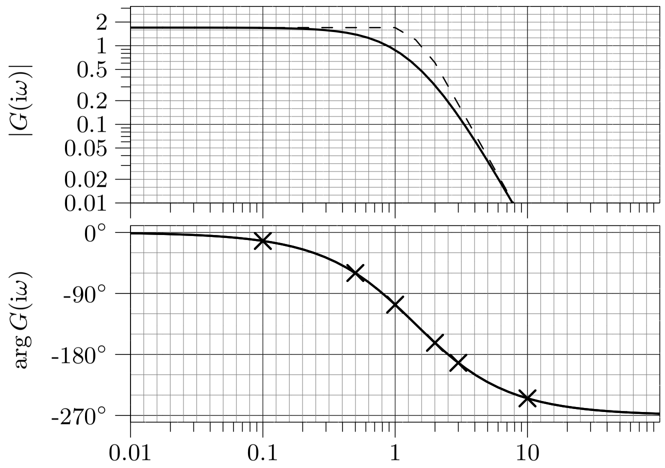


Figure 4.11a

b) This is a lead-lag design task. The amplitude and phase of G at $\omega_{\mathrm{c,d}} = 0.1$ is 0.6325 and $-100°$. Thus we have a phase margin of 80° which is sufficient, and hence no lead controller is needed. To remove the steady-state error we need a lag controller with $M = \infty$. This results in the controller structure

$$F(s) = K \underbrace{\frac{s+a}{s}}_{F_{\mathrm{lag}}}$$

Chose $a = 0.1\omega_{\mathrm{c,d}} = 0.01$ (a bigger value on a makes the error go to zero faster) and $K = \frac{1}{|G(\mathrm{i}\omega_{\mathrm{c,d}})F_{\mathrm{lag}}(\mathrm{i}\omega_{\mathrm{c,d}})|} = \frac{1}{0.6325} = 1.58$. This gives the controller $F(s) = 1.58\frac{s+0.01}{s}$.

4.13 Ansätt $G(s) = \frac{b}{s+a}$ (med $a > 0$ och $b > 0$).

Utsignalen ges av (ekvation 4.2 i boken)

$$y(t) = |G(i\omega)| \sin\left(\omega t + \arg G(i\omega)\right), \quad \omega = 2$$

$$|G(i\omega)| = \frac{b}{\sqrt{\omega^2 + a^2}} = 2$$

$$\arg G(i\omega) = \arg b - \arg\left(i\omega + a\right) = -\arctan\frac{\omega}{a} = -\frac{\pi}{4}.$$

Och alltså för $\omega = 2$ fås $a = 2$ och $b = 4\sqrt{2}$, samt initialvärdet $y_0 = 2\sin(0 - \pi/4) = -\sqrt{2}$.

5 Compensation

5.1 The compensator is constructed using lead-lag design. “Twice as fast” is interpreted as a doubling of the bandwidth, which, in turn, is approximated by a doubling of the gain crossover frequency. “Same damping” is interpreted as maintaining the old phase margin, which is accomplished using a lead compensator in the controller. The error in static reference following is controlled by adjusting the static gain of the open loop system, which is accomplished using a lag compensator in the controller. Sensitivity to measurement disturbances is given by the complementary sensitivity function, $1-(1+G_o)^{-1}$. It is small where the open loop gain is small. Thus, to make it small at high frequencies, the high frequency gain of the controller should be kept as low as possible.

First, the open loop system when $F(s) = 1 \quad \Rightarrow \quad G_o = G$ is examined in order to quantify the requirements.

$$G(s) = \frac{0.4}{(s+0.1)(s+0.5)(s^2+0.4s+4)} = \frac{2}{(1+s/0.1)(1+s/0.5)(1+2\cdot 0.1\cdot s/2+(s/2)^2)}$$

which implies that

$$|G(i\omega)| = \frac{2}{\sqrt{1+(\frac{\omega}{0.1})^2}\sqrt{1+(\frac{\omega}{0.5})^2}\sqrt{(1-(\frac{\omega}{2})^2)^2+4\cdot 0.01(\frac{\omega}{2})^2}}$$

with low frequency asymptote

$$|G(i\omega)| \to 2, \ \omega \to 0$$

and

$$\arg G(i\omega) = -\arctan\frac{\omega}{0.1} - \arctan\frac{\omega}{0.5} - \arctan\frac{2\cdot 0.1\frac{\omega}{2}}{1-(\frac{\omega}{2})^2}$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote, 2 (at any point), and the breakpoints and slopes of the asymptotes:

Frequency [rad/s]		0.1		0.5		2	
Slope	0		−1		−2		−4

The system has two complex conjugated poles which implies that the amplitude curve has a resonance peak. The approximative amplitude curve must be modified at the resonance peak. An exact calculation of the gain gives

Frequency [rad/s]	1	1.5	2	2.5
Gain	0.12	0.09	0.12	0.025

The phase curve is drawn based on a couple of samples:

Frequency [rad/s]	0.01	0.1	1	1.5
Phase	−7°	−57°	−155°	−177°
Frequency [rad/s]	2	2.5	10	
Phase	−253°	−322°	−354°	

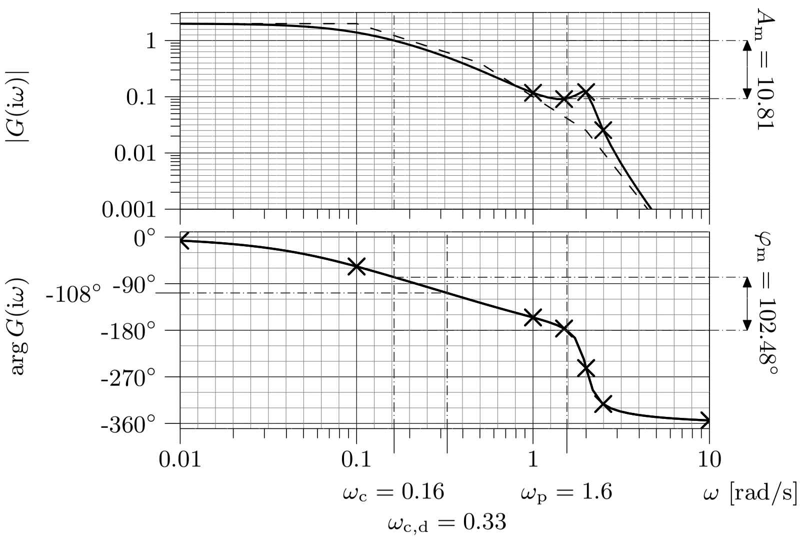


Figure 5.1a

The Bode plot in Figure 5.1a gives

$$\omega_c = 0.16 \text{ rad/s} \quad \varphi_m = 102° \quad A_m = 10.6$$

and hence

$$\omega_{c,d} = 0.32 \text{ rad/s} \quad \varphi_{m,d} = 102°$$

The phase of G at the $\omega_{c,d}$ is $-108°$. Hence, in order to obtain the desired phase margin of $102° = -78° - (-180°)$, a phase advance of approximately

$(-78°)-(-108°) = 30°$ is required. To this end, introduce a lead compensator in the controller:

$$F_{\text{lead}} = N\frac{s+b}{s+bN}$$

See the discussion of lead compensators in Glad&Ljung! To keep the high frequency gain of the controller as small as possible, N should be chosen as small as possible. The desired phase advance is obtained with $N = 3$. This phase lead is obtained at the desired crossover frequency if

$$b = \frac{\omega_{\text{c,d}}}{\sqrt{N}} = 0.185$$

The desired crossover frequency is obtained by adjusting the gain of the open loop system by introducing a factor, K, in the controller:

$$1 = K\,|F_{\text{lead}}| \cdot |G(\text{i}0.32)| = K\sqrt{N}\cdot 0.52 \quad \Rightarrow \quad K = 1.11$$

With

$$F(s) = 3.33\frac{s+0.185}{s+0.555}$$

and $\omega_{\text{ref}}(s) = A/s$, where A is constant, it follows that (using the notation $e(t) = \omega_{\text{ref}}(t) - \omega(t)$)

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+F(s)G(s)}\frac{A}{s}$$
$$= \frac{A}{1+1.1\cdot 2\cdot F_{\text{lag}}(0)} \leq 0.05A.$$

This is equivalent to

$$F_{\text{lag}}(0) \geq 8.63$$

If the low frequency gain of $F(s)$ is increased approximately 9 times the stationary error will be smaller than 5%. To this end, introduce a lag (phase-retarding) compensator

$$F_{\text{lag}} = \frac{s+a}{s+\frac{a}{M}}$$

in the controller, where M should be kept as small as possible to avoid unnecessary high gain at low frequencies. See the discussion of lag compensators in Glad&Ljung! With $M = 9$ and $a = 0.1\omega_{\text{c,d}} = 0.032$ the desired low frequency gain increase is obtained without altering the phase margin too much.

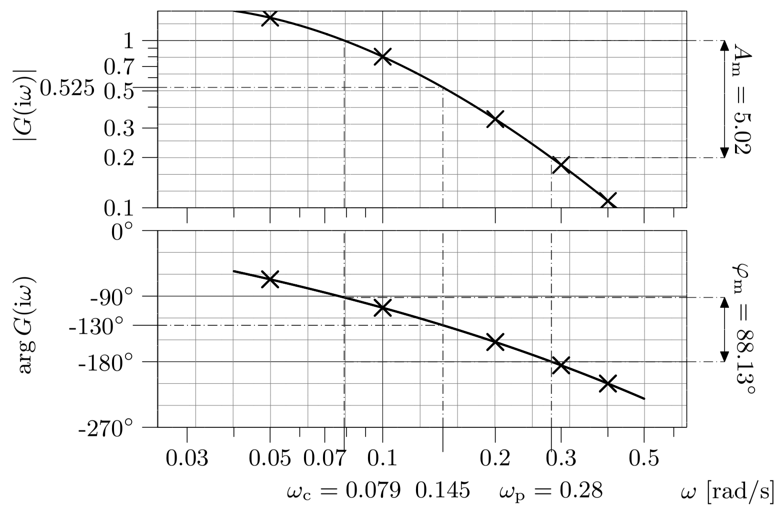


Figure 5.2a

This gives the controller

$$F(s) = K\cdot F_{\text{lead}}(s)\cdot F_{\text{lag}}(s) = 3.33\frac{(s+0.185)}{(s+0.555)}\frac{(s+0.032)}{(s+0.0036)}$$

5.2 Let G denote the heat exchanger's transfer function.

a) Draw the Bode plot using the given table. From the diagram in Figure 5.2a it follows that

$$\omega_{\text{c}} = 0.079 \text{ rad/s} \quad \varphi_{\text{m}} = 88° \quad A_{\text{m}} = 5.0$$

b) A proportional controller does not change the phase curve. According to Figure 5.2a, the phase curve crosses $-130°$ at the frequency 0.15 rad/s. A gain crossover at this frequency will yield exactly the required phase margin, and any higher crossover frequency will yield one that is too small.*

*The controller gain that yields the desired gain crossover frequency can be computed as

$$K = \frac{1}{|G(0.15\text{i})|} = \frac{1}{0.525} = 1.9$$

c) Twice as large crossover frequency is desired:

$$\omega_{c,d} = 0.29 \text{ rad/s} \quad \varphi_{m,d} = 50^\circ$$

At the frequency 0.29 rad/s the phase margin is approximately 0° (actually a little less, since $\omega_p = 0.28$ rad/s). Hence, a phase lead of 50° is needed. To this end, use a lead-compensator (using standard notation of parameters) with $\beta = 0.13$ (according to the diagram in Glad&Ljung) in order to achieve this. To obtain the maximum phase lead at the desired crossover frequency, let

$$\tau_D = \frac{1}{\omega_{c,d}\sqrt{\beta}} = 9.47$$

Finally, K is chosen so that $\omega_{c,d}$ is obtained:

$$1 = K\,|F_{PD}(i\omega_{c,d})| \cdot |G(i\omega_{c,d})| \approx K \cdot \frac{1}{\sqrt{\beta}} \cdot \frac{1}{A_m} \quad \Leftrightarrow$$

$$K \approx \sqrt{\beta} A_m = 1.83$$

Answer:

$$F(s) = 1.83 \frac{(9.47s+1)}{(0.13 \cdot 9.47s + 1)}$$

5.3 a)

$$G(s) = \frac{20}{s(1 + 2 \cdot 0.1 \cdot \frac{s}{150} + (\frac{s}{150})^2)}$$

which implies that

$$|G(i\omega)| = \frac{20}{\omega\sqrt{(1-(\frac{\omega}{150})^2)^2 + 4 \cdot 0.01 \cdot (\frac{\omega}{150})^2}}$$

with low frequency asymptote

$$|G(i\omega)| \to \frac{20}{\omega}, \ \omega \to 0$$

and

$$\arg G(i\omega) = -90^\circ - \arctan \frac{2 \cdot 0.1 \cdot \omega}{(1 - (\frac{\omega}{150})^2)}$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote, $|\frac{20}{20}| = 1$, and the breakpoints and slopes of the asymptotes:

Frequency [rad/s]		150	
Slope	−1		−3

The system has two complex conjugated poles which implies that the amplitude curve has a resonance peak. The approximative amplitude curve must be modified at the resonance peak. An exact calculation of the gain gives

Frequency [rad/s]	100	150	200
Gain	0.35	0.67	0.12

The phase curve is drawn based on a couple of samples:

Frequency [rad/s]	10	50	100	150	200
Phase	−91°	−94°	−103°	−180°	−251°

In addition, one can also use

$$\arg G(i\omega) \to -90^\circ, \ \omega \to 0$$
$$\arg G(i\omega) \to -270^\circ, \ \omega \to \infty$$

The Bode plot with the gain curve labeled "A" in Figure 5.3a gives

$$\omega_c = 20 \text{ rad/s} \quad \varphi_m = 88^\circ \quad A_m = 1.5$$

b) If K would be chosen to the gain margin, $A_m = 1.5$, the new gain margin would be 1. Thus, if

$$K = \frac{A_m}{2} = 0.75$$

the resulting gain margin becomes 2. With this amplification the final value theorem gives the ramp error

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s \to 0} \frac{1}{sKG(s)} = \frac{1}{0.75 \cdot 20} = 0.067$$

Note that the system is stable by construction (the new gain margin is greater than 1).

c) The new gain crossover frequency obtained in part b is 15 rad/s, see the gain curve labeled "B" in Figure 5.3a. The low frequency gain of $F(s)$

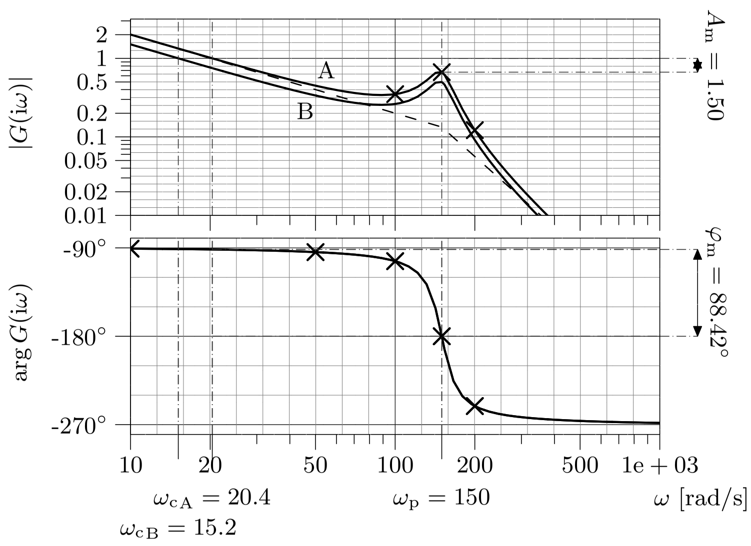


Figure 5.3a

must be increased at least 15 times. A lag-compensator with $M = 15$ can be used. Choose, according to the rule of thumb, $a = 0.1 \cdot \omega_{c,d}$, where $\omega_{c,d} = 15$, and hence $a = 1.5$.

Answer:

$$F(s) = 0.75 \cdot \frac{(s + 1.5)}{(s + 0.1)}$$

5.4 We begin by drawing a Bode plot of the system.

$$G(s) = \frac{10}{s(1 + \frac{s}{10})(1 + \frac{s}{100})}$$

which implies that

$$|G(i\omega)| = \frac{10}{\omega\sqrt{1 + (\frac{\omega}{10})^2}\sqrt{1 + (\frac{\omega}{100})^2}}$$

with low frequency asymptote

$$|G(i\omega)| \to \frac{10}{\omega}, \ \omega \to 0$$

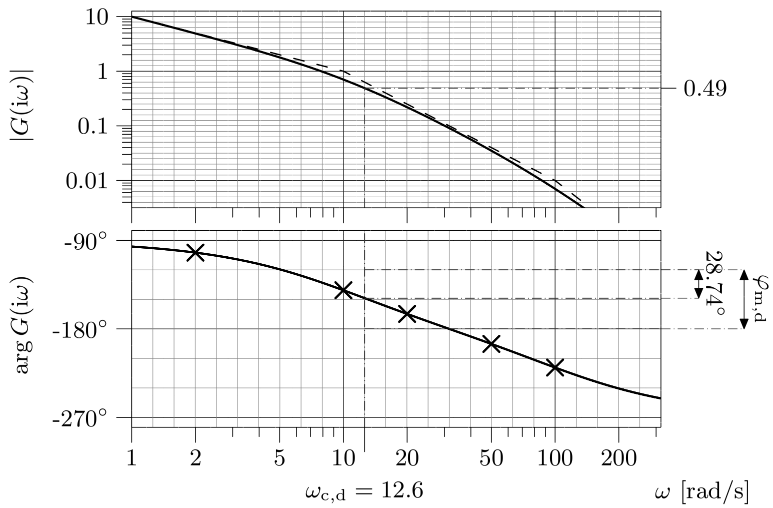


Figure 5.4a

and

$$\arg G(i\omega) = -90° - \arctan\frac{\omega}{10} - \arctan\frac{\omega}{100}$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote, $\left|\frac{10}{1}\right| = 10$, and the breakpoints and slopes of the asymptotes:

Frequency [rad/s]		10		100	
Slope	−1		−2		−3

The phase curve is drawn based on a couple of samples:

Frequency [rad/s]	2	10	20	50	100
Phase	−102°	−141°	−165°	−195°	−219°

In addition, one can also use

$$\arg G(i\omega) \to -90°, \ \omega \to 0$$
$$\arg G(i\omega) \to -270°, \ \omega \to \infty$$

From the Bode digram in Figure 5.4a it follows that $\omega_c = 7.8$ rad/s, $\varphi_m = 47°$ and $A_m = 11$. However, these values are not used by the solution to this problem.

Figure 5.4b (the figure can also be found in Glad&Ljung) gives that the overshoot is acceptable if $\zeta \geq 0.58$. Choose for instance $\zeta = 0.6$. This results in a desired phase margin $\varphi_{\mathrm{m,d}} = 60°$. According to Figure 5.4c (the figure can also be found in Glad&Ljung), this also implies a desired gain crossover frequency:

$$\omega_0 T_\mathrm{r} = 1.8 \text{ and } \frac{\omega_{\mathrm{c,d}}}{\omega_0} = 0.7 \quad \Rightarrow$$

$$\omega_{\mathrm{c,d}} = 0.7\frac{1.8}{T_\mathrm{r}} = 0.7\frac{1.8}{0.1} = 12.6$$

At 12.6 rad/s a phase advance of approximately 30° is needed in order to get the desired phase margin. To this end, use a lead compensator (with the usual notation of parameters) with $N = 4$ and $b = \omega_{\mathrm{c,d}}/\sqrt{N} = 6.3$. K is adjusted to get the desired gain crossover frequency:

$$1 = K\,|F_{\mathrm{lead}}(\mathrm{i}\omega_{\mathrm{c,d}})| \cdot |G(\mathrm{i}\omega_{\mathrm{c,d}})| = K\sqrt{N} \cdot 0.49 \quad \Rightarrow \quad K = 1.02$$

The transfer function from the reference input to the control error is given by

$$E(s) = \frac{1}{1 + F(s)G(s)}\theta_{\mathrm{ref}}(s)$$

When $\theta_{\mathrm{ref}}(t)$ is a step signal, the final value theorem gives

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = 0$$

even without a lag compensator thanks to the integration in G. Here, the final value theorem may be used since the system by construction is stable (the phase margin is 60°).

In order to handle errors for ramp references, introduce a lag compensator (with the usual notation of parameters) in the controller. Then $|F_{\mathrm{lag}}(0)| = M$, and if $\theta_{\mathrm{ref}}(t) = 10 \cdot t$, that is, if

$$\theta_{\mathrm{ref}}(s) = \frac{10}{s^2}$$

one obtains

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1 + F(s)G(s)}\frac{10}{s^2} = \frac{10}{k_\mathrm{m} \cdot K \cdot M} < 0.1$$

which gives $M > \frac{1}{0.1K} = 9.8$. Take $M = 9.8$ to avoid excessively high low frequency loop gain. According to the rule of thumb, let $a = 0.1 \cdot \omega_{\mathrm{c,d}} = 1.26$.

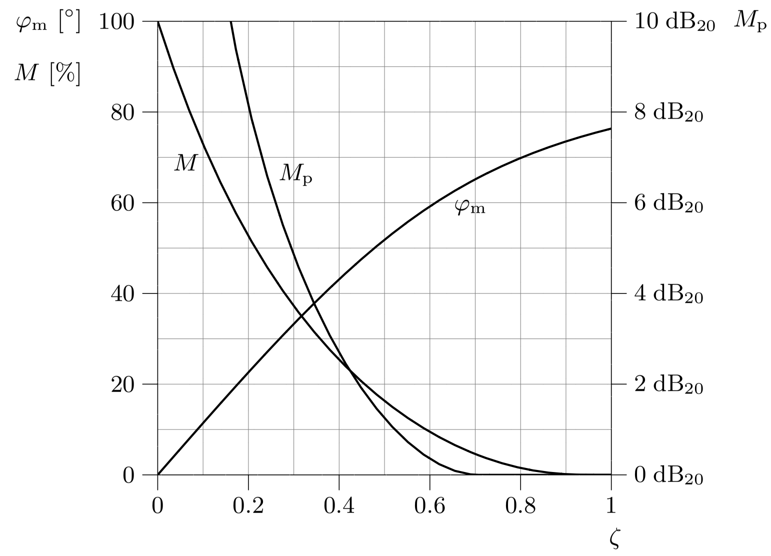


Figure 5.4b. Relations between overshoot, M, phase margin, φ_m, resonance gain, M_p, and relative damping, ζ, for a second order system with no zeros and static gain 1.

Answer:

$$F(s) = 1.02 \cdot 4 \cdot \frac{s + 6.3}{s + 25} \cdot \frac{s + 1.26}{s + 0.13}$$

5.5 *Notation.* The notation "A – B – C" is used to say that the system with open loop Bode plot in row A has its closed loop Bode plot in row B, and its step response in row C.

A good start is often to look at the static gain and the final value of the step responses. The static gain of the open loop system and the closed loop system are related as $|G_\mathrm{c}(0)| = \frac{|G_\mathrm{o}(0)|}{|1+G_\mathrm{o}(0)|}$. Systems with the same static gain can then be separated by looking at stability margins, resonance peek, overshoot, bandwidth, and speed. Three of the combinations are easy to identify:

A – E – C: Finite but non-zero open loop static gain matches non-zero closed loop static gain less than 1. Infinite stability margins matches step response without overshoot.

B – C – E: Infinite open loop static gain matches closed loop static gain equal

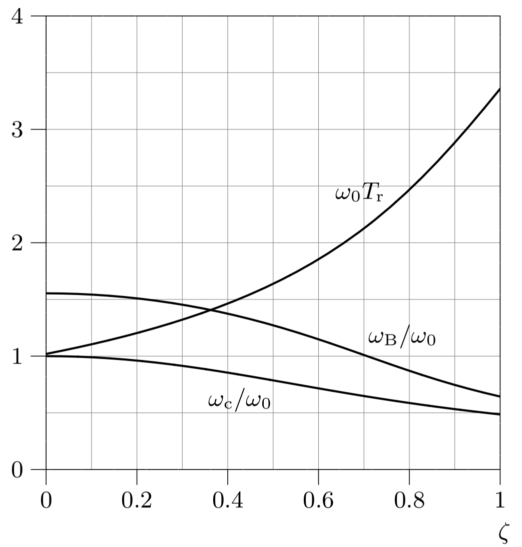


Figure 5.4c. Relations between gain crossover frequency, ω_c, bandwidth, ω_B, raise time, T_r, and relative damping, ζ, for a second order system with no zeros and static gain 1.

1, which in turn matches a step response that settles at amplitude 1.
C – A – B: Zero static open loop gain matches zero closed loop gain, which in turn matches a step response that settles at amplitude 0.

The remaining open loop Bode plots are D and E. These should be matched with the closed loop gain curves B and D, and step responses A and D. Both open loop Bode plots show a static gain near 1, which will make it hard (albeit possible) to use that feature for identification. Easier is to approximately locate the (closed loop) resonance frequency, which will be near the frequency where the Nyquist curve minimizes its distance to −1. That is, the magnitude shall be near 1, and the phase near −180° in the open loop Bode plot. This happens at a lower frequency in open loop Bode plot D than in E. The resonance peak in the closed loop gain curve B is located at a higher frequency than that in D. Finally, a higher resonance frequency gives faster oscillations in the step response, and the oscillations in step response A are much quicker than those in D. Alternatively the bandwidth's relation to response speed may be used; the bandwidth is higher in closed loop B than in D, and step response A is quicker than D. Anyway, the last two combinations are D–D–D, E–B–A.

5.6

$$G(s) = \frac{10}{s(1+\frac{s}{20})(1+\frac{s}{40})(1+\frac{s}{100})}$$

gives

$$|G(\mathrm{i}\omega)| = \frac{10}{\omega\sqrt{1+(\frac{\omega}{20})^2}\sqrt{1+(\frac{\omega}{40})^2}\sqrt{1+(\frac{\omega}{100})^2}}$$

with low frequency asymptote

$$|G(\mathrm{i}\omega)| \to \frac{10}{\omega},\ \omega \to 0$$

and

$$\arg G(\mathrm{i}\omega) = -90^\circ - \arctan\frac{\omega}{20} - \arctan\frac{\omega}{40} - \arctan\frac{\omega}{100}$$

The gain is drawn approximately based on a known gain at some point of the low frequency asymptote, $\left|\frac{10}{10}\right| = 1$, and the breakpoints and slopes of the asymptotes:

Frequency [rad/s]		20		40		100	
Slope	−1		−2		−3		−4

The phase curve is drawn based on a couple of samples:

Frequency [rad/s]	10	20	50
Phase	−136°	−173°	−236°

In addition, one can also use

$$\arg G(\mathrm{i}\omega) \to -90^\circ,\ \omega \to 0$$
$$\arg G(\mathrm{i}\omega) \to -360^\circ,\ \omega \to \infty$$

The Bode plot in Figure 5.6a gives that $\omega_c = 8.9$ rad/s, $\varphi_m = 48^\circ$ and $A_m = 3.9$. However, it is only the gain crossover frequency which directly interests us here; an increase of the speed with a factor of two and a preserved damping imply $\omega_{c,d} = 18$ rad/s and $\varphi_{m,d} = \varphi_m$. From the figure, we have $\varphi_\Delta = \arg G(\mathrm{i}\omega_c) - \arg G(\mathrm{i}\omega_{c,d}) = 35^\circ$. The required phase lead is thus at least $\varphi_\Delta + 6^\circ = 41^\circ$. To this end, use a lead compensator (with standard notation of the parameters) with $\beta = 0.21$ and $\tau_D = \frac{1}{\omega_{c,d}\sqrt{\beta}} = 0.12$. K is adjusted to get the desired crossover frequency:

$$K\,|G(\mathrm{i}\omega_{c,d})| \cdot |F_{\mathrm{lead}}(\mathrm{i}\omega_{c,d})| = 1 \quad\Rightarrow\quad K = \frac{\sqrt{\beta}}{0.37} = 1.2$$

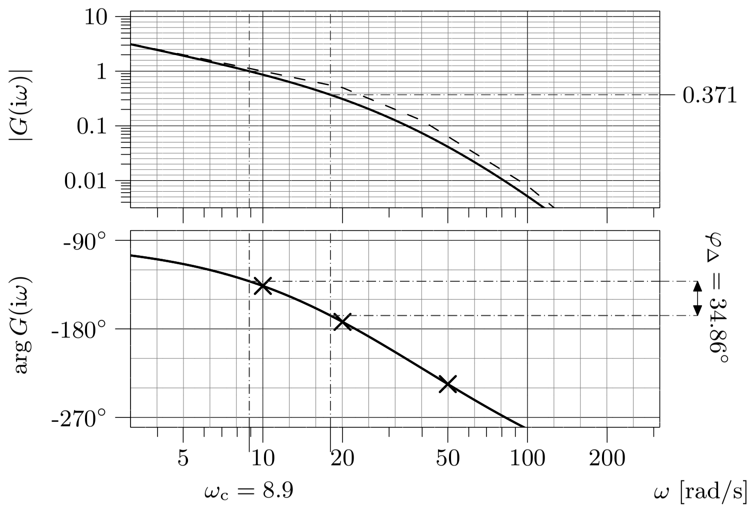


Figure 5.6a

The transfer function from the reference input to the control error is given by

$$E(s) = \frac{1}{1+F(s)G(s)}\theta_{\mathrm{ref}}(s)$$

When $\theta_{\mathrm{ref}}(t)$ is a step, the final value theorem gives

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = 0$$

even without a lag compensator thanks to the integration in $G(s)$. Here, the final value theorem may be used since the system by construction is stable (the phase margin is positive).

In order to handle errors for ramp references, introduce a lag compensator (with the usual notation of parameters) in the controller. Then $|F_{\mathrm{lag}}(0)| = 1/\gamma$, and if $\theta_{\mathrm{ref}}(t) = 10\cdot t$, that is, if

$$\theta_{\mathrm{ref}}(s) = \frac{10}{s^2}$$

one obtains

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+F(s)G(s)}\frac{10}{s^2} = \frac{10\gamma}{k_{\mathrm{m}}\cdot K} < 0.01$$

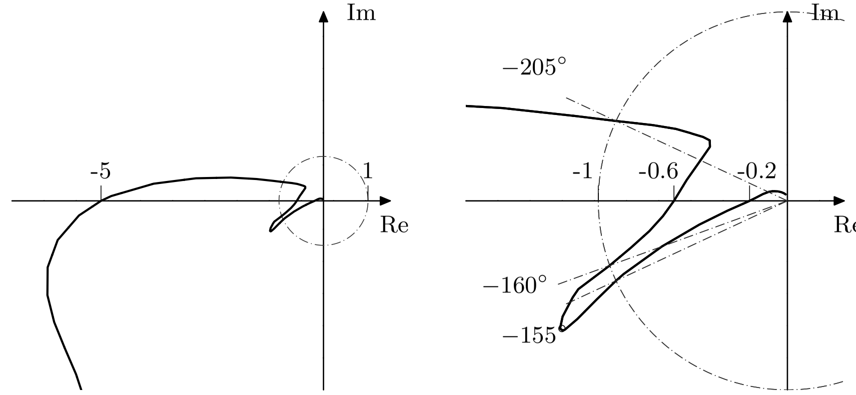


Figure 5.7a. Nyquist curve in two scales. Left: small scale. Right: big scale.

which gives $\gamma < 0.01K = 0.012$. Take $\gamma = 0.012$ to avoid excessively high low frequency loop gain. According to the rule of thumb, let $\tau_{\mathrm{I}} = 10/\omega_{\mathrm{c,d}} = 0.56$.

Answer:

$$F(s) = 1.2\frac{0.12s+1}{0.21\cdot 0.12s+1}\cdot\frac{0.56s+1}{0.56s+0.012}$$

5.7 Based on the Bode plot we plot the Nyquist curve, see Figure 5.7a. The system is stable when the point -1 is not encircled by the Nyquist curve. This gives

$$K < \frac{1}{5} \quad \text{or} \quad \frac{1}{0.6} < K < \frac{1}{0.2}$$

5.8 A time delay of T seconds changes the phase curve with $-\omega T$ rad at frequency ω. The amplitude curve is not affected.

a) The crossover frequency is $\omega = 1$ rad/s and the phase margin is 0.698 rad. This gives the stability condition

$$0.698 \text{ rad} - 1 \text{ rad/s}\cdot T > 0$$

that is, $T < 0.698$ s.

b) Plot the Nyquist curve as in Figure 5.8a. The point -1 is not encircled if the phase is decreased at least 40° at $\omega = 7$ rad/s but not more than 80°

at $\omega = 5$ rad/s. This gives the following conditions

$$7 \text{ rad/s} \cdot T > 40° = 0.698 \text{ rad} \quad \text{and} \quad 5 \text{ rad/s} \cdot T < 80° = 1.396 \text{ rad}$$

that is, $0.1 \text{ s} < T < 0.28$ s.

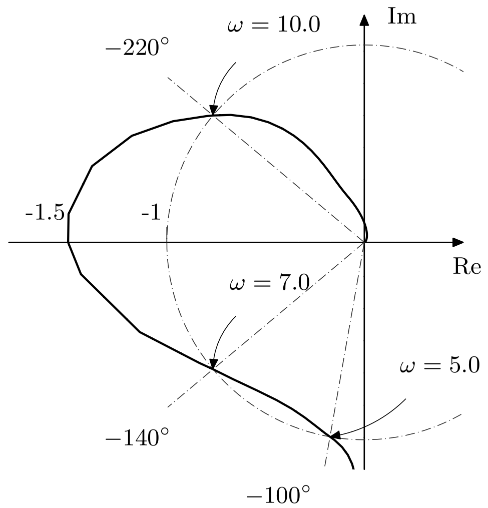


Figure 5.8a

5.9 a) The step response of G_A is

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{k_A}{a}(1 - e^{-at}) \to \frac{k_A}{a}, \ t \to \infty$$

From the figure it is seen that $k_A/a = 0.5$. At time $t = 1/a$ we have

$$y(1/a) = \frac{k_A}{a}(1 - e^{-1}) = 0.5 \cdot 0.63 = 0.315 = y(2)$$

Thus $a = 0.5$, which gives $k_A = 0.25$:

$$G_A(s) = \frac{0.25}{s + 0.5}$$

which is rewritten to make apparent the amplitude and phase

$$G_A(i\omega) = \frac{0.25}{\sqrt{\omega^2 + 0.25}} e^{-i \arctan 2\omega}$$

The corresponding Bode plot is shown in Figure 5.9a. To see how G_A modifies the Bode plot of G_m, consider for instance the frequency 0.1 rad/s. When computing the new gain, the logarithmic scale in the diagrams is used to do directly obtain the logarithm of the product of the two systems' gains:

$$|G_m(0.1i)| = 10^{0.15}$$
$$|G_A(0.1i)| = 10^{-0.31}$$
$$|G_A(0.1i)G_m(0.1i)| = 10^{0.15} \cdot 10^{-0.31} = 10^{0.15+(-0.31)} = 10^{-0.16}$$

The new phase is obtained by adding the arguments of the two transfer functions:

$$\arg G_m(0.1i) = -135°$$
$$\arg G_A(0.1i) = -11°$$
$$\arg G_A(0.1i)G_m(0.1i) = \arg G_A(0.1i) + \arg G_m(0.1i) = -146°$$

Carrying out the procedure of "adding Bode plots" at a range of selected frequencies results in the Bode plot in Figure 5.9b, where $G_o = G_A G_m$.

b) In Figure 5.9b it can be seen that the crossover frequency is 0.078 rad/s. Hence, let $\omega_{c,d} = 0.4$ to obtain a 5 times as fast system. At the desired crossover frequency, the phase must be advanced by 68° to maintain the phase margin. To this end, employ two equal lead compensators (using standard notation of the parameters), each advancing the phase by 34°; take $N = 4$, and $b = \frac{\omega_{c,d}}{\sqrt{N}} = 0.2$.

The controller gain is adjusted by the factor K to get the desired crossover frequency:

$$K\,|G(i\omega_{c,d})| \cdot |F_{lead}(i\omega_{c,d})|^2 = 1 \quad \Rightarrow$$
$$K = \frac{1}{0.047 \cdot \sqrt{N}^2} = 10.6$$

Answer:

$$F(s) = 10.6 \cdot \left(4 \frac{(s + 0.2)}{(s + 0.2 \cdot 4)}\right)^2$$

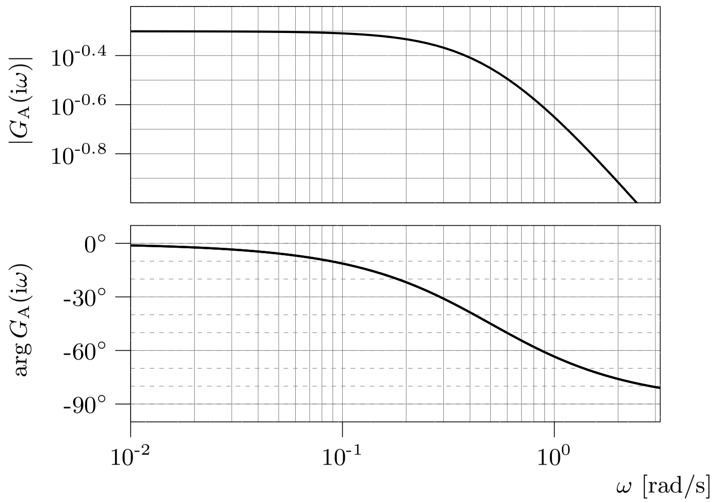


Figure 5.9a

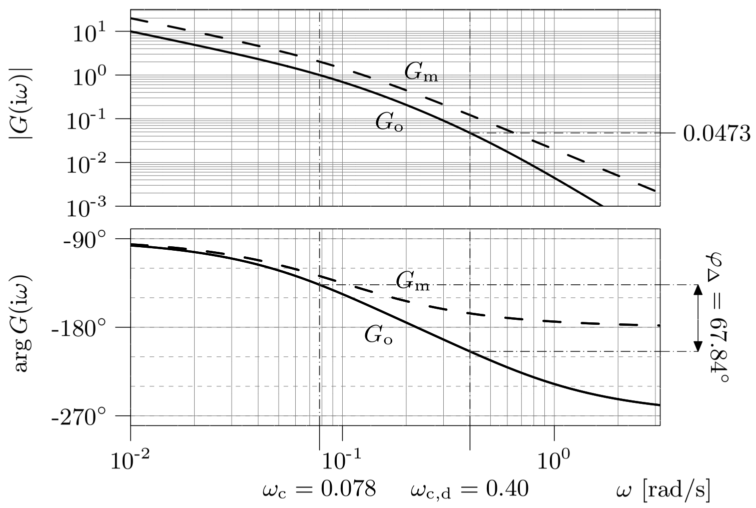


Figure 5.9b

5.10

$$G(s) = \frac{1}{s}G_1(s)$$

gives

$$|G(i\omega)| = \frac{|G_1(i\omega)|}{\omega}$$
$$\arg G(i\omega) = G_1(i\omega) - 90°$$

A P controller gives a phase margin of 40° when

$$\arg G(i\omega) = -140° \quad \Rightarrow \quad \arg G_1(i\omega) = -50°$$

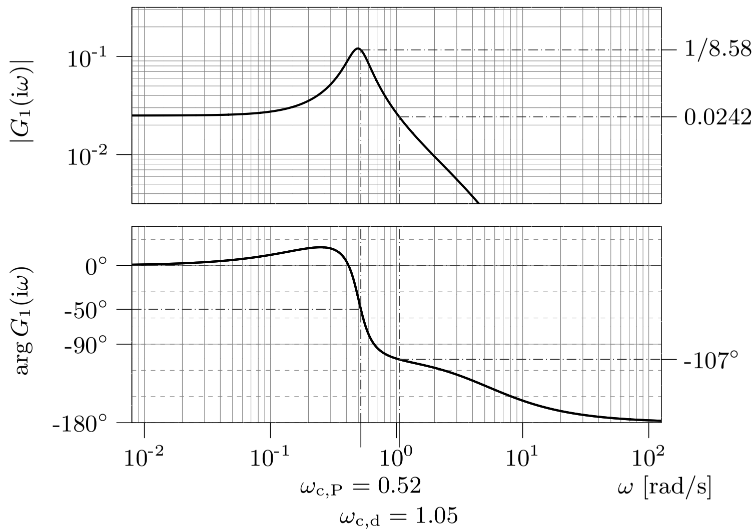


Figure 5.10a

From Figure 5.10a it is seen (although not easily) that this occurs at $\omega_{c,P} = 0.52$ rad/s, which is also the highest possible gain crossover frequency possible to obtain with P control. The desired increase in speed by a factor of two is thus achieved by a new gain crossover $\omega_{c,d} = 1.05$ rad/s. Figure 5.10a gives

$$\arg G_1(i\omega_{c,d}) = -107° \quad \Rightarrow \quad \arg G(i\omega_{c,d}) = -197°$$

A desired phase margin of 40° requires that the phase be advanced by $57° + 6° = 63°$. To this end, employ a two equal lead compensators (using standard notation of parameters), each advancing the phase by 32°; take $\beta = 0.31$ and $\tau_D = \frac{1}{\omega_{c,d}\sqrt{\beta}} = 1.72$. The controller gain is adjusted by the factor K to get the desired crossover frequency:

$$K\left|F_{\text{lead}}(i\omega_{c,d})\right|^2 \cdot |G(i\omega_{c,d})| = 1 \quad \Rightarrow \quad K\frac{1}{\sqrt{0.31}^2}\frac{0.024}{1.05} = 1 \quad \Rightarrow \quad K = 13.3$$

In order to handle errors for ramp references, introduce a lag compensator (with the usual notation of parameters) in the controller. Then $|F_{\text{lag}}(0)| = 1/\gamma$, and $|F(0)| = K/\gamma$. To choose γ, consider the Laplace transform of the control error,

$$E(s) = \frac{1}{1 + F(s)G(s)}R(s)$$

If $r(t) = A \cdot t$ (a ramp), that is, if

$$R(s) = \frac{A}{s^2}$$

one obtains

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1 + F(s)G_1(s)/s}\frac{A}{s^2} = \lim_{s\to 0}\frac{A}{s + F(s)G_1(s)}$$
$$= \frac{A}{|F(0)| \cdot |G_1(0)|}$$

This shows that the ramp error is inversely proportional to the static gain of the controller. According to Figure 5.10a, the highest possible controller gain when using a P controller and a phase margin of 40° is required, is 8.6. Hence, to reduce the ramp error to 1% of that of the P controller, the static gain of the new controller has to be at least 860. Therefore, take $\gamma = K/860 = 0.0155$, and, according to the rule of thumb, let $\tau_I = 10/\omega_{c,d} = 9.52$.

Answer:

$$F(s) = 13.3\frac{1.72s + 1}{0.31 \cdot 1.72s + 1}\frac{9.52s + 1}{9.52s + 0.0155}$$

5.11 a) The Nyquist curve is drawn based on the following observations: First, as $\omega \to 0$, $|G(i\omega)|$ increases and $\arg G(i\omega) \to -90°$. Then, as $\omega \to \infty$,

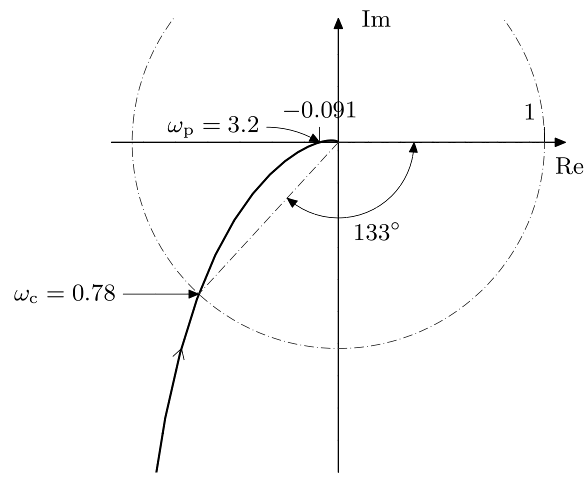


Figure 5.11a

$|G(i\omega)| \to 0$ and $\arg G(i\omega)$ decreases. We also have, $\omega_c = 0.78$ rad/s with $\arg G(i\omega_c) = -133°$, and finally $\omega_p = 3.2$ rad/s with $|G(i\omega_p)| = 0.091$. The resulting Nyquist curve is shown in Figure 5.11a.

b) The gain margin is $1/|G(i\omega_p)| = 11$, which is also the highest possible proportional gain that preserves closed loop asymptotic stability.

c) The Laplace transform of the control error is related to the reference as follows:

$$E(s) = \frac{1}{1 + KG(s)}R(s)$$

With

$$r(t) = 10t \quad \Rightarrow \quad R(s) = \frac{10}{s^2}$$

and using the final value theorem (from b we have that the system is stable), this yields

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{10}{2\lim_{s\to 0} sG(s)}$$

For small ω we have

$$G(s) \approx \frac{1}{s} \quad \Rightarrow \quad sG(s) \to 1,\ s \to 0 \quad \Rightarrow \quad \lim_{t\to\infty} e(t) = 5$$

d) Raising the gain curve in the Bode plot by $K = 2$ results in

$$\omega_c = 1.24 \text{ rad/s} \quad \varphi_m = 32^\circ$$

The closed loop system becomes unstable when the phase margin is eaten up by the phase lag of the delay,

$$\arg e^{-i\omega T} = -\omega T$$

so in order to get an asymptotically stable closed loop system it is thus required that

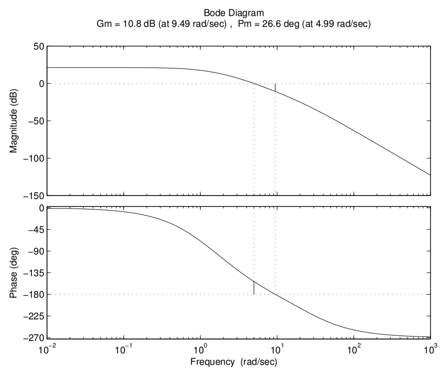
$$\omega_c T < 32^\circ \quad \Rightarrow \quad T < \frac{32^\circ}{1.24 \text{ rad/s}} = \frac{0.55 \text{ rad}}{1.24 \text{ rad/s}} = 0.44 \text{ s}$$

5.12 a) For this amplitude curve we cannot say anything about the stability since the system can contain an arbitrarily large time delay which could make the gain margin less than 1.

b) It is stable, since the gain is less than 1 for all frequencies; there is no risk that the Nyquist curve could encircle -1 under these circumstances.

5.13 a) Enter the system and the regulator. Draw the Bode plot. This gives $\omega_c = 5$ rad/s, $\omega_p = 9.5$ rad/s, $A_m = 3.5$ and $\varphi_m = 27^\circ$.

```
>> s = tf( 's' );
>> G = 725 / ...
        ( ( s + 1 ) * ( s + 2.5 ) * ( s + 25 ) );
>> F = 1;
>> margin( F * G )
```



b) From a) we know that at $\omega_{c,d} = 5$ rad/s the phase margin is 27°. In order to have $\varphi_m \geq 60^\circ$ we need to increase the phase by approximately 40°, including 6° extra to compensate for a future lag compensator. This is obtained using a lead compensator (using standard notation of parameters) with $\beta = 0.21$. The phase compensation is located at the correct frequency by taking $\tau_D = \frac{1}{\omega_{c,d}\sqrt{\beta}} = 0.43$.

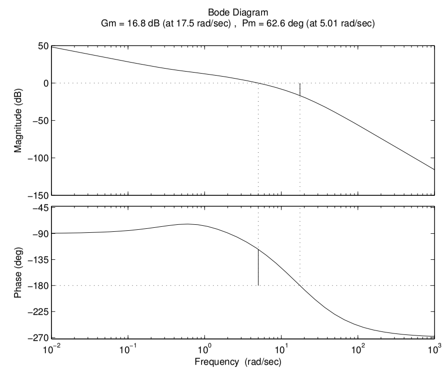
The controller gain is adjusted by the factor K to get the desired crossover frequency:

$$K \cdot \frac{1}{\sqrt{\beta}} \cdot |G(i5)| = K \cdot \frac{1}{\sqrt{0.21}} \cdot 1 = 1 \quad \Rightarrow$$
$$K = 0.46$$

The requirement $e_0 = 0$, that is, no steady state error for a unit step reference signal, is achieved by incorporating a lag compensator (using standard notation of parameters) with $\gamma = 0$, and, using the rule of thumb for the choice of τ_I, we take $\tau_I = 10/5 = 2$.

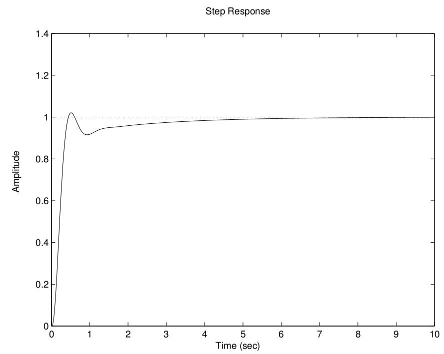
Generate a lead-lag regulator and make a Bode plot of the open loop system. Both the crossover frequency and the phase margin requirements are satisfied.

```
>> wc = 5;
>> b = 0.21;
>> tD = 1 / ( wc * sqrt( b ) );
>> K = sqrt( b ) / 1;
>> Flead = ( tD * s + 1 ) / ( b * tD * s + 1 );
>> g = 0;
>> tI = 10 / wc;
>> Flag = ( tI * s + 1 ) / ( tI * s + g );
>> F = K * Flead * Flag;
>> margin( F * G )
```



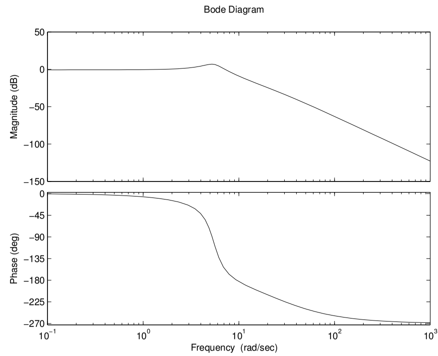
Plot the step response of the closed loop system.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 10 )
```



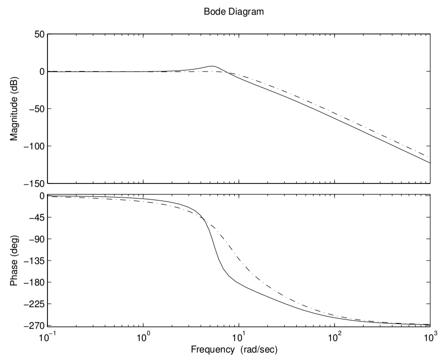
c) Compute the transfer function of the closed loop system for $F(s) = 1$. Draw the Bode plot.

```
>> Gc1 = feedback( G, 1 );
>> bode( Gc1 )
```



Add the Bode plot of the compensated closed loop system to the previous figure. The curves of the compensated system are dash-dotted.

```
>> hold on
>> bode( Gc, '-.' )
>> hold off
```



Comparing the two Bode plots we see that the main difference is that the height of the resonance peak has been reduced, that is, the damping of the closed loop system has been increased due to the increased phase margin. We also see that the bandwidth is approximately the same, since we have not changed the gain crossover frequency.

d) Calculate the transfer function from the reference signal to the error:

$$E(s) = R(s) - F(s)G(s)E(s) \quad \Rightarrow \quad E(s) = \frac{1}{1+F(s)G(s)}R(s)$$

Let

$$S(s) = \frac{1}{1+F(s)G(s)}$$

Enter the transfer function S.

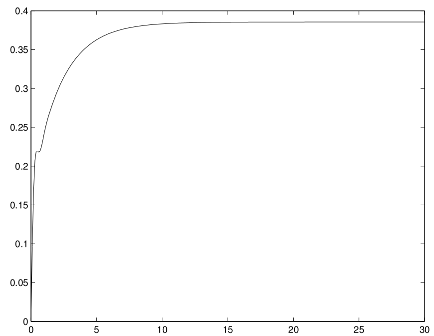
```
>> S = 1 / ( 1 + F * G );
```

Create a time vector between 0 and 30 with step 0.1, and a reference signal vector $r(t) = t$.

```
>> t = ( 0 : 0.1 : 30 ).';
>> r = t;
```

Plot the result. Even though the steady state error for a step reference signal is zero (due to $\gamma = 0$), the steady state error for a ramp reference signal is non-zero.

```
>> y = lsim( S, r, t );
>> plot( t, y )
```



5.14 The amplitude and phase at $\omega = 0.2$ rad/s is 0.0162 and $-140°$. We need a phase lift of 20° to obtain a phase margin of 60°. A lag part is needed to remove the steady state error. Hence we need 6° more in phase lift, all in all a 26° phase lift. This is obtain by employing lead and lag compensators (using standard notation of the parameters). First, $N = 3$ and $b = \omega_{c,d}/\sqrt{N} = 0.12$ give the required phase lead at the desired gain crossover frequency. Then $K = \frac{1}{0.0162\sqrt{N}} = 35.7$ achieves that gain crossover frequency. Finally, $a = 0.1\omega_{c,d} = 0.02$ and $M = \infty$ remove the steady-state error.

The resulting controller is:

$$F(s) = 35.7 \cdot 3 \frac{s+0.12}{s+3\cdot 0.12} \frac{s+0.02}{s}$$

5.15 a) Combining the system's transfer function with the controller K, the loop gain becomes

$$G_o(s) = \frac{0.25K}{(\tau_1 s+1)(\tau_2 s+1)s}$$

which leads to the error coefficients

$$e_0 = \frac{1}{1+\lim_{s\to 0} G_o(s)} = 0, \qquad e_1 = \frac{1}{\lim_{s\to 0} sG_o(s)} = \frac{4}{K}$$

provided that G_c is stable. The Bode plot shows that stability of G_c under proportional control may be evaluated via the gain margin A_m, that is, G_c is stable if $K < A_m$. The Bode plot gives $A_m = 4000$, so the condition under which the error coefficients are defined is

$$K < 4000$$

b) The problem formulation suggests the use of a lead-lag compensator.

Let $\omega_{c,d}$ denote the desired gain crossover frequency 100 rad/s. The Bode plot gives $|G(i\omega_{c,d})| = 5\cdot 10^{-4}$ and $\arg G(i\omega_{c,d}) = -175°$. To obtain the desired phase margin, a phase lead of $((-180°)+50°+6°)-(-175°) = 51°$ is needed, where 6° has been added to ensure that the phase margin is kept even if a lag compensator is used. To this end, introduce a lead compensator in the controller:

$$F_{lead} = N\frac{s+b}{s+bN}$$

See the discussion of lead compensators in Glad&Ljung! To keep the high frequency gain of the controller as small as possible, N should be chosen as small as possible. The desired phase advance is obtained with $N = 8$. This phase lead is obtained at the desired crossover frequency if

$$b = \frac{\omega_{c,d}}{\sqrt{N}} = 35.4$$

The desired crossover frequency is obtained by adjusting the gain of the open loop system by introducing a factor, K, in the controller:

$$1 = K\,|F_{lead}(i\omega_{c,d})| \cdot |G(i\omega_{c,d})| = K\sqrt{N}\cdot 5\cdot 10^{-4} \quad \Rightarrow \quad K = 707$$

Since the system contains an integrator, the step error coefficient e_0 is zero. The ramp error coefficient requrement is

$$e_1 = \frac{1}{\lim_{s\to 0} sF(s)G(s)} < 0.001 \iff \frac{4}{\lim_{s\to 0} F(s)} < 0.001 \iff 4000 < \lim_{s\to 0} F(s)$$

but the controller KF_{lead} doesn't fulfill this requrement since

$$\lim_{s\to 0} KF_{\text{lead}}(s) = 707$$

Hence, the static gain of the controller must be increased by the factor $\frac{4000}{707} = 5.7$. To this end, introduce a lag compensator in the controller,

$$F_{\text{lag}} = \frac{s+a}{s+a/M}$$

with $M = 5.7$ and $a = 0.1\omega_{\text{c,d}} = 10$ (see the discussion of lag compensators in Glad&Ljung!).

The resulting controller is

$$F(s) = 707 \cdot 8\frac{s+35.4}{s+282.8} \cdot \frac{s+10}{s+1.78}$$

5.16 Systemet $G(s) = \frac{2}{s+1}e^{-0.25s}$ regleras med en P-regulator med $K = 1/\sqrt{2}$. Skärfrekvensen ω_c ges av

$$1 = |KG(i\omega_c)| = \frac{2K}{\sqrt{\omega_c^2+1}} \Rightarrow 4K^2 = \omega_c^2 + 1 \Rightarrow \omega_c = \sqrt{4K^2-1} = 1.$$

Fasmarginalen φ_m ges av

$$\varphi_m = \pi + \arg(KG(i\omega_c)) = \pi - 0.25\omega_c - \arctan\omega_c = \\ = \pi - 0.25 - \arctan 1 = \frac{3\pi - 1}{4} \approx 121°.$$

Vi vill bestämma en lead-lag-regulator $F(s)$ som ger dubbla skärfrekvensen och samma fasmarginal. Vid $\omega_{c,ny} = 2$ är fasmarginalen

$$\varphi_{m,ny} = \pi + \arg G(2i) = \pi - 0.5 - \arctan 2 \approx 88°.$$

Det innebär att vi måste höja fasen med

$$\Delta\varphi_m = 121° - 88° + 6° = 39°,$$

med 6° för lag-länk. Det ger

$$\beta = \frac{1-\sin(\Delta\varphi_m)}{1+\sin(\Delta\varphi_m)} = 0.2, \quad \tau_D = \frac{1}{\omega_{c,ny}\sqrt{\beta}} = 1.1.$$

Då har vi $F_{lead} = \frac{1+\tau_D s}{1+\beta\tau_D s}$.

För att få rätt skärfrekvens bestämmer vi ett K' så att

$$1 = |K'F_{lead}(i\omega_c)G(i\omega_c)| = 2K'\frac{1}{\sqrt{\beta(\omega_{c,ny}^2+1)}} = 2K'\sqrt{\frac{1}{5\beta}} = 2K' \Rightarrow K' = \frac{1}{2}$$

Det stationära felet måste vara mindre än 0.05 när referensen är ett steg. Vi lägger till en lag-länk $F_{lag} = \frac{1+\tau_I s}{\tau_I s+\gamma}$, där $\tau_I = \frac{10}{\omega_{c,ny}} = 5$ och γ bestäms så att

$$\frac{1}{1+K'F_{lead}(0)F_{lag}(0)G(0)} = \frac{1}{1+2K'/\gamma} \leq 0.05 \Rightarrow \gamma \leq \frac{2K'}{19} = 0.05$$

5.17 a) Vi söker $F(s) = F_{lead}(s)F_{lag}(s)$.

Vi börjar med den fasavancerande länken

$$F_{lead}(s) = K\frac{\tau_D s+1}{\beta\tau_D s+1}.$$

Den nya skärfrekvensen är $w_{c,d} = 30$ rad/s.

Eftersom $\varphi_m = 40°$ och $\varphi_m = \arg(F(iw_{c,d})G(iw_{c,d})) + 180° = \arg(F_{lead}(iw_{c,d})) + \arg(F_{lag}(iw_{c,d})) + \arg(G(iw_{c,d})) + 180°$, så får vi $\arg(F_{lead}(iw_{c,d})) = -140° - \arg(F_{lag}(iw_{c,d})) - \arg(G(iw_{c,d}))$. Från bode-diagrammet har vi $\arg(G(iw_{c,d})) \approx -180°$ och från tumregeln om fas-retarderande länkar, vet vi att den minskar fasen med 6° för lämpliga parameterval. Alltså $\arg(F_{lead}(iw_{c,d})) = -140° + 6° + 180° = 46°$ och $\beta = 0.17$. Med detta β får vi $\tau_D = (w_{c,d}\sqrt{\beta})^{-1} = 0.0812$.

Vi väljer K så att $w_{c,d} = 30$: $|F(iw_{c,d})G(iw_{c,d})| = 1$. Detta ger

$$|F_{lead}(iw_{c,d})||F_{lag}(iw_{c,d})||G(iw_{c,d})| = 1.$$

Från tumregeln följer $|F_{lag}(iw_{c,d})| \approx 1$, och

$$\frac{K}{\sqrt{\beta}} \frac{|k_1|}{|iw_{c,d}(iw_{c,d}+a)(iw_{c,d}+b)|} = 1,$$

vilket ger $K = 395.17$.

Den fasretarderande länken ges av

$$F_{lag}(s) = \frac{\tau_I s + 1}{\tau_I s + \gamma},$$

och enligt tumregeln ska $\tau_I = 10/w_{c,d} = 0.33$. Vi vill välja γ så att statiska felet vid steginsignaler är noll. Enligt slutvärdesteoremet (slutna systemet är asymptotiskt stabilt, se ovan)

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{1}{1 + F_{lead}(s)F_{lag}(s)G(s)} \frac{1}{s},$$

vilket ger

$$\lim_{s\to 0} \frac{(s+a)(s+b)(s+c)(\beta\tau_D s+1)(\tau_I s+\gamma)}{(s+a)(s+b)(s+c)(\beta\tau_D s+1)(\tau_I s+\gamma) + Kk_1(\tau_D s+1)(\tau_I s+1)} =$$

$$= \frac{abc\gamma}{abc\gamma + Kk_1}.$$

Alltså ska vi välja $\gamma = 0$.

Den resulterande regulatorn ges av

$$F(s) = F_{lead}(s)F_{lag}(s) = 395.17\frac{0.0812s+1}{0.0137s+1}\frac{0.33s+1}{0.33s}.$$

b) Den verkliga öppna loopen ges av $F(s)G^0(s) = F(s)G(s)e^{-T_d s}$. Notera att $|F(jw)G^0(jw)| = |F(jw)G(jw)e^{-jT_d w}| = |F(jw)G(jw)||e^{-jT_d w}| = |F(jw)G(jw)|$, medan $\arg(F(jw)G^0(jw)) = \arg(F(jw)G(jw)) - T_d w$. Eftersom tidsfördröjningen bara påverkar fasen tittar vi på fasmarginalen. Regulatorn är designad så att $\varphi_m = 40° = \frac{40°}{180°}\pi$ rad. Alltså $\varphi_m^0 = \varphi_m - T_d w_c$. Slutna systemet är stabilt om $\varphi_m^0 > 0$, vilket ger $\varphi_m - T_d w_c > 0 \Leftrightarrow T_d < \frac{\varphi_m}{w_c} = 0.0233$ s.

c) Slutna systemets (G_c) snabbhet ges av dess bandbredd, vilken är $w_B \approx 50$ rad/s. Ett lågpassfilter $F_r(s) = \frac{1}{1+\tau s}$ uppfyller $|F_r(jw)| \approx 1$ för $w < \tau^{-1}$, medan för $w > \tau^{-1}$ avtar förstärkningen med lutning -1 i ett bodediagram. Approximativt gäller då att F_r bara reducerar hela systemets bandbredd om $w_B > \tau^{-1}$, vilket ger $\tau > w_B^{-1} = \frac{1}{50}$.

6 Sensitivity and Robustness

6.1 The sensitivity function is the transfer function from v to y. The block diagram gives

$$Y(s) = \frac{1}{1+\frac{K}{s(s+1)}}V(s) = \underbrace{\frac{s^2+s}{s^2+s+K}}_{S(s)}V(s)$$

$$|S(\mathrm{i}\omega)| = \frac{\omega\sqrt{\omega^2+1}}{\sqrt{(K-\omega^2)^2+\omega^2}}$$

For $\omega = 1$ we get

$$|S(1\mathrm{i})| = \frac{\sqrt{2}}{\sqrt{(K-1)^2+1}}$$

The amplitude of $y(t)$ is less than the amplitude of $v(t)$ if $|S(1\mathrm{i})| < 1$, that is,

$$\frac{\sqrt{2}}{\sqrt{(K-1)^2+1}} < 1 \quad \Leftrightarrow \quad 2 < (K-1)^2+1 \quad \overset{K\geq 0}{\Leftrightarrow} \quad K > 2$$

6.2 Determine the upper limit of the relative model error

$$G_\Delta(s) = \frac{G^0(s)-G(s)}{G(s)} = s \quad \Rightarrow \quad |G_\Delta(\mathrm{i}\omega)| = \omega$$

The stability is then guaranteed if

$$|G_c(\mathrm{i}\omega)| = \left|\frac{F(\mathrm{i}\omega)G(\mathrm{i}\omega)}{1+F(\mathrm{i}\omega)G(\mathrm{i}\omega)}\right| < \frac{1}{\omega} \qquad \forall\omega$$

No steady state error for steps implies $G_c(0) = 1$ and the bandwidth ω_B is thus defined by the smallest value that satisfies

$$|G_c(\mathrm{i}\omega)| < \frac{1}{\sqrt{2}}, \qquad \omega > \omega_B$$

The curve $1/\omega$ crosses $1/\sqrt{2}$ at $\omega = \sqrt{2}$. Thus, the bandwidth must be less than $\sqrt{2}$. However, the curve $|G_c(\mathrm{i}\omega)|$ asymptotically approaches a line with slope -20 dB_{20}/decade, which implies that ω_B cannot be arbitrarily close to $\sqrt{2}$.

For example, if G_c is a first order system, then the breakpoint of the asymptote must be 1 rad/s if it shall coincide with $1/\omega$. The first order system with that asymptote is $\frac{1}{1+s/1}$, which has a bandwidth of 1 rad/s. If G_c would be a higher order system, the bandwidth could be made slightly higher, but the limited information about G_c excludes this possibility.

Answer: The maximum bandwidth is $\omega_B = 1$.

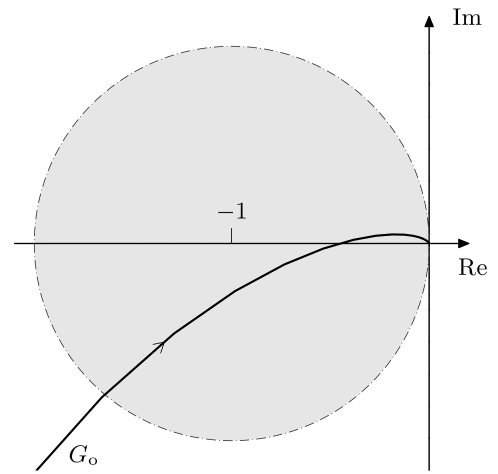


Figure 6.3a

6.3 The disturbance is amplified when the magnitude of the sensitivity function exceeds one, that is, when

$$\left|\frac{1}{1+G_o(\mathrm{i}\omega)}\right| > 1$$

that is

$$|1+G_o(\mathrm{i}\omega)| < 1$$

which corresponds to the part of $G_o(\mathrm{i}\omega)$ that is within a circle with center at -1 and radius 1, see Figure 6.3a.

6.4 Let

$$g(\omega) = \frac{0.9}{\sqrt{1+\omega^2}}$$

denote the upper bound on the norm of the relative model error. Robustness condition:

$$|T(i\omega)| = \left|\frac{F(i\omega)G(i\omega)}{1+F(i\omega)G(i\omega)}\right| < \frac{1}{g(\omega)} \qquad \forall\omega$$

Now,

$$F(s)G(s) = \frac{s+10}{s}\,\frac{1}{s+10} = \frac{1}{s} \quad \Rightarrow$$

$$\left|\frac{F(i\omega)G(i\omega)}{1+F(i\omega)G(i\omega)}\right| = \left|\frac{1}{i\omega+1}\right| = \frac{1}{\sqrt{\omega^2+1}}$$

so the robustness condition becomes

$$\forall\omega: \ \frac{1}{\sqrt{\omega^2+1}} < \frac{\sqrt{\omega^2+1}}{0.9} \quad \Leftrightarrow$$
$$\forall\omega: \ 0.9 < \omega^2+1$$

which is satisfied.

Answer: Yes.

6.5 a) Using notation similar to that in Glad&Ljung, we have

$$G_\Delta(s) = e^{-sT} - 1$$

that is, $G_\Delta(i\omega) = \cos\omega T - 1 - i\sin\omega T$. This implies

$$|G_\Delta(i\omega)| = \sqrt{2 - 2\cos\omega T}$$

and in particular

$$|G_\Delta(i\omega)| = \begin{cases} 0, & \text{when } \cos\omega T = 1 \\ 2, & \text{when } \cos\omega T = -1 \end{cases}$$

In Figure 6.5a, $|G_\Delta(i\omega)|^{-1}$ is plotted as a function of ωT.

b) The robustness criterion results in

$$\forall\omega: \ \left|\frac{F(i\omega)G(i\omega)}{1+F(i\omega)G(i\omega)}\right| < \frac{1}{|G_\Delta(i\omega)|}$$

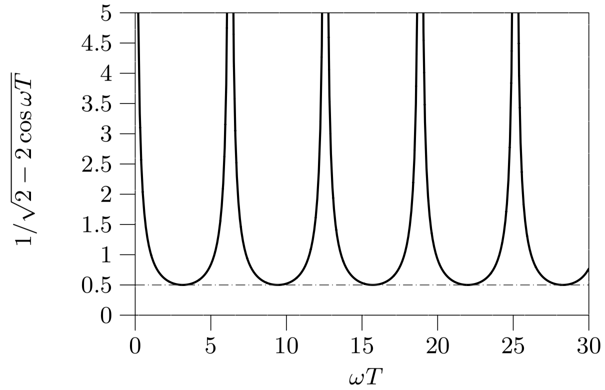


Figure 6.5a

Figure 6.5a therefore provides the answer.

Answer:

$$\left|\frac{F(i\omega)G(i\omega)}{1+F(i\omega)G(i\omega)}\right| < \frac{1}{2}$$

6.6 a) First identify the relative model error:

$$G^0(s) = G(s) + \tilde{G}(s) = G(s)\left(1 + \frac{\tilde{G}(s)}{G(s)}\right)$$

that is,

$$G_\Delta(s) = \frac{\tilde{G}(s)}{G(s)}$$

The robustness criterion

$$\forall\omega: \ \left|\frac{1}{G_\Delta(i\omega)}\right| = \left|\frac{G(i\omega)}{\tilde{G}(i\omega)}\right| > \left|\frac{KG(i\omega)}{1+KG(i\omega)}\right|$$

gives

$$|\tilde{G}(i\omega)| < \frac{|i\omega(i\omega+5)+K|}{|Ki\omega(i\omega+5)|} = \frac{2}{25}\cdot\sqrt{\frac{(25/2-\omega^2)^2+25\omega^2}{\omega^2(\omega^2+25)}} =$$
$$= \frac{2}{25}\cdot\sqrt{\frac{\omega^4+(25/2)^2}{\omega^2(\omega^2+25)}} =: g(\omega)$$

Because $g(\omega) \to 2/25$ as $\omega \to \infty$ stability cannot be guaranteed when $\tilde{G}(s) = 1$. Also note that the requirement that $G^0(\mathrm{i}\omega)F(\mathrm{i}\omega) \to 0$ as $\omega \to \infty$ fails, since $G^0(\mathrm{i}\omega) \to 1,\ \omega \to \infty$.

b) When $\tilde{G}(s) = \alpha$ the closed loop system becomes

$$\frac{KG^0(s)}{1+KG^0(s)} = \frac{K(1+\alpha s(s+5))}{s(s+5)+K(1+\alpha s(s+5))}$$

with characteristic equation

$$s^2(2+25\alpha)+5s(2+25\alpha)+25=0$$

Rouths algorithm gives the condition

$$2+25\alpha > 0 \quad \Leftrightarrow \quad \alpha > -2/25$$

This is not contradictory since the robustness criterion is a *sufficient* but not *necessary* condition.

6.7 a) The characteristic equation can be determined for a generic nominal loop gain. Let

$$G_\mathrm{o}(s) = \frac{b(s)}{a(s)}$$

denote the nominal loop gain. The true closed loop system becomes

$$G_\mathrm{c}(s) = \frac{\frac{b(s)}{a(s)}\frac{\alpha}{s+\alpha}}{1+\frac{b(s)}{a(s)}\frac{\alpha}{s+\alpha}} = \frac{b(s)\alpha}{a(s)(s+\alpha)+b(s)\alpha} = \frac{b(s)\alpha}{a(s)s+(a(s)+b(s))\alpha}$$

and has the same root locus with respect to α as the open loop system

$$\frac{a(s)+b(s)}{a(s)s} = \frac{G_\mathrm{o}+1}{s}$$

has with respect to a proportional feedback. This can be used to draw the root locus using MATLAB. However, to draw the root locus by hand, we use that here $G_\mathrm{o}(s) = KG(s)$, so

$$b(s) = 4 \qquad a(s) = s(s+1)$$

which lets us identify the polynomials P and Q in the characteristic equation $P(s)+\alpha Q(s) = 0$ as

$$P(s) = a(s)s = s^2(s+1) \qquad Q(s) = a(s)+b(s) = s^2+s+4$$

- ◇ Starting points ⇒ zeros of $P(s)$: 0 (double), and -1
  End points ⇒ zeros of $Q(s)$: $-\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{15}}{2}$
- ◇ Number of asymptotes: $3-2=1$.
  Direction of asymptote: $\frac{1}{1}\cdot\pi$, that is, the negative real axis.
- ◇ Part of the real axis that belongs to the root locus: $(-\infty, -1]$.
- ◇ Intersection with the imaginary axis: Set $s = \mathrm{i}\omega$ and solve the characteristic equation:

$$-\omega^2(\mathrm{i}\omega+1)+\alpha(-\omega^2+\mathrm{i}\omega+4)=0$$

Isolate real and imaginary parts:

$$\begin{cases} -\omega^2(1+\alpha)+4\alpha = 0 \\ -\omega^3+\alpha\omega = 0 \end{cases}$$

with solutions

$$(\,\alpha=0,\ \omega=0\,) \quad \text{or} \quad (\,\alpha=3,\ \omega=\pm\sqrt{3}\,)$$

The root locus is shown in Figure 6.7a, from which the conclusion immediately follows.

Answer: Asymptotically stable for $\alpha > 3$.

b) Begin by identifying the relative model error:

$$G^0(s) = G(s)\frac{\alpha}{(s+\alpha)} = G(s)\Big(1+\underbrace{\frac{\alpha}{(s+\alpha)}-1}_{G_\Delta(s)}\Big)$$

Thus

$$\frac{1}{|G_\Delta(\mathrm{i}\omega)|} = \left|\frac{s+\alpha}{-s}\right| = \frac{\sqrt{\omega^2+\alpha^2}}{\omega} =: f(\omega)$$

The robustness criterion $\forall\omega:\ |G_\mathrm{c}(\mathrm{i}\omega)| < f(\omega)$ is fulfilled if the low frequency asymptote of $f(\omega)$ exceeds the resonance peak at $\omega = 2$, where $|G_\mathrm{c}(\mathrm{i}2)| = 2$. This gives the condition

$$\frac{\sqrt{4+\alpha^2}}{2} > 2 \quad \overset{\alpha \geq 0}{\Leftrightarrow} \quad \alpha > \sqrt{12}$$

Answer: $\alpha > \sqrt{12}$

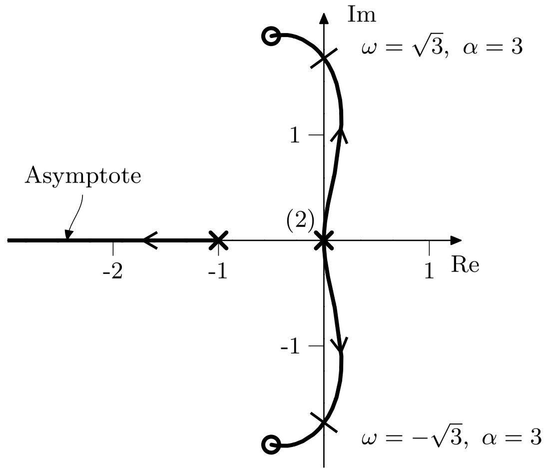


Figure 6.7a

c) The robustness criterion gives a sufficient but not necessary condition, that is, the system can be stable even if the criterion is not satisfied. In this case for $3 < \alpha < \sqrt{12}$. With a root locus we obtain an exact characterization of the stabilizing parameter values, that is, a necessary and sufficient condition.

6.8 It can be shown that both $F(\mathrm{i}\omega)G(\mathrm{i}\omega)$ and $F(\mathrm{i}\omega)G^0(\mathrm{i}\omega)$ tend to 0 as $\omega \to \infty$. The robustness criterion guarantees stability if

$$|G_c(\mathrm{i}\omega)| < \frac{1}{\gamma\omega}$$

since

$$|G_\Delta(\mathrm{i}\omega)| < \gamma\omega \quad \Rightarrow \quad \frac{1}{\gamma\omega} < \frac{1}{|G_\Delta(\mathrm{i}\omega)|}$$

The transfer function G_c has a resonance peak at $\omega = 1$ with $|G_c(\mathrm{i}1)| = 35$, which leads to the condition

$$35 < \frac{1}{\gamma \cdot 1} \quad \Leftrightarrow \quad \gamma < \frac{1}{35}$$

Trivially, γ must also be positive.

Answer: $0 \leq \gamma < \frac{1}{35}$

6.9 The closed loop system becomes

$$Y(s) = V(s) + G_o(s)(R(s) - N(s) - Y(s)) \quad \Rightarrow$$

$$Y(s) = \frac{G_o(s)}{1+G_o(s)}(R(s) - N(s)) + \frac{1}{1+G_o}V(s)$$

where we can identify

$$T(s) = \frac{G_o(s)}{1+G_o(s)} \qquad S(s) = \frac{1}{1+G_o(s)}$$

Notice that $S(s) + T(s) = 1$. In the problem formulation we have $Y(s) = S(s)V(s)$ since the other inputs are zero. Hence, for $v(t) = \sin t$, we have

$$\mathcal{L}^{-1}\{SV\}(t) = \frac{1}{\sqrt{2}}\sin(t - \frac{\pi}{4})$$

and thus for $n(t) = \sin t$

$$Y(s) = -T(s)N(s) = -(1 - S(s))N(s) = S(s)N(s) - N(s) \quad \Rightarrow$$

$$y(t) = \frac{1}{\sqrt{2}}\sin(t - \frac{\pi}{4}) - \sin(t)$$

6.10 a) Putting

$$G^0(s) = G(s)\frac{1}{(s+1)} = G(s)(1 + G_\Delta(s))$$

gives

$$G_\Delta(s) = -\frac{s}{s+1}$$

and

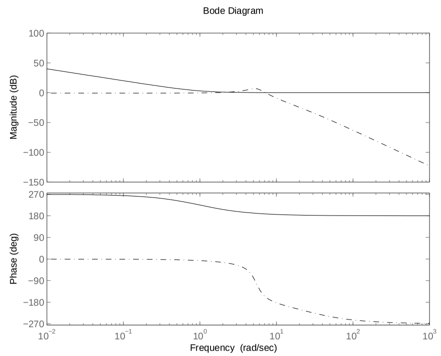
$$\frac{1}{G_\Delta(s)} = -\frac{s+1}{s}$$

b) Enter the system and the regulator from Problem ??.

```
>> s = tf( 's' );
>> G = 725 / ...
       ( ( s + 1 ) * ( s + 2.5 ) * ( s + 25 ) );
>> wc = 5;
>> N = 5;
>> b = wc / sqrt( N );
>> K = 1 / sqrt( N );
>> Flead = N * ( s + b ) / ( s + b * N );
>> a = 0.1 * wc;
>> Flag = ( s + a ) / s;
>> F = K * Flead * Flag;
```

Enter the inverse relative model error and the complementary sensitivity function obtained when $G(s)$ is controlled by $F(s) = 1$. Plot the amplitude curve of the inverse relative model error in the same diagram as the amplitude curve of the complementary sensitivity function.

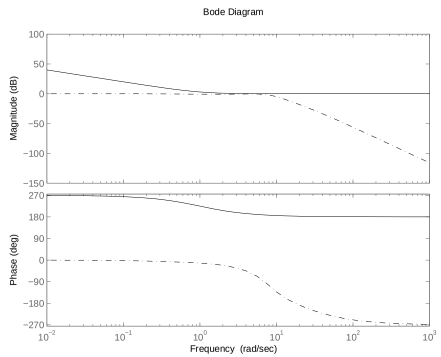
```
>> IDG = - ( s + 1 ) / s;
>> T = feedback( 1 * G, 1 );
>> bode( IDG, 'k-', ...
        T, 'k-.' );
```



Since the absolute value of the complementary sensitivity function goes above the inverse relative model error over a frequency interval, we cannot guarantee that the closed loop system obtained when $G^0(s)$ is controlled by $F(s)$ is asymptotically stable.

Enter the complementary sensitivity function obtained when $G(s)$ is controlled by the lead-lag regulator designed in Problem **??**. Plot the amplitude curve of the inverse relative model error in the same diagram as the the amplitude curve of the complementary sensitivity function.

```
>> T = feedback( F * G, 1 );
>> bode( IDG, 'k-', ...
        T, 'k-.' );
```



In this case $|T(i\omega)|$ stays below the inverse relative model error, and hence we can guarantee that the closed loop system obtained when the lead-lag regulator is applied to $G^0(s)$ will be asymptotically stable.

6.11 The transfer function between the reference and the error is the sensitivity function. When the reference signal is a sinus the error signal will also be a sinus with the same frequency and with an amplitude modified by the gain of the transfer function at that frequency, $|S(0.1i)| = -20\ \mathrm{dB}_{20} = 0.1$. This gives that the amplitude of the error is 0.2.

6.12 A way to see if the controller also stabilizes the system at 400 r/min is to look at the phase and amplitude margin of

$$F(s)G(s) = 35.7 \cdot 3 \frac{s + 0.116}{s + 0.116 \cdot 3} \frac{s + 0.02}{s} \frac{0.02}{s + 0.02} \frac{e^{-2s}}{1 + 20s}$$

A bode plot of this system is given in Figure 6.12a were it can be seen that the phase margin is 9.54° and that the amplitude margin is 1.3. The closed loop system is stable but the margin is small.

6.13 The sensitivity function is given by

$$S(s) = \frac{1}{1 + F(s)G(s)}$$

which in this case means

$$S(s) = \frac{(s+1)^2}{(s+1)^2 + K}$$

The demand that the amplification of the sensitivity function should be less than 1 at $\omega = 1$ gives

$$|S(i1)| = \frac{2}{\sqrt{4 + K^2}} \leq 0.1$$

that is, $K \geq \sqrt{396} \approx 19.9$.

To illustrate, the condition is verified in MATLAB.

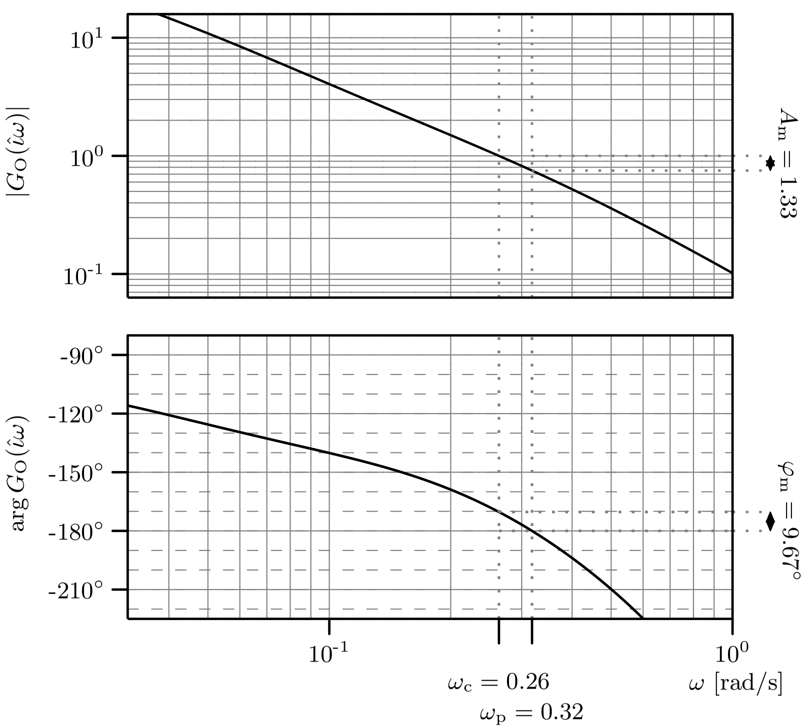
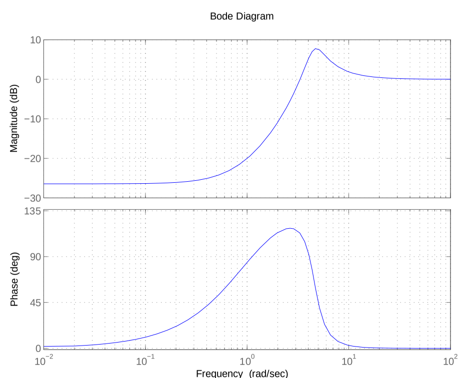


Figure 6.12a

Enter the system and create the sensitivity function. Plot with a grid.

```
>> s = tf( 's' );
>> G = 1 / ( s + 1 )^2;
>> K = 20;
>> S = minreal( 1 / ( 1 + K * G ) );
>> bode( S );
>> grid;
```



6.14 a) **Hitta** $G_c(s)$

Härledning av överföringsfunktionen för det slutna systemet,

$$G_c(s) = \frac{G_o(s)}{1+G_o(s)} = \frac{FG(s)}{1+FG(s)} = \frac{\frac{3s+1}{s}\frac{1}{(s-1)(\epsilon s+1)}}{1+\frac{3s+1}{s}\frac{1}{(s-1)(\epsilon s+1)}} = \frac{3s+1}{s(s-1)(\epsilon s+1)+3s+1}.$$

Identifiera $P(s)$ **och** $Q(s)$

Skriv om nämaren till $G_c(s)$ som

$$s(s-1)(\epsilon s+1)+3s+1 = \epsilon(s^3-s^2)+(s^2+2s+1).$$

Detta ger

$$Q(s) = s^3 - s^2, \quad P(s) = s^2+2s+1.$$

Dock måste gradtalet för $P(s)$ vara större än gradtalet för $Q(s)$. Så är ej fallet. Vi hanterar detta genom att rita rotorten för $K = 1/\epsilon$ istället. Följaktligen blir

$$P(s) = s^3 - s^2, \quad Q(s) = s^2+2s+1.$$

Hitta startpunkter

Startpunkter är de s där $P(s) = 0$. De är $s_1 = 0$, $s_2 = 0$ och $s_3 = 1$.

Hitta ändpunkter

Ändpunkter är de s där $Q(s) = 0$. De är $s_1 = -1$, $s_2 = -1$.

Antal asymptoter

Antalet asymptoter är $n - m = 1$, där n är gradtalet för $P(s)$ och m är gradtalet för $Q(s)$.

Hitta riktningar

Asymptotens riktning ges av

$$\frac{\pi}{n-m} = \pi.$$

Följaktligen kommer asymptoten ej att skära reella axeln.

Hitta eventuell skärning med imaginära axeln
Sätt in $s = i\omega$ i $P(s) + KQ(s) = 0$ och lös ekvationen för reella ω och icke-negativa K. Vi får

$$P(i\omega) + KQ(i\omega) = -i\omega^3 + \omega^2 - K\omega^2 + 2Ki\omega + K = 0$$
$$\Rightarrow \omega = 0,\ K = 0 \text{ och } \omega = \sqrt{3},\ K = 3/2.$$

Bestäm de delar av reella axeln som tillhör rotorten
Den del av reella axeln som tillhör rotorten är $-\infty < s \leq 1$.

Rita rotorten
Se figur 6.14a.

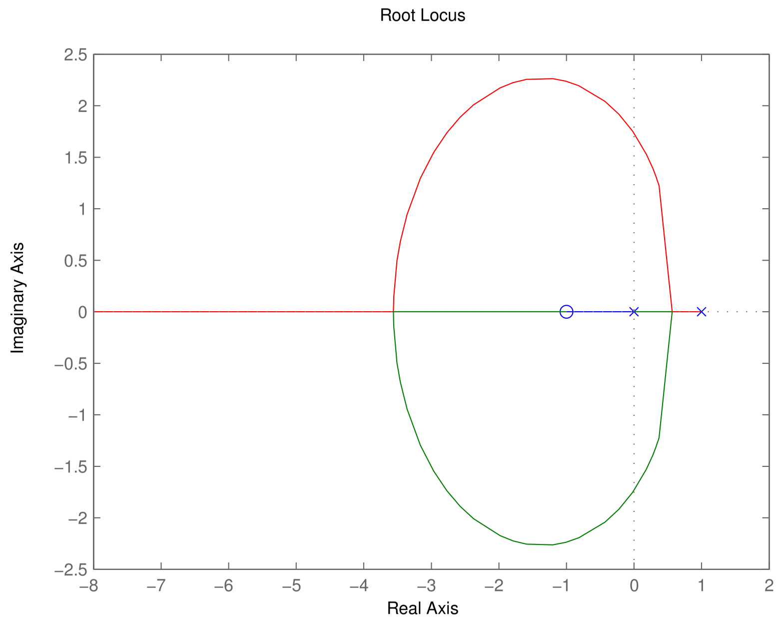


Figure 6.14a

Följaktligen så är det slutna systemet stabilt för $K > 3/2 \Rightarrow 0 \leq \epsilon < 2/3$.

Svar: $0 \leq \epsilon < 2/3$.

b) **Identifiera det relativa modellfelet**
Enligt definitionen så har vi

$$G^0(s) = G(s)[1 + \Delta_G(s)].$$

I vårt fall så är

$$G^0(s) = \frac{1}{(s-1)(\epsilon s + 1)} = G(s)(1 - \frac{\epsilon s}{\epsilon s + 1}) \Rightarrow \Delta_G(s) = \frac{-\epsilon s}{\epsilon s + 1}$$

Enligt robusthetskriteriet så är det slutna systemet stabilt om $|T(i\omega)| < 1/|\Delta_G(i\omega)|$ för alla ω. Den asymptotiska amplitudkurvan för

$$\frac{1}{\Delta_G(s)} = \frac{1 + s/(1/\epsilon)}{-s/(1/\epsilon)}$$

har lutning -1 fram till $\omega = 1/\epsilon$ och sedan lutning 0. Den har förstärkning 1 för frekvenser $\omega > 1/\epsilon$. Enligt figuren har $|T(i\omega)|$ förstärkning mindre än 1 för ungefär $\omega > 3$, så olikheten är uppfylld om $\epsilon < 0.33$.

Svar: $0 \leq \epsilon < 0.33$.

6.15 a) $S(s)$ är överföringsfunktionen från störsignal till utsignal. För att undertrycka en störning av frekvens ω ska $|S(i\omega)| < 1$. $-G_c(s)$ är överföringsfunktionen från mätbruset till systemets utsignal. För att undertrycka mätbrus av frekvens ω ska $|G_c(i\omega)| < 1$. Vi har följande samband mellan $S(s)$ och $G_c(s)$

$$S(s) + G_c(s) = \frac{1}{1 + G_0(s)} + \frac{G_0(s)}{1 + G_0(s)} = 1.$$

På grund av detta samband så kan inte både $S(s)$ och $G_c(s)$ göras små oberoende av varandra. Således kan vi inte både undertrycka störningen och mätbruset godtyckligt mycket samtidigt.

b) $S(s)$ är stabil så vi kan använda slutvärdessatsen:

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} sS(s)\frac{1}{s} = \lim_{s \to 0} S(s) = \{\text{nollställe i origo}\} = 0.$$

7 Special Controller Structures

7.1 a) Derive the transfer function:

$$\theta(s) = \frac{1}{(1+30s)(1+3s)}\theta_{\mathrm{m}}(s)$$
$$\theta_{\mathrm{m}}(s) = \frac{G_{\mathrm{R2}}(s)}{(1+10s)+G_{\mathrm{R2}}(s)}W(s)$$

$G_{\mathrm{R2}}(s) = K_2 = 9$ gives

$$\theta(s) = \frac{0.9}{(1+\frac{s}{0.033})(1+\frac{s}{0.33})(1+s)}W(s) =: G(s)W(s)$$

Thus,

$$|G(\mathrm{i}\omega)| = \frac{0.9}{\sqrt{1+(\frac{\omega}{0.033})^2}\sqrt{1+(\frac{\omega}{0.33})^2}\sqrt{1+\omega^2}}$$

with low frequency asymptote

$$|G(\mathrm{i}\omega)| \to 0.9,\ \omega \to 0$$

and

$$\arg G(\mathrm{i}\omega) = -\arctan\frac{\omega}{0.033} - \arctan\frac{\omega}{0.33} - \arctan\omega$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote, 0.9, and the breakpoints and slopes of the asymptotes:

Frequency [rad/s]		0.033		0.33		1	
Slope	0		−1		−2		−3

The phase curve is drawn based on a couple of samples:

Frequency [rad/s]	0.033	0.1	0.2	0.5	1.0
Phase	−52°	−94°	−123°	−169°	−205°

The Bode plot in Figure 7.1a gives that the gain crossover frequency and the phase margin are undefined, but we have a gain margin:

$$\omega_{\mathrm{p}} = 0.61\ \mathrm{rad/s} \quad A_{\mathrm{m}} = 50.5$$

A gain margin of 2 is obtained when

$$K_1 \cdot \frac{1}{50.5} = \frac{1}{2}$$

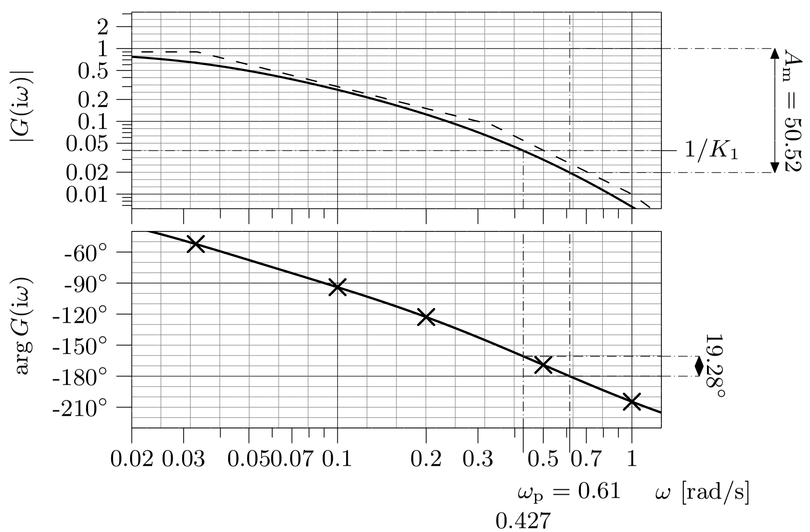


Figure 7.1a

that is, $K_1 = 25.25$. This results in a new gain crossover of 0.43 rad/s (and new phase margin of 19°). To find the steady state error, study how the Laplace transforms of the controll error relates to that of the reference:

$$E(s) = \frac{1}{1+K_1G(s)}\theta_{\mathrm{ref}}(s)$$

which with $\theta_{\mathrm{ref}}(s) = \frac{a}{s}$ gives

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{a}{1+K_1\cdot 0.9} = 0.042\cdot a$$

b) Without the internal feedback we get the transfer function defined by

$$\theta(s) = \frac{1}{(1+\frac{s}{0.033})(1+\frac{s}{0.1})(1+\frac{s}{0.33})}W(s) =: G(s)W(s)$$

and thus

$$|G(\mathrm{i}\omega)| = \frac{1}{\sqrt{1+(\frac{\omega}{0.033})^2}\sqrt{1+(\frac{\omega}{0.33})^2}\sqrt{1+(\frac{\omega}{0.1})^2}}$$

with low frequency asymptote

$$|G(\mathrm{i}\omega)| \to 1,\ \omega \to 0$$

and

$$\arg G(\mathrm{i}\omega) = -\arctan\frac{\omega}{0.033} - \arctan\frac{\omega}{0.33} - \arctan\frac{\omega}{0.1}$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote, 1, and the breakpoints and slopes of the asymptotes:

Frequency [rad/s]		0.033		0.1		0.33	
Slope	0		−1		−2		−3

The phase curve is drawn based on a couple of samples:

Frequency [rad/s]	0.033	0.1	0.2	0.4
Phase	−69°	−134°	−174°	−212°

The Bode plot in Figure 7.1b gives that, again, the gain crossover frequency and phase margin are undefined, but we have a gain margin:

$$\omega_\mathrm{p} = 0.22 \text{ rad/s} \quad A_\mathrm{m} = 19$$

A gain margin of 2 is obtained when $K_1 \cdot \frac{1}{19} = \frac{1}{2}$, which leads to $K_1 = 9.5$. This results in a new gain crossover of 0.15 rad/s (and a new phase margin of 21°). As above, we get the controll error for step references:

$$\lim_{t\to\infty} e(t) = \frac{1}{1+9.5\cdot 1}a = 0.095a$$

We conclude that due to the internal feedback, the system in a) is faster (higher bandwidth) as well as more precise (smaller stationary error).

7.2 Consider the block diagram in Figure 7.2a. The change in tank volume per time unit is given by

$$A\frac{\mathrm{d}}{\mathrm{d}t}h(t) = x(t) - v(t)$$

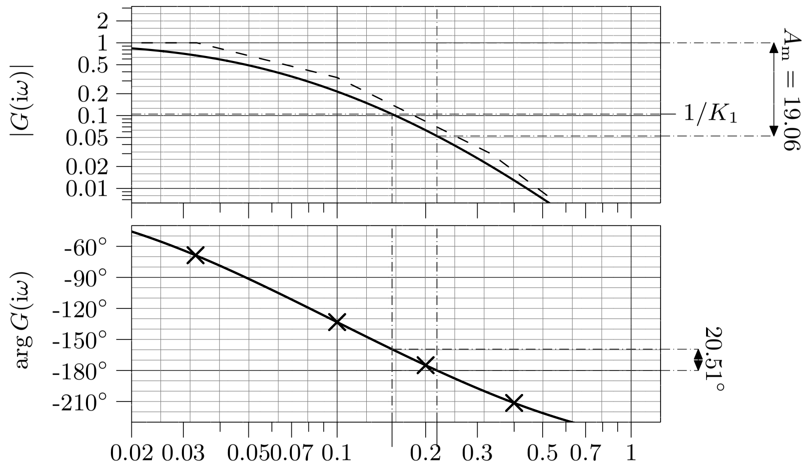


Figure 7.1b

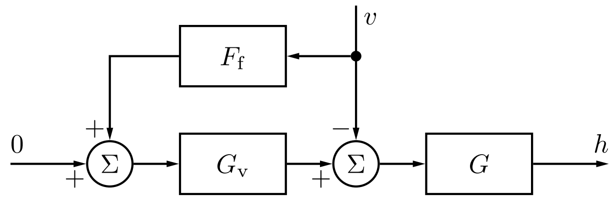


Figure 7.2a

or, equivalently,

$$A \cdot s \cdot H(s) = X(s) - V(s)$$

which gives

$$H(s) = \frac{1}{As}(X(s) - V(s))$$

Furthermore,

$$X(s) = G_\mathrm{v}(s)U(s)$$

where

$$G_\mathrm{v}(s) = \frac{1}{1+s/2}$$

a) We let the input $u(t)$ be a function of $v(t)$ only, that is,

$$U(s) = F_f(s)V(s)$$

The level $h(t)$ as a function of $v(t)$ then becomes

$$H(s) = \frac{1}{As}(G_v(s)F_f(s) - 1)V(s)$$

If we choose

$$F_f(s) = \frac{1}{G_v(s)} = 1 + s/2$$

the level becomes independent of $v(t)$, but to get the controller Stu uses, we remove the derivative term:

$$F_f(s) = 1$$

The level as a function of $v(t)$ then becomes

$$H(s) = \frac{1}{As}(\frac{1}{1+s/2} - 1)V(s) = -\frac{1}{2A}\frac{1}{1+s/2}V(s)$$

With $V(s) = 0.1/s$ this yields

$$H(s) = -\frac{0.1}{2A}\frac{1}{s(1+s/2)} = -\frac{0.1}{2A}\left(\frac{1}{s} - \frac{1}{2+s}\right)$$

that is

$$h(t) = -\frac{0.1}{A \cdot 2}(1 - e^{-2t})$$

which gives the steady state error $-0.05/A$.

b) We now choose the input $u(t)$ to be a function of both $h(t)$ and $v(t)$, that is, we add the term $-Kh(t)$ to the control law from a). (See Figure 7.2b.) Thus

$$u(t) = -Kh(t) + v(t)$$

or, equivalently,

$$U(s) = -KH(s) + V(s)$$

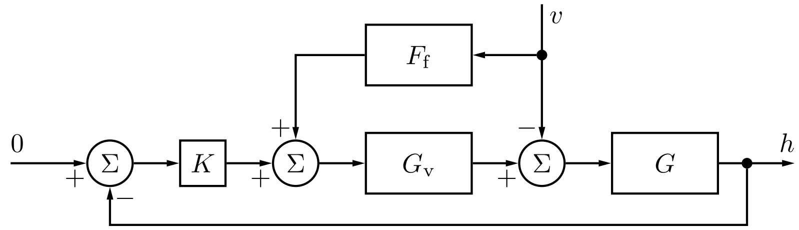


Figure 7.2b

This gives

$$\begin{aligned} AsH(s) &= G_v(s)(-KH(s) + V(s)) - V(s) \\ (As + KG_v(s))H(s) &= (G_v(s) - 1)V(s) \\ \frac{H(s)}{V(s)} &= \frac{-s/2}{A/2 \cdot s^2 + As + K} = \frac{-s}{A(s^2 + 2s + 2K/A)} \end{aligned}$$

To select K, we may compare*

$$s^2 + 2s + 2K/A = 0$$

with the standard equation

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

which gives

$$\omega_0^2 = 2 \cdot K/A \qquad \zeta\omega_0 = 1$$

To obtain approximately 5% overshoot we choose $\zeta = 0.707$, and from

$$\sqrt{A/(2K)} = \zeta = 0.707$$

we get $K = A$. Hence,

$$\frac{H(s)}{V(s)} = \frac{-s}{A(s^2 + 2s + 2)}$$

If $v(t)$ is a step of amplitude 0.1, the final level becomes

$$\lim_{t \to \infty} h(t) = 0$$

*Note that any $K > 0$ results in a stable closed loop system, and that the steady state error computations below are independent of the particular value of K. Hence, selecting K is not necessary for the solution of this problem.

that is, there will be no steady state error in the level for a step disturbance.

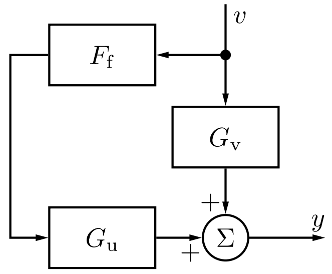


Figure 7.3a

7.3 a) A block diagram of the system is given in Figure 7.3a. The output is given by

$$Y = (G_v + G_u F_f)V$$

where

$$G_u(s) = \frac{2}{s+3} \qquad G_v(s) = \frac{3}{s+4}$$

Chose F_f such that $(G_v + G_u F_f)V = 0$:

$$F_f = -\frac{G_v}{G_u} = -\frac{3(s+3)}{2(s+4)}$$

Compute the controller.

```
>> s = tf( 's' );
>> Gu = 2 / ( s + 3 );
>> Gv = 3 / ( s + 4 );
>> F = - Gv / Gu;
```

b) If $v(t) = 2\sin\omega t$ then

$$u(t) = 2\,|F_f(i\omega)|\sin(\omega t + \arg F_f(i\omega))$$

The amplitude is then

$$A(\omega) = 2\,|F_f(i\omega)| = 2\cdot\frac{3}{2}\sqrt{\frac{\omega^2+9}{\omega^2+16}} \le 3$$

$A(\omega) \to 3,\ \omega \to \infty$

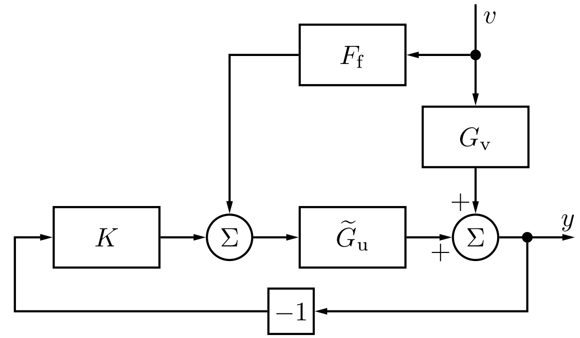


Figure 7.3b

c) A block diagram of the system with both feedforward and feedback is shown in Figure 7.3b. The output is now given by

$$Y = G_v V + \widetilde{G}_u U = (G_v + \widetilde{G}_u F_f)V - \widetilde{G}_u K Y$$

where

$$\widetilde{G}_u(s) = \frac{b}{s+3}$$

The transfer function from V to Y is given by

$$Y(s) = \frac{G_v + \widetilde{G}_u F_f}{1 + \widetilde{G}_u K}V(s) = \frac{\frac{3}{s+4} - \frac{3b}{2(s+4)}}{1 + K\frac{b}{s+3}}V(s)$$
$$= \frac{3(1-b/2)(s+3)}{(s+4)(s+3) + Kb(s+4)}V(s)$$

This is stable for $K \ge 0$ and $b \ge 0$. The final value theorem can therefore be used (with $V(s) = \frac{1}{s}$):

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{3(1-b/2)(s+3)}{(s+4)(s+3)+Kb(s+4)}\cdot\frac{1}{s} = \frac{9(1-b/2)}{12+4Kb}$$

7.4 a) The output is given by

$$Y = (G_v + G_u F_f)V$$

where

$$G_u(s) = \frac{3}{s+1} \qquad G_v(s) = \frac{4}{(s+2)(s+5)}$$

Chose F_f such that $(G_\mathrm{v} + G_\mathrm{u}F_\mathrm{f})V = 0$:

$$F_\mathrm{f}(s) = -\frac{G_\mathrm{v}(s)}{G_\mathrm{u}(s)} = -\frac{4(s+1)}{3(s+2)(s+5)}$$

Create the system and the feedforward controller.

```
>> s = tf( 's' );
>> Gv = 4 / ( s + 2 ) / ( s + 5 );
>> Gu = 3 / ( s + 1 );
>> F = - Gv / Gu;
```

b) The constant to replace $F_\mathrm{f}(s)$ is given by

$$\widetilde{F}_\mathrm{f} = F_\mathrm{f}(0) = -\frac{4}{30}$$

The output is then given by

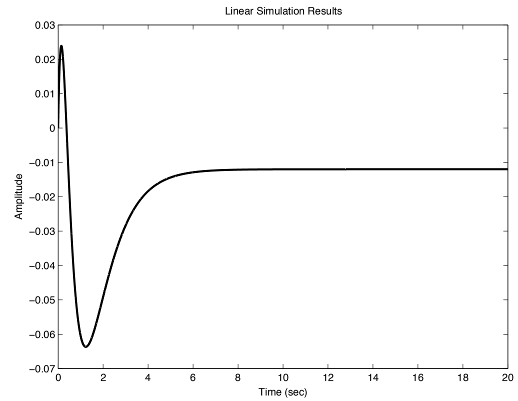
$$\begin{aligned} Y(s) &= \left(-\frac{12}{30(s+1)} + \frac{4}{(s+2)(s+5)}\right)V(s) = \frac{40(s+1) - 4(s+2)(s+5)}{10(s+1)(s+2)(s+5)}V(s) \\ &= \frac{-4s^2 + 12s}{10(s+1)(s+2)(s+5)}V(s) \end{aligned}$$

Taking the Laplace transform of $v(t) = -1 - 0.1t$ we get $V(s) = -\frac{1}{s} - \frac{0.1}{s^2}$. The final value theorem then gives (verify that the system is stable)

$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{s\to 0} s\frac{-4s^2 + 12s}{10(s+1)(s+2)(s+5)}\left(-\frac{1}{s} - \frac{0.1}{s^2}\right) \\ &= \frac{12}{100}\cdot(-0.1) = -0.012 \end{aligned}$$

Create the system with the controller and create the disturbance signal.

```
>> F = -4/30;
>> G = F * Gu + Gv;
>> t = ( 0 : 0.001 : 20 ).';
>> v = -1 - 0.1*t;
>> lsim( G, v, t )
```



c) With the P controller the output is given by

$$Y(s) = -\frac{3}{(s+1)}KY(s) + \left(-\frac{12}{30(s+1)} + \frac{4}{(s+2)(s+5)}\right)V(s)$$

which means that

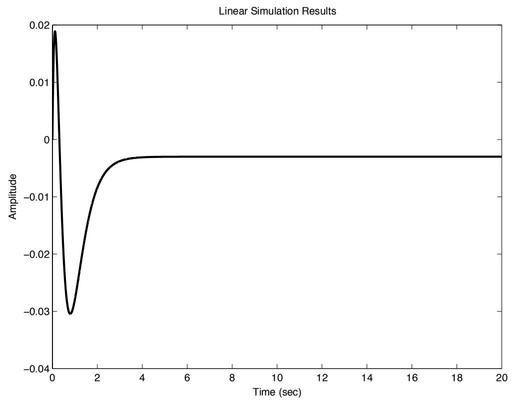
$$Y(s) = \frac{\frac{40(s+1)-4(s+2)(s+5)}{10(s+1)(s+2)(s+5)}}{1 + \frac{3K}{s+1}}V(s) = \frac{-0.4s^2 + 1.2s}{(s+3K+1)(s+2)(s+5)}V(s)$$

Using the same disturbance, $V(s) = -\frac{1}{s} - \frac{0.1}{s^2}$, the final value theorem gives (verify that the system is stable)

$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{s\to 0} s\frac{-0.4s^2 + 1.2s}{(s+3K+1)(s+2)(s+5)}\left(-\frac{1}{s} - \frac{0.1}{s^2}\right) \\ &= \frac{1.2}{(3K+1)\cdot 10}\cdot(-0.1) = -\frac{0.012}{3K+1} \end{aligned}$$

Create the new closed loop system with different values on K.

```
>> K = 1;
>> Gc = minreal( G / ( 1 + K * Gu ) );
>> lsim( Gc, v, t )
```



d) When only a P controller is used we have the following relationship between the disturbance and the output

$$Y(s) = -\frac{3}{(s+1)}KY(s) + \frac{4}{(s+2)(s+5)}V(s)$$

which means that

$$Y(s) = \frac{4(s+1)}{(s+2)(s+5)(s+3K+1)}V(s)$$

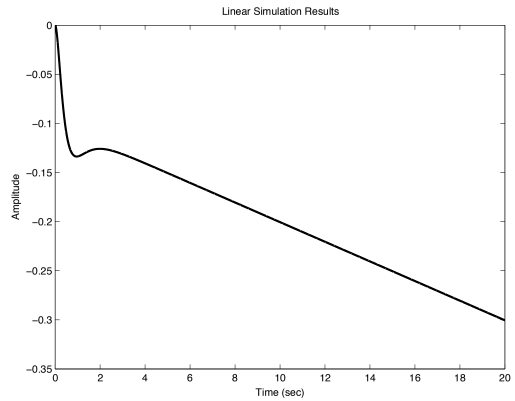
Again using the same disturbance, $V(s) = -\frac{1}{s} - \frac{0.1}{s^2}$, a careful inspection of $Y(s)$ gives that there is no final value of y, hence the final value theorem does not apply.* However, the possibility to simulate the system remains.

*If it is assumed that the final value exists, a contradiction follows since then the final value theorem would apply, but give

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\frac{4(s+1)}{(s+2)(s+5)(s+3K+1)}\left(-\frac{1}{s} - \frac{0.1}{s^2}\right)$$
$$= -\lim_{s\to 0}\frac{4(s+1)}{(s+2)(s+5)(s+3K+1)}\frac{s+0.1}{s} = -\infty$$

Simulate the output.

```
>> Gc = minreal( Gv / ( 1 + K * Gu ) );
>> lsim( Gc, v, t )
```



7.5 a) i) $Y = G_1G_2(F_rR - F_yY) \Rightarrow Y = \dfrac{G_1G_2F_r}{1+G_1G_2F_y}R.$

Svar: $\dfrac{G_1G_2F_r}{1+G_1G_2F_y}$

ii) $Y = G_2(D + G_1F_fD - G_1F_yY) \Rightarrow Y = \dfrac{G_2(1+G_1F_f)}{1+G_1G_2F_y}D.$

Svar: $\dfrac{G_2(1+G_1F_f)}{1+G_1G_2F_y}$

a) Enligt boken (eller så inses det från överföringsfunktionen ovan) så elimineras d om $F_f(s) = -1/G_1(s)$. I detta fallet alltså $F_f(s) = -\dfrac{s^2+2s+1}{s+2}$.

(Detta val av $F_f(s)$ kan dock ej implementeras eftersom det har deriverande verkan för höga frekvenser.) För att eliminera konstanta störningar räcker det att framkoppla med den statiska förstärkningen av $F_f(s)$, d.v.s. $F_f(0) = -1/2$.

8 State Space Description

8.1 According to Solution 2.1 the differential equation for the motor is

$$\ddot{\theta} + \frac{1}{\tau}\dot{\theta} = Ku$$

where

$$\frac{fR_\mathrm{a} + k_a k_v}{JR_\mathrm{a}} = \frac{1}{\tau} \quad \frac{k_a}{JR_\mathrm{a}} = K$$

Introduce the state variables x_1 and x_2 according to

$$x_1 = \theta \quad x_2 = \dot{\theta}$$

This gives the state space equations

$$\begin{aligned} \dot{x}_1 &= \dot{\theta} = x_2 \\ \dot{x}_2 &= \ddot{\theta} = -\frac{1}{\tau}\dot{\theta} + Ku = -\frac{1}{\tau}x_2 + Ku \end{aligned}$$

In matrix form we get

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & -1/\tau \end{pmatrix} x + \begin{pmatrix} 0 \\ K \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

where $x^\mathrm{T} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}$.

8.2 We start with the differential equations

$$\ell\ddot{\theta} + g\sin\theta + \ddot{z}\cos\theta = 0$$

The state variables

$$x_1 = \theta \quad x_2 = \dot{\theta}$$

input

$$u = \frac{\ddot{z}}{\ell}$$

and output

$$y = \theta$$

gives the (nonlinear) state space description

$$\begin{aligned} \dot{x}_1 &= x_2 =: f_1(x,u) \\ \dot{x}_2 &= \ddot{\theta} = -\frac{g}{\ell}\sin\theta - \frac{\ddot{z}}{\ell}\cos\theta = -\omega_0^2 \sin x_1 - u\cos x_1 =: f_2(x,u) \end{aligned}$$

where $\omega_0^2 = g/\ell$. We get that

$$\begin{aligned} \frac{\partial f_1}{\partial x} &= \begin{pmatrix} 0 & 1 \end{pmatrix} \\ \frac{\partial f_1}{\partial u} &= 0 \\ \frac{\partial f_2}{\partial x} &= \begin{pmatrix} -\omega_0^2\cos x_1 + u\sin x_1 & 0 \end{pmatrix} \\ \frac{\partial f_2}{\partial u} &= -\cos x_1 \end{aligned}$$

Introduce $x_{1\Delta} = x_1 - \pi$, $x_{2\Delta} = x_2$, $u_\Delta = u$, and $y_\Delta = y - \pi$. Linearization around $x_1 = \pi$, $x_2 = 0$ and $u = 0$ gives

$$\begin{aligned} \dot{x}_{1\Delta} &= x_{2\Delta} \\ \dot{x}_{2\Delta} &= \omega_0^2 x_{1\Delta} + u_\Delta \\ y_\Delta &= x_{1\Delta} \end{aligned}$$

8.3 Introduce the state variables

$$x_1 = y \quad x_2 = \theta \quad x_3 = z$$

According to the figure, the variables are related as

$$\begin{aligned} X_1(s) &= Y(s) = \frac{1}{s}(M_\mathrm{l}(s) + K_2X_2(s)) \\ X_2(s) &= \theta(s) = \frac{1}{s}(X_3(s) - X_1(s)) \\ X_3(s) &= Z(s) = \frac{1}{s}(K_1I(s) - K_2X_2(s)) \end{aligned}$$

Inverse Laplace transformation gives, in the time domain,

$$\begin{aligned}\dot{x}_1(t) &= K_2 x_2(t) + M_l(t)\\ \dot{x}_2(t) &= -x_1(t) + x_3(t)\\ \dot{x}_3(t) &= -K_2 x_2(t) + K_1 i(t)\end{aligned}$$

In matrix notation this becomes

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 0 & K_2 & 0\\ -1 & 0 & 1\\ 0 & -K_2 & 0\end{pmatrix} x(t) + \begin{pmatrix}0\\0\\K_1\end{pmatrix} i(t) + \begin{pmatrix}1\\0\\0\end{pmatrix} M_l(t)\\ y(t) &= \begin{pmatrix}1 & 0 & 0\end{pmatrix} x(t)\end{aligned}$$

8.4 a)

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}y(t) + 6\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + 11\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 6y(t) = 6u(t)$$

The state variables

$$x_1(t) = y \quad x_2(t) = \dot{y} \quad x_3(t) = \ddot{y}$$

gives

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_3(t)\\ \dot{x}_3(t) &= \frac{\mathrm{d}^3}{\mathrm{d}t^3}y(t) = -6\ddot{y}(t) - 11\dot{y}(t) - 6y(t) + 6u(t)\\ &= -6x_3(t) - 11x_2(t) - 6x_1(t) + 6u(t)\end{aligned}$$

In matrix form we get

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{pmatrix} x(t) + \begin{pmatrix}0\\0\\6\end{pmatrix} u(t)\\ y(t) &= \begin{pmatrix}1 & 0 & 0\end{pmatrix} x(t)\end{aligned}$$

b)

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}y(t) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + 5\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 3y(t) = 4\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(t) + \frac{\mathrm{d}}{\mathrm{d}t}u(t) + 2u(t)$$

If we introduce $x_1(t) = y(t)$ in the equation and collect all terms without differentiation on the right hand side we get

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}x_1(t) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}x_1(t) + 5\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) - 4\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(t) - \frac{\mathrm{d}}{\mathrm{d}t}u(t) = -3x_1(t) + 2u(t)$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_1(t) + \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) + 5x_1(t) - 4\frac{\mathrm{d}}{\mathrm{d}t}u(t) - u(t)\right) = -3x_1(t) + 2u(t)$$

Now introduce the expression within the parenthesis as a new state variable

$$x_2(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}x_1(t) + \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) + 5x_1(t) - 4\frac{\mathrm{d}}{\mathrm{d}t}u(t) - u(t)$$

that is

$$\dot{x}_2(t) = -3x_1(t) + 2u(t) \tag{8.1}$$

Repeating this procedure yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) + x_1(t) - 4u(t)\right) = x_2(t) - 5x_1(t) + u(t) \tag{8.2}$$

and we can introduce

$$x_3(t) = \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) + x_1(t) - 4u(t)$$

that is

$$\dot{x}_1(t) = x_3(t) - x_1(t) + 4u(t) \tag{8.3}$$

Equation (8.1), (8.2), and (8.3) define the state space equations

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} -1 & 0 & 1\\ -3 & 0 & 0\\ -5 & 1 & 0\end{pmatrix} x(t) + \begin{pmatrix}4\\2\\1\end{pmatrix} u(t)\\ y(t) &= \begin{pmatrix}1 & 0 & 0\end{pmatrix} x(t)\end{aligned}$$

c) Partial fraction expansion of

$$Y(s) = \frac{2s+3}{s^2+5s+6}U(s)$$

gives

$$Y(s) = -\frac{1}{s+2}U(s) + \frac{3}{s+3}U(s)$$

Introducing the state variables

$$X_1(s) = -\frac{1}{s+2}U(s) \quad X_2(s) = \frac{3}{s+3}U(s)$$

gives

$$\begin{aligned}\dot{x}_1(t) &= -2x_1(t) - u(t)\\ \dot{x}_2(t) &= -3x_2(t) + 3u(t)\end{aligned}$$

in the time domain. Furthermore, we have

$$y(t) = x_1(t) + x_2(t)$$

In matrix form

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix}-2 & 0\\ 0 & -3\end{pmatrix}x(t) + \begin{pmatrix}-1\\ 3\end{pmatrix}u(t)\\ y(t) &= \begin{pmatrix}1 & 1\end{pmatrix}x(t)\end{aligned}$$

8.5 The impulse response

$$g(t) = 2e^{-t} + 3e^{-4t}$$

gives the transfer function

$$G(s) = \frac{2}{s+1} + \frac{3}{s+4}$$

The output can then be written

$$Y(s) = \underbrace{\frac{2}{s+1}U(s)}_{X_1(s)} + \underbrace{\frac{3}{s+4}U(s)}_{X_2(s)}$$

Defining the state variables as above gives

$$\begin{aligned}sX_1(s) + X_1(s) &= 2U(s)\\ sX_2(s) + 4X_2(s) &= 3U(s)\end{aligned}$$

which in time domain can be written as

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + 2u(t)\\ \dot{x}_2(t) &= -4x_2(t) + 3u(t)\\ y(t) &= x_1(t) + x_2(t)\end{aligned}$$

8.6 The transfer function is given by

$$\begin{aligned}G(s) &= C(sI - A)^{-1}B\\ &= \begin{pmatrix}-1 & 2\end{pmatrix}\begin{pmatrix}s+2 & -1\\ 0 & s+3\end{pmatrix}^{-1}\begin{pmatrix}1\\ 1\end{pmatrix}\\ &= \frac{1}{(s+2)(s+3)}\begin{pmatrix}-1 & 2\end{pmatrix}\begin{pmatrix}s+3 & 1\\ 0 & s+2\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}\\ &= \frac{s}{(s+2)(s+3)}\end{aligned}$$

8.7

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

The state space equations have the general solution

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-s)}Bu(s)\,\mathrm{d}s$$

The input signal u is constant, that is, $u(t) = u_0$, on the interval $(t_0, t_0 + T)$. This implies

$$x(t_0 + T) = e^{AT}x(t_0) + \left(\int_{t_0}^{t_0+T} e^{A(t_0+T-s)}\,\mathrm{d}s\right)Bu_0$$

where

$$e^{AT} \quad and \quad \int_{t_0}^{t_0+T} e^{A(t_0+T-s)}\,\mathrm{d}s\, B$$

are constant matrices.

8.8 a) Introduce the state variables $x_1 = h$, $x_2 = \int_0^t (h_{\text{ref}} - h)\,\mathrm{d}\tau$ and $x_3 = \int_0^t (h_{\text{ref}} - h)\,\mathrm{d}\tau$. This gives the following expressions for the control signals

$$\begin{aligned} u_1 &= h_{\text{ref}} - x_1 + x_2 \\ u_2 &= h_{\text{ref}} - x_1 + x_3 \end{aligned}$$

by using these expressions we can eliminate u_1 and u_2 form $\dot{h} + h = u_1 + u_2$. This gives

$$\dot{x}_1 = -x_1 + h_{\text{ref}} - x_1 + x_2 + h_{\text{ref}} - x_1 + x_3$$

By taking the Laplace transform on the expressions for x_2 and x_3 we obtain

$$\begin{aligned} X_2(s) &= \frac{H_{\text{ref}}(s) - H(s)}{s} \\ X_3(s) &= \frac{H_{\text{ref}}(s) - H(s)}{s} \end{aligned}$$

Inverse Laplace transformation gives

$$\begin{aligned} \dot{x}_2 &= h_{\text{ref}} - x_1 \\ \dot{x}_3 &= h_{\text{ref}} - x_1 \end{aligned}$$

In matrix notation this becomes

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -3 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} h_{\text{ref}}(t) \\ h(t) &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t) \end{aligned}$$

b) The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ 7 & -3 & -3 \end{pmatrix}$$

A vector which span the null space of a matrix must satisfy $\mathcal{O}x = 0$.

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ 7 & -3 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This means in practise that you can't say if it is u_1 or u_2 or a combination of the two which fills the tank.

c) With $h_{\text{ref}} = 0$ and $u_1 = -h - n + \int_0^t -h - n\,\mathrm{d}\tau$ we get

$$\begin{aligned} \dot{x}_2 &= -x_1 - n \\ \dot{x}_3 &= -x_1 \end{aligned}$$

and

$$\dot{x}_1 = -x_1 - x_1 - n + x_2 - x_1 + x_3$$

this gives in matrix form

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -3 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} n(t) \\ h(t) &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t) \end{aligned}$$

8.9 The controllability matrix is

$$\mathcal{S} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

Since $\det \mathcal{S} = -1 \neq 0$ the system is controllable and it is possible to control the system from the origin to $x^{\mathrm{T}} = \begin{pmatrix} 1 & 3 \end{pmatrix}$ within 4 seconds.

8.10 a) The controllability matrix becomes

$$\mathcal{S} = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 \\ -1 & 3 & -9 \\ 2 & -6 & 18 \end{pmatrix}$$

and $\det \mathcal{S} = 0$ since $\operatorname{rank} \mathcal{S} = 2$. The controllable subspace is spanned by

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 3 \\ -6 \end{pmatrix}$$

The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1.5 \\ -2 & -3 & -1.5 \\ 4 & 3 & 1.5 \end{pmatrix}$$

with $\det \mathcal{O} = 0$. Solving for the unobservable subspace

$$\mathcal{O}x = 0$$

gives (Gauss elimination)

$$\begin{aligned} x_1 + 3x_2 + 1.5x_3 &= 0 \\ 3x_2 + 1.5x_3 &= 0 \\ x_1 \qquad\qquad &= 0 \end{aligned}$$

Introducing $x_3 = a$ gives $x_2 = -0.5a$ and $x^{\mathrm{T}} = \begin{pmatrix} 0 & -0.5a & a \end{pmatrix}$, that is, the silent (unobservable) subspace is spanned by

$$\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

b) The controllability matrix becomes

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & -8 & 16 \\ -2 & 8 & -32 \end{pmatrix}$$

with $\operatorname{rank} \mathcal{S} = 2$. The controllable subspace is spanned by, for example,

$$\begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} \quad \begin{pmatrix} 0 \\ -8 \\ 8 \end{pmatrix}$$

The observability matrix is

$$\mathcal{O} = \begin{pmatrix} 0 & 3 & 0 \\ 3 & -6 & 0 \\ -9 & 12 & 0 \end{pmatrix}$$

Solving for the unobservable subspace $\mathcal{O}x = 0$ gives

$$x = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}$$

The unobservable subspace is spanned by

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

8.11 a)

$$\begin{aligned} \dot{x}_1 &= -x_1 + u \quad \Rightarrow \quad x_1 = 1 - e^{-t} \\ \dot{x}_2 &= 2x_2 + u \quad \Rightarrow \quad x_2 = 0.5(e^{2t} - 1) \end{aligned}$$

b) The system is not asymptotically stable since $x_2 \to \infty$ as $t \to \infty$, but input-output stable because the transfer function has its pole in the complex left hand plane.

c)

$$\mathcal{S} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \qquad \det \mathcal{S} = 3$$

The system is controllable.

$$\mathcal{O} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \qquad \det \mathcal{O} = 0$$

The system is not observable. $\mathcal{O}x = 0$ has solutions

$$x = \begin{pmatrix} 0 \\ a \end{pmatrix}$$

This implies that the second component of the state vector cannot be seen in the output.

d) Because the second component of the state vector has unconstrained growth and this is not reflected in the output, the system will finally collapse.

8.12

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} s-1 & 1 \\ -2 & s-1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{s+1}{(s-1)^2 + 2} \end{aligned}$$

This gives poles in $1 \pm \mathrm{i}\sqrt{2}$ and zeros in -1.

8.13 a) For pendulum 1 we have

$$\ddot{z}\cos(\phi_1) + \alpha\ddot{\phi}_1 = \sin(\phi_1)$$

and for pendulum 2

$$\ddot{z}\cos(\phi_2) + \ddot{\phi}_2 = \sin(\phi_2)$$

Linearization gives

$$\begin{aligned} \ddot{z} + \alpha\ddot{\phi}_1 &= \phi_1 \\ \ddot{z} + \ddot{\phi}_2 &= \phi_2 \end{aligned}$$

Consider $\ddot{z}$ as an input to the system (the acceleration of the trolley $\sim$ the force applied to the system). Introduce the state variables

$$x_1 = \phi_1 \quad x_2 = \dot{\phi}_1 \quad x_3 = \phi_2 \quad x_4 = \dot{\phi}_2$$

This gives the state space equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{\alpha}x_1 - \frac{u}{\alpha} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= x_3 - u \end{aligned}$$

In matrix form

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/\alpha \\ 0 \\ -1 \end{pmatrix} u$$

b) The controllability matrix becomes

$$\mathcal{S} = \begin{pmatrix} 0 & -1/\alpha & 0 & -1/\alpha^2 \\ -1/\alpha & 0 & -1/\alpha^2 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \qquad \det\mathcal{S} = \frac{1}{\alpha^2}(1 - \frac{1}{\alpha})^2$$

Thus, the system is controllable except for the case $\alpha = 1$, that is, when the two pendulums have the same lengths. If the pendulums have different lengths they react differently to the input, but if they have the same length there is no possibility to act upon them separately using the input.

8.14 The figure gives

$$X_1(s) = \frac{1}{(s+1)}U(s) \quad \Rightarrow \quad sX_1(s) = -X_1(s) + U(s)$$

and

$$X_2(s) = \frac{1}{(s+3)}(U(s) + X_1(s)) \quad \Rightarrow \quad sX_2(s) = -3X_2(s) + U(s) + X_1(s)$$

Inverse Laplace transformation gives

$$\begin{aligned} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= -3x_2 + x_1 + u \end{aligned}$$

In matrix form this becomes

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 0 \\ 1 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 1 \end{pmatrix} x \end{aligned}$$

8.15 a) Mass balance gives

$$\begin{aligned} \frac{\mathrm{d}\,(Vc_\mathrm{A})}{\mathrm{d}t} &= Vr_\mathrm{A} + qc_\mathrm{A,in} - qc_\mathrm{A} \\ \frac{\mathrm{d}\,(Vc_\mathrm{B})}{\mathrm{d}t} &= Vr_\mathrm{B} + qc_\mathrm{B} \end{aligned}$$

By using $r_\mathrm{A} = -k_1c_\mathrm{A}^3$ and $r_\mathrm{B} = \frac{-r_A}{3}$ the following expression is obtained

$$\begin{aligned} V\frac{\mathrm{d}c_\mathrm{A}}{\mathrm{d}t} &= -Vk_1c_\mathrm{A}^3 + qc_\mathrm{A,in} - qc_\mathrm{A} \\ V\frac{\mathrm{d}c_\mathrm{B}}{\mathrm{d}t} &= \frac{Vk_1c_\mathrm{A}^3}{3} - qc_\mathrm{B} \end{aligned}$$

b) Linearization around c_A^*, c_B^*, and $c_\mathrm{A,in}^*$ gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} c_{\mathrm{A},\Delta} \\ c_{\mathrm{B},\Delta} \end{pmatrix} &= \begin{pmatrix} \frac{-q-3k_1c_\mathrm{A}^*V}{V} & 0 \\ k_1c_\mathrm{A}^* & \frac{-q}{V} \end{pmatrix} \begin{pmatrix} c_{\mathrm{A},\Delta} \\ c_{\mathrm{B},\Delta} \end{pmatrix} + \begin{pmatrix} \frac{q}{V} \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} c_{\mathrm{A},\Delta} \\ c_{\mathrm{B},\Delta} \end{pmatrix} \end{aligned}$$

8.16 a) Linjärisera systemet runt jämvikspunkten $y(t) = y_0$. Stationärt innebär $\dot{x} = 0$, alltså $0 = -y_0 u_0 + v$, eller $u_0 = \frac{v}{y_0}$.

Taylorutveckling av $\dot{y} = f(y, u)$ runt jämviktspunkten $y = y_0 + \Delta y, u = u_0 + \Delta u$, där alltså $f(y_0, u_0) = 0$ ger

$$\begin{aligned} \dot{y} = \Delta\dot{y} &= f(y_0 + \Delta y, u_0 + \Delta y) \\ &\approx f(y_0, u_0) + \frac{\partial f(y_0, u_0)}{\partial y}\Delta y + \frac{\partial f(y_0, u_0)}{\partial u}\Delta u \\ &= 0 - u_0\Delta y - y_0\Delta u = -\frac{v}{y_0}\Delta y - y_0\Delta u. \end{aligned}$$

b) Laplacetransformera det linjäriserade systemet från a).

$s\Delta Y(s) = -\frac{v}{y_0}\Delta Y(s) - y_0\Delta U(s)$, dvs.
$\Delta Y(s) = \frac{-y_0^2}{y_0 s + v}\Delta U(s) = G(s)\Delta U(s)$.

Det återkopplade systemet fås från

$$\begin{aligned} U(s) &= F(s)(Y_0 - Y(s)) \\ Y(s) &= G(s)U(s) \end{aligned}$$

vilket ger

$$\begin{aligned} G_c(s) &= \frac{F(s)G(s)}{1 + F(s)G(s)} = \frac{K\frac{\tau_i s+1}{\tau_i s}\frac{-y_0^2}{y_0 s+v}}{1 + K\frac{\tau_i s+1}{\tau_i s}\frac{-y_0^2}{y_0 s+v}} \\ &= \frac{-Ky_0^2(\tau_i s + 1)}{(\tau_i s)(y_0 s + v) - K\tau_i s y_0^2 - Ky_0^2} \\ &= \frac{-Ky_0^2(\tau_i s + 1)}{y_0\tau_i s^2 + v\tau_i s - K\tau_i y_0^2 s - Ky_0^2}. \end{aligned}$$

Enligt t.ex. Routh's algoritm, krav för stabilitet hos det återkopplade systemet är $K < 0$ samt att $v - Ky_0^2 > 0$ vilket då är uppfyllt för alla $v > 0$ då $\tau_I > 0$.

8.17 a) Systemet kan skrivas som

$$\begin{aligned} \dot{x} &= A(\alpha)x + Bu \\ y &= Cx \end{aligned}$$

där

$$\begin{aligned} x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ A(\alpha) &= \begin{bmatrix} 1 & 0 \\ -2 & \alpha \end{bmatrix} \\ B &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ C &= \begin{bmatrix} 0 & 1 \end{bmatrix} \end{aligned}$$

Egenvärdena av systemmatrisen A ges av lösningarna till den karakteristiska ekvationen

$$\begin{aligned} \det\left(A(\alpha) - sI\right) &= \det\left(\begin{bmatrix} 1 & 0 \\ -2 & \alpha \end{bmatrix} - s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-s & 0 \\ -2 & \alpha - s \end{bmatrix}\right) \\ &= (1-s)(\alpha - s) = 0 \end{aligned}$$

alltså $s_1 = 1$ och $s_2 = \alpha$. Då egenvärdet $s_1 > 0$ är systemet instabilt för alla α.

OBS! Notera att

$$\det(A(\alpha)) - sI) = 0 \Leftrightarrow \det(sI - A(\alpha)) = 0,$$

och att den senare formen är den som vi använt oftast i kursen för att räkna ut den karakteristiska ekvationen. Av ekvivalensen följer att båda formerna är rätt.

b) Systemet är observerbart då observerbarhetsmatrisen

$$\mathcal{O} = \begin{bmatrix} C \\ CA(\alpha) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & \alpha \end{bmatrix}$$

ej är singulär. Då

$$\det\left(\begin{bmatrix} 0 & 1 \\ -2 & \alpha \end{bmatrix}\right) = 2 \neq 0$$

är systemet observerbart för alla α.

c) Systemet är styrbart då styrbarhetsmatrisen

$$\mathcal{S} = \begin{bmatrix} B & A(\alpha)B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2-\alpha \end{bmatrix}$$

ej är singulär. Då

$$\det\left(\begin{bmatrix} 1 & 1 \\ -1 & -2-\alpha \end{bmatrix}\right) = 1 \cdot (-2-\alpha) - (-1) = -1 - \alpha$$

är systemet styrbart precis då $\alpha \neq -1$.

9 State Feedback

9.1 a) The control law

$$u = -Lx + y_{\text{ref}}$$

gives the closed loop system

$$\dot{x} = (A - BL)x + By_{\text{ref}}$$

and the poles of the closed loop system are given by the eigenvalues of $A - BL$.

$$A - BL = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 \end{pmatrix} = \begin{pmatrix} -2 - l_1 & -1 - l_2 \\ 1 & 0 \end{pmatrix}$$

The characteristic equation is given by

$$\det(sI - A + BL) = s^2 + (2 + l_1)s + 1 + l_2 = 0$$

Poles in $\{-3, -5\}$ implies that we will have the equation

$$(s + 3)(s + 5) = s^2 + 8s + 15 = 0$$

Identification of the coefficients gives

$$l_1 = 6 \quad l_2 = 14$$

This gives the control law

$$u = -6x_1 - 14x_2 + y_{\text{ref}}$$

Similarly, poles in $\{-10, -15\}$ gives

$$l_1 = 23 \quad l_2 = 149$$

corresponding to the control law

$$u = -23x_1 - 149x_2 + y_{\text{ref}}$$

One observes that the coefficients in the control law increase when the poles are placed further into the left half plane. In a physical system, this means that larger forces are required to realize to the control law.

b) Employ an observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

By combining the differential equations for the system and the observer we obtain an equation for the estimation error, $\tilde{x} = x - \hat{x}$,

$$\dot{\tilde{x}} = Ax + Bu - A\hat{x} - Bu - K(Cx - C\hat{x}) = (A - KC)\tilde{x}$$

If K is chosen so that $A - KC$ gets eigenvalues in the complex left hand plane, then $\tilde{x}(t) \to 0$ as $t \to \infty$. It is desirable that the estimation error approaches zero faster than the dynamics of the system. Thus, one should place the eigenvalues of the observer to the left of the poles of the closed loop system, for example, in -20. Regarding the influence of the pole placement, placing the poles too far into the left half plane will make the observer unneccessary sensitive to measurement noise. The characteristic equation is given by

$$\det(sI - A + KC) = s^2 + (2 + k_1)s + 1 - k_2 = 0$$

Two poles in -20 corresponds to the equation

$$s^2 + 40s + 400 = 0$$

Identification of the coefficients gives

$$k_1 = 38 \quad k_2 = -399$$

The resulting observer becomes

$$\dot{\hat{x}} = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 38 \\ -399 \end{pmatrix} (y - \begin{pmatrix} 1 & 0 \end{pmatrix} \hat{x})$$

9.2 a) Introduce the state variables

$$x_1 = \dot{z} \quad x_2 = \theta \quad x_3 = \dot{\theta}$$

The figure gives the state equations

$$X_1(s) = \frac{1}{s} K_2 X_2(s)$$
$$X_2(s) = \frac{1}{s} X_3(s)$$
$$X_3(s) = \frac{1}{s} K_1 U(s)$$

Inverse Laplace transformation gives

$$\dot{x}_1(t) = K_2 x_2(t) \quad \dot{x}_2(t) = x_3(t) \quad \dot{x}_3(t) = K_1 u(t)$$

In matrix form we get

$$\dot{x}(t) = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ K_1 \end{pmatrix} u(t)$$

b) Since it is assumed that all states are measurable we apply a state feedback

$$u = -Lx + y_{\text{ref}}$$

which gives the closed loop system

$$\dot{x} = (A - BL)x + B y_{\text{ref}}$$

where

$$A - BL = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 1 \\ -K_1 l_1 & -K_1 l_2 & -K_1 l_3 \end{pmatrix}$$

The characteristic equation

$$\det(sI - A + BL) = s^3 + K_1 l_3 s^2 + K_1 l_2 s + K_2 K_1 l_1 = 0$$

All three poles in -0.5 implies that we will have the equation

$$(s + 0.5)^3 = s^3 + 1.5s^2 + 0.75s + 0.125 = 0$$

Identification of the coefficients gives

$$l_1 = \frac{1}{8K_1K_2} \quad l_2 = \frac{3}{4K_1} \quad l_3 = \frac{3}{2K_1}$$

c) If only x_1 is measurable we have

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x$$

Employ the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where

$$K = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

The characteristic equation is

$$\det(sI - A + KC) = s^3 + k_1 s^2 + k_2 K_2 s + k_3 K_2 = 0$$

To get a similar behavior as in a), the poles of the observer are placed to the left of the poles of the closed loop system, for example, in -2. This pole placement corresponds to the equation

$$s^3 + 6s^2 + 12s + 8 = 0$$

Identification of the coefficients gives

$$k_1 = 6 \quad k_2 = 12/K_2 \quad k_3 = 8/K_2$$

9.3 Introduce the state variables

$$x_1 = \theta \quad x_2 = \omega$$

This gives the state equations

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -1/\tau \end{pmatrix} x + \begin{pmatrix} 0 \\ c_1 \end{pmatrix} u + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} T$$

a) The feedback

$$u = -Lx + l_0\theta_{\text{ref}} = -l_1\theta - l_2\omega + l_0\theta_{\text{ref}}$$

gives

$$A - BL = \begin{pmatrix} 0 & 1 \\ -c_1 l_1 & -(c_1 l_2 + 1/\tau) \end{pmatrix}$$

The characteristic equation

$$\det(sI - A + BL) = s^2 + (l_2 c_1 + \frac{1}{\tau})s + c_1 l_1 = 0$$

Poles in $1/\tau(-1 \pm \mathrm{i})$ corresponds to

$$(s + \frac{1-\mathrm{i}}{\tau})(s + \frac{1+\mathrm{i}}{\tau}) = s^2 + \frac{2}{\tau}s + \frac{2}{\tau^2} = 0$$

Identification of the coefficients gives

$$l_1 = \frac{2}{c_1\tau^2} \quad l_2 = \frac{1}{\tau c_1}$$

This gives the closed loop system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2/\tau^2 & -2/\tau \end{pmatrix} x + \begin{pmatrix} 0 \\ c_1 \end{pmatrix} l_0\theta_{\text{ref}} + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} T$$

At steady state, that is, when $\dot{x}_1 = \dot{x}_2 = 0$, we should have $\theta = \theta_{\text{ref}}$ when $T = 0$. $\dot{x}_1 = 0$ implies that $x_2 = 0$, and $\dot{x}_2 = 0$ then gives

$$\frac{-2}{\tau^2}x_1 + c_1 l_0 \theta_{\text{ref}} = 0$$

so that

$$l_0 = \frac{2}{c_1\tau^2}$$

The resulting control law becomes

$$u = -\frac{2}{c_1\tau^2}\theta - \frac{1}{\tau c_1}\omega + \frac{2}{c_1\tau^2}\theta_{\text{ref}}$$

b) Introduce the integrated control error as an extra state:

$$\dot{x}_3 = \theta_{\text{ref}} - \theta$$

The new state equations become

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1/\tau & 0 \\ -1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ c_1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ c_2 \\ 0 \end{pmatrix} T + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \theta_{\text{ref}}$$

Using the feedback law

$$u = -l_1\theta - l_2\omega - l_3 x_3$$

we get the state derivative term

$$\begin{pmatrix} 0 \\ c_1 \\ 0 \end{pmatrix} u = \begin{pmatrix} 0 & 0 & 0 \\ -c_1 l_1 & -c_1 l_2 & -c_1 l_3 \\ 0 & 0 & 0 \end{pmatrix} x$$

and hence the closed loop system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ -c_1 l_1 & -1/\tau - c_1 l_2 & -c_1 l_3 \\ -1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ c_2 \\ 0 \end{pmatrix} T + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \theta_{\text{ref}}$$

The poles of the closed loop system are the eigenvalues of the "A" matrix, that is, they are given by the characteristic equation

$$\det \begin{pmatrix} -\lambda & 1 & 0 \\ -c_1 l_1 & -1/\tau - c_1 l_2 - \lambda & -c_1 l_3 \\ -1 & 0 & -\lambda \end{pmatrix} = 0$$

Writing out and changing sign yields

$$\lambda^3 + (c_1 l_2 + \frac{1}{\tau})\lambda^2 + c_1 l_1 \lambda - c_1 l_3 = 0$$

Poles in $\{ \frac{1}{\tau}(-1 \pm \mathrm{i}), \frac{1}{\tau}(-2) \}$ correspond to the equation

$$\lambda^3 + \frac{4}{\tau}\lambda^2 + \frac{6}{\tau^2}\lambda + \frac{4}{\tau^3} = 0$$

where the coefficients may be identified as:

$$l_1 = \frac{6}{c_1\tau^2} \quad l_2 = \frac{3}{c_1\tau} \quad l_3 = -\frac{4}{c_1\tau^3}$$

The resulting control law becomes (note that the static gain is 1 by construction, so there is no "l_0" in this controller)

$$\dot{x_3} = \theta_{\text{ref}} - \theta$$
$$u = -\frac{6}{c_1\tau^2}\theta - \frac{3}{c_1\tau}\omega + \frac{4}{c_1\tau^3}x_3$$

9.4 The feedback $u = -Lx + y_{\text{ref}}$ gives the closed loop system

$$\dot{x} = (A - BL)x + By_{\text{ref}}$$

with characteristic equation

$$s^2 + (1 + l_1 + l_2)s + l_1 = 0$$

Poles in $\{-2, -3\}$ implies that we will have the equation

$$(s+3)(s+2) = s^2 + 5s + 6 = 0$$

Identification of the coefficients gives

$$l_1 = 6 \quad l_2 = -2$$

and the control law becomes

$$u = -6x_1 + 2x_2 + y_{\text{ref}}$$

Introduce the observer

$$\dot{\hat{x}}(t) = A\hat{x} + Bu(t) + K(y(t) - C\hat{x}(t))$$

It is desirable that the estimation error converges to zero faster than the dynamics of the system. Thus, we should place the eigenvalues of the observer to the left of the poles of the closed loop system, for example, in -4. The characteristic equation of the observer is

$$s^2 + (1 + k_1 - k_2)s + k_1 = 0$$

and poles in -4 corresponds to the equation

$$s^2 + 8s + 16 = 0$$

Identification of coefficients gives

$$k_1 = 16 \quad k_2 = 9$$

The complete system, that is, the closed loop system with reconstructed states, will have poles in $\{-2, -3\}$, and the observer will have poles in $\{-4, -4\}$.

9.5 The system has the observability matrix

$$\mathcal{O} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

that is, $\det \mathcal{O} \neq 0$. The system is observable and thus the poles of the observer may be placed arbitrarily.

9.6 The system is described in matrix form by

$$\dot{x}(t) = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u(t)$$

a) Arbitrary values of the states can be obtained if the system is controllable. The controllability matrix becomes

$$\mathcal{S} = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

and since $\det \mathcal{S} = 1$ the system is controllable and an arbitrary temperature profile can be obtained.

b) How the state decays depends on the poles of the closed loop system. Poles in -3 will yield the desired result. The closed loop system,

$$\dot{x} = (A - BL)x + By_{\text{ref}}$$

$$A - BL = \begin{pmatrix} -2 - l_1 & 1 - l_2 & -l_3 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

has the characteristic equation

$$s^3 + (6 + l_1)s^2 + (10 + 4l_1 + l_2)s + 4 + 3l_1 + 2l_2 + l_3 = 0$$

Poles in -3 implies that this coincide with the equation

$$(s+3)^3 = s^3 + 9s^2 + 27s + 27 = 0$$

Identification of the coefficients gives

$$l_1 = 3 \quad l_2 = 5 \quad l_3 = 4$$

Thus, the control law is given by

$$u = -3x_1 - 5x_2 - 4x_3 + y_{\text{ref}}$$

c) Check when the system is observable. The sensor at x_1 corresponds to $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, and results in

$$\mathcal{O} = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{pmatrix} \qquad \det \mathcal{O} = 1$$

The sensor at x_2 corresponds to $C = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$, and results in

$$\mathcal{O} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ -4 & 6 & -4 \end{pmatrix} \qquad \det \mathcal{O} = 0$$

The sensor at x_3 corresponds to $C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$, and results in

$$\mathcal{O} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -4 & 5 \end{pmatrix} \qquad \det \mathcal{O} = -1$$

The system is hence observable when the sensor is placed at x_1 or x_3, but not with the sensor placed at x_2. That is, the specifications may be fulfilled with the sensor placed at x_1 or x_3. If the sensor is placed at x_1, the characteristic equation of the observer is given by

$$s^3 + (6 + k_1)s^2 + (10 + 4k_1 + k_2)s + 4 + 3k_1 + 2k_2 + k_3 = 0$$

Placing the poles in -4 (which is somewhat faster than the nominal closed loop system) corresponds to the equation

$$(s + 4)^3 = s^3 + 12s^2 + 48s + 64 = 0$$

Identification of coefficients gives

$$k_1 = 6 \quad k_2 = 14 \quad k_3 = 14$$

9.7 From Solution 9.2 we have the state space description

$$\dot{x}(t) = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ K_1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t)$$

Introduce a reduced observer to estimate x_3 from m_2. The last row in the state space description implies

$$\dot{\hat{x}}_3 = K_1 u + K(x_3 - \hat{x}_3) = K_1 u + K(\dot{x}_2 - \hat{x}_3)$$

The estimation error becomes

$$\dot{\tilde{x}}_3 = x_3 - \hat{x}_3 = -K\tilde{x}_3$$

With a suitable choice of K, the estimation error can be made to decrease arbitrarily fast. To avoid differentiation of x_2 we introduce

$$z = \hat{x}_3 - Kx_2$$

which implies

$$\dot{z} = \dot{\hat{x}}_3 - K\dot{x}_2 = -K(z + Kx_2) + K_1 u$$

This gives

$$\hat{X}_3(s) = \frac{K_1}{s + K} U(s) + \frac{K^2 s}{s + K} X_2(s)$$

which results in the block diagram in Figure 9.7a.

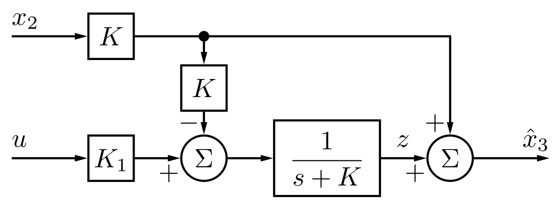


Figure 9.7a

9.8 a) The equations

$$T\dot{q} = -q + k_1 u$$
$$A\dot{h} = q - v$$

with $k_1 = 1$, $T = 0.5$ and $A = 1$ give, in state space form,

$$\begin{pmatrix} \dot{q} \\ \dot{h} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ h \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ -1 \end{pmatrix} v$$

The feedback

$$u = -l_1 q - l_2 h + r$$

gives the closed loop system

$$\begin{pmatrix} \dot{q} \\ \dot{h} \end{pmatrix} = \begin{pmatrix} -2-2l_1 & -2l_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ h \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} r + \begin{pmatrix} 0 \\ -1 \end{pmatrix} v$$

with characteristic equation

$$(s+2+2l_1)s + 2l_2 = s^2 + (2+2l_1)s + 2l_2 = 0$$

Comparison with the desired characteristic equation

$$(s+2)^2 = s^2 + 4s + 4 = 0$$

gives

$$l_1 = 1 \quad l_2 = 2$$

b) At steady state we have $\dot{q} = 0$ and $\dot{h} = 0$. With $v = 0.1$ and $r = 0$ we get

$$\begin{aligned} 0 &= -4q - 4h \\ 0 &= q - 0.1 \end{aligned}$$

which gives $h = -0.1$.

c) In order to determine the feedforward controller we start from the description

$$Y(s) = G_1(s)R(s) + H(s)V(s)$$

The state space description

$$\begin{aligned} \begin{pmatrix} \dot{q} \\ \dot{h} \end{pmatrix} &= \begin{pmatrix} -4 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ h \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} r + \begin{pmatrix} 0 \\ -1 \end{pmatrix} v \\ y &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ h \end{pmatrix} \end{aligned}$$

gives

$$\begin{aligned} H(s) &= \frac{1}{s^2+4s+4} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s & -4 \\ 1 & s+4 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ &= -\frac{(s+4)}{s^2+4s+4} \end{aligned}$$

and

$$\begin{aligned} G_1(s) &= \frac{1}{s^2+4s+4} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s & -4 \\ 1 & s+4 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ &= \frac{2}{s^2+4s+4} \end{aligned}$$

To eliminate v completely we shall choose the feedforward controller

$$R(s) = F_{\mathrm{f}}(s)V(s)$$

where

$$F_{\mathrm{f}}(s) = -\frac{H(s)}{G_1(s)}$$

The computations above give

$$F_{\mathrm{f}}(s) = \frac{(s+4)}{2} = \frac{1}{2}s + 2$$

Removing the differentiation term yields $F_{\mathrm{f}}(s) = 2$ or

$$r = 2v$$

At steady state this gives

$$\begin{aligned} 0 &= -4q - 4h + 4v \\ 0 &= q - v \end{aligned}$$

that is $h = 0$.

d) Because $k_1 \neq 1$ the feedback $u = -q - 2h + 2v$ gives, at steady state,

$$\begin{aligned} 0 &= -2(1+k_1)q - 4k_1 h + 4k_1 v \\ 0 &= q - v \end{aligned}$$

which gives

$$h = \frac{k_1 - 1}{2k_1} v$$

Because $k_1 \neq 1$ we get a steady state control error. In order to determine when the expression for h is valid we consider the stability. The characteristic equation

$$s^2 + (2 + 2k_1)s + 4k_1 = 0$$

has both roots in the complex left hand plane for $k_1 > 0$, that is, the expression is valid for all $k_1 > 0$.

e) Introduce the integral of the height as a new state

$$z(t) = \int_0^t h(s)\,\mathrm{d}s \quad \Rightarrow \quad \dot{z} = h$$

With the state vector

$$x(t) = \begin{pmatrix} q(t) & h(t) & z(t) \end{pmatrix}^{\mathrm{T}}$$

this gives

$$\dot{x} = \begin{pmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 2k_1 \\ 0 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} v$$

The state feedback $u = -Lx$ gives

$$\dot{x} = \begin{pmatrix} -2 - 2k_1 l_1 & -2k_1 l_2 & -2k_1 l_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} v$$

The third equation gives $h = 0$ at steady state, independent of k_1 provided L stabilizes the system.

9.9 The transfer function u to y is given by

$$Y(s) = C(sI - A)^{-1} BU(s) = \frac{1}{s^2} U(s)$$

In order to study the effect of the time delay we consider the block diagram in Figure 9.9a. The block diagram corresponds to the situation where the

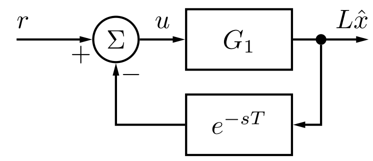


Figure 9.9a

observer uses the measured input (not the computed input). To determine the effect of the time delay, we study the loop gain, $G_1(s)e^{-sT}$, where $G_1(s)$ is the transfer function from $U(s)$ to $Z(s) = L\hat{X}(s)$.

The equation for the observer

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

gives

$$\begin{aligned} \hat{X}(s) &= (sI - A + KC)^{-1}(BU(s) + KY(s)) \\ &= (sI - A + KC)^{-1}(BU(s) + K\frac{1}{s^2}U(s)) \end{aligned}$$

Using this together with $Z(s) = L\hat{X}(s)$ gives

$$\begin{aligned} Z(s) &= G_1(s)U(s) \\ &= L(sI - A + KC)^{-1}(B + K\frac{1}{s^2})U(s) \\ &= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} s+4 & -1 \\ 4 & s \end{pmatrix}^{-1} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} \frac{1}{s^2} \right) U(s) \\ &= \frac{1 + 2s}{s^2} U(s) \end{aligned}$$

We shall analyze the stability using the Nyquist curve* for $G_o = G_1(s)e^{-sT}$, that is,

$$G_1(\mathrm{i}\omega)e^{-\mathrm{i}\omega T} = \frac{1 + \mathrm{i}2\omega}{-\omega^2} e^{-\mathrm{i}\omega T}$$

The crossover frequency is obtained from

$$\left| G_1(\mathrm{i}\omega_c)e^{-\mathrm{i}\omega_c T} \right| = \frac{\sqrt{1 + 4\omega_c^2}}{\omega_c^2} = 1$$

*Using a Bode plot instead of the Nyquist curve would perhaps be more straightforward. However, for no particular reason, we use the Nyquist curve here.

or

$$\omega_c = \sqrt{2+\sqrt{5}}$$

The phase of G_o is

$$\arg\left(G_1(i\omega)e^{-\omega T}\right) = -\pi + \arctan 2\omega - \omega T$$

In order to obtain a stable closed loop system it is required that

$$-\pi + \arctan 2\omega_c - \omega_c T > -\pi$$

which gives

$$T < \frac{\arctan 2\omega_c}{\omega_c} = 0.65 \text{ s}$$

9.10 a) The observability matrix:

$$\mathcal{O} = \begin{pmatrix} 2 & 1 \\ -2+a & 0 \end{pmatrix} \qquad \det \mathcal{O} = 2 - a$$

The system is observable (and the poles of the observer can be placed arbitrarily) when $a \neq 2$.

$$A - KC = \begin{pmatrix} -1-2k_1 & 1-2k_2 \\ 1-k_1 & -2-k_2 \end{pmatrix}$$

We desire that the eigenvalues be $\{-5, -10\}$. Use that the determinant is the product of the eigenvalues and the trace* is the sum of the eigenvalues:

$$\begin{aligned} 5k_1 + 3k_2 + 1 &= 50 \\ -2k_1 - k_2 - 3 &= -15 \end{aligned}$$

which gives

$$k_1 = -13 \quad k_2 = 38$$

b) The equation for the estimation error is

$$\dot{\tilde{x}}(t) = (A - KC)\tilde{x}(t) - Kv(t)$$

The transfer function from v to $\tilde{x}_1$ is

$$-C_1(sI - A + KC)^{-1}K = \frac{13s - 12}{s^2 + 15s + 50}$$

where $C_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

*The trace of a matrix is the sum of its diagonal elements.

9.11 a) According to the initial value theorem we have that

$$y(0) = \lim_{s\to\infty} sG(s)U(s)$$

For a step input, that is, $U(s) = 1/s$, we get

$$\dot{y}(0) = \lim_{s\to\infty} s \cdot sG(s)U(s) = \lim_{s\to\infty} \frac{s(1-s/\alpha)}{(1+\frac{s}{\beta})^2} = -\frac{\beta^2}{\alpha}$$

Hence $\dot{y}(0)$ decreases as α decreases, that is, as the zero of the system approaches the origin.

b) No. This problem is caused by a RHP zero and it is impossible to move the zeros with state feedback.

9.12 A very fast closed loop system:

- implies that the poles are far into the LHP which implies a need for generating large input signals.
- easily becomes unstable in case of model uncertainties.
- becomes sensitive to measurement noise.
- has a sensitivity function with a large peak.

9.13 a) The system $G(s) = C(sI - A + BL)^{-1}B$ has poles where

$$\det(sI - A + BL) = s^2 + (5 - l_1 + 2l_2)s + 5 + 6l_2 = 0$$

The poles in $-2 \pm i$ implies the characteristic equation

$$(s + 2 + i)(s + 2 - i) = s^2 + 4s + 5 = 0$$

Identification of coefficients gives

$$l_1 = 1 \quad l_2 = 0$$

b) The closed loop system is given by

$$\dot{x}(t) = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} -1 \\ 2 \end{pmatrix} r(t)$$
$$y(t) = \begin{pmatrix} 1 & 1 \end{pmatrix} x(t)$$

The condition $y(t) = 0$ gives $x_1 + x_2 = 0$, and hence $\dot{x}_1 = -\dot{x}_2$. From the state equations we get

$$-2x_1 + x_2 - r = x_1 + 2x_2 - 2r \qquad \Leftrightarrow$$
$$-3x_1 = x_2 - r$$

Together with $x_1 + x_2 = 0$ we get $x_1 = -x_2 = r/2$ and

$$\dot{r} = 2\dot{x}_1 = 2(-2x_1 + x_2 - r) = -5r$$

Since $r(t) = e^{\alpha t}$ we have $\alpha = -5$. Moreover, for $y(t)$ to be zero for all t, the system must start in the initial condition $x_1(0) = -x_2(0) = r(0)/2$.

9.14 a) Enter the transfer function and generate the state space model.

```
>> s = tf( 's' );
>> G = ss( 1 / ( s * ( s + 1 ) ) )
a =
         x1  x2
     x1  -1  -0
     x2   1   0

b =
         u1
     x1   1
     x2   0

c =
         x1  x2
     y1   0   1

d =
         u1
     y1   0

Continuous-time model.
```

We hence have the state space representation

$$\dot{x}(t) = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} x(t)$$

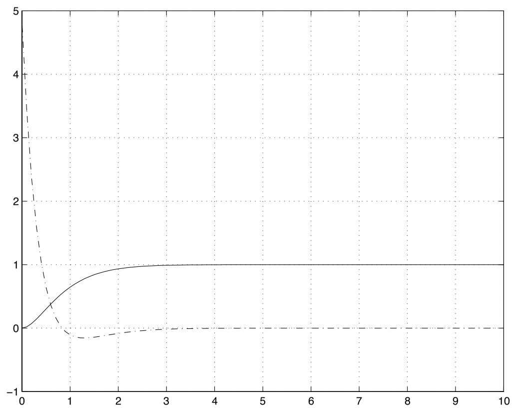
From the last equation we have $x_2(t) = y(t)$, that is, x_2 is the motor angle. From the first equation we have $\dot{x}_2(t) = x_1(t)$, that is, x_1 is the angular velocity.

b) Compute feedback gains. This time, the gain l_0 is computed by explicitly constructing a system with $l_0 = 1$ first, and then correcting by the inverse of that system's static gain. Note that if we don't need l_0, this approach simplifies to `Gc = Gc0 / dcgain( Gc0 )`. However, we do need l_0 in order to compute the control signal.

```
>> L = acker( G.a, G.b, [ -2.2 -2.2 ] );
>> Gc0 = ss( G.a - G.b * L, G.b, G.c, 0 );
>> l_0 = 1 / dcgain( Gc0 );
>> Gc = l_0 * Gc0;
```

Calculate the step response and the corresponding control signal of the closed loop system. To calculate the control signal magnitude use `[ y, t, x ] = step( Gc )`. The function `step` will in this case return y, the output of the closed loop system, t the time vector, and x the states of the system. To compute the control signal, use that $u(t) = l_0 r(t) - Lx(t)$, where $r(t) = 1$. Then plot the result.

```
>> [ y, t, x ] = step( Gc, 10 );
>> u = l_0 - x * L.';
>> plot( t, y, '-', ...
        t, u, '-.' );
>> grid
```

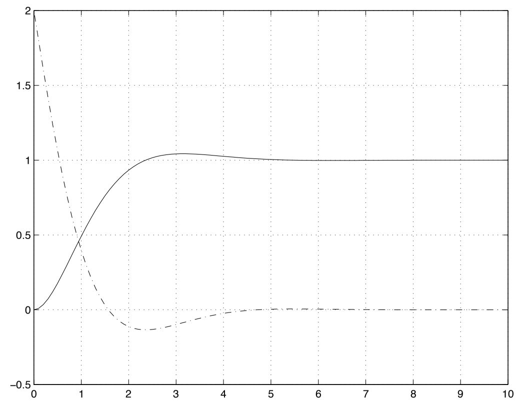


Compute a new feedback. This time, we compute the gain l_0 by using the formula for the static gain of the system with $l_0 = 1$ (put $s = 0$ in the generic expression for the transfer function).

```
>> L = acker( G.a, G.b, [ -1+i -1-i ] );
>> l_0 = 1 / ( G.c * inv( -G.a + G.b*L ) * G.b );
>> Gc = ss( G.a - G.b * L, G.b * l_0, G.c, 0 );
```

Calculate the step response and the corresponding control signal. Plot the result.

```
>> [ y, t, x ] = step( Gc, 10 );
>> u = l_0 - x * L.';
>> plot( t, y, '-', ...
        t, u, '-.' );
>> grid
```



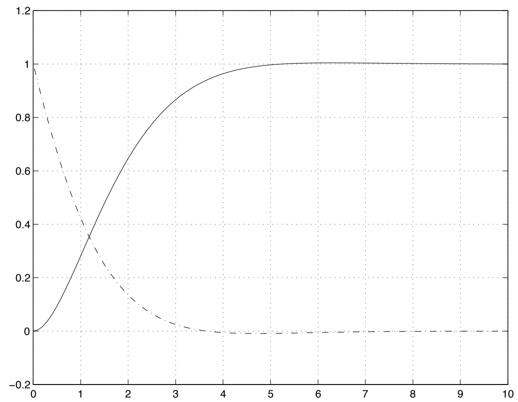
The step responses have approximately the same rise and settling times. By choosing the closed loop poles complex, and hence allowing a small overshoot in the step response, we have however reduced the maximum value of the input signal significantly.

c) Case (i): Compute the feedback gain L, l_0, and the closed loop system.

```
>> L = lqr( G.a, G.b, diag([ 0 1 ]), 1 );
>> l_0 = 1 / ( G.c * inv( -G.a + G.b*L ) * G.b
>> Gc = ss( G.a - G.b * L, G.b * l_0, G.c, 0 );
```

Simulate the system and plot the result.

```
>> [ y, t, x ] = step( Gc, 10 );
>> u = l_0 - x * L.';
>> plot( t, y, '-', ...
        t, u, '-.' );
>> grid
```



Compute the closed loop poles. This time, via the eigenvalues of the "A" matrix.

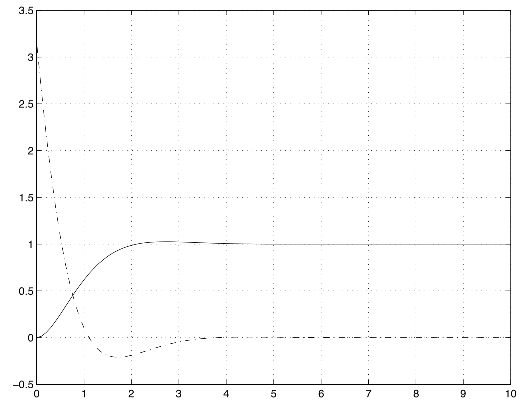
```
>> eig( Gc.a )
ans =
  -0.8660 + 0.5000i
  -0.8660 - 0.5000i
```

Case (ii): Repeat, this time with larger weight on the motor angle.

```
>> L = lqr( G.a, G.b, diag([ 0 10 ]), 1 );
>> l_0 = 1 / ( G.c * inv( -G.a + G.b*L ) * G.b );
>> Gc = ss( G.a - G.b * L, G.b * l_0, G.c, 0 );
```

Simulate the system and plot the result. The step response is now significantly faster.

```
>> [ y, t, x ] = step( Gc, 10 );
>> u = l_0 - x * L.';
>> plot( t, y, '-', ...
        t, u, '-.' );
>> grid
```



Compute the closed loop poles. This time using a dedicated command from the toolbox. The poles are now further away from the origin and the relative damping is slightly reduced.

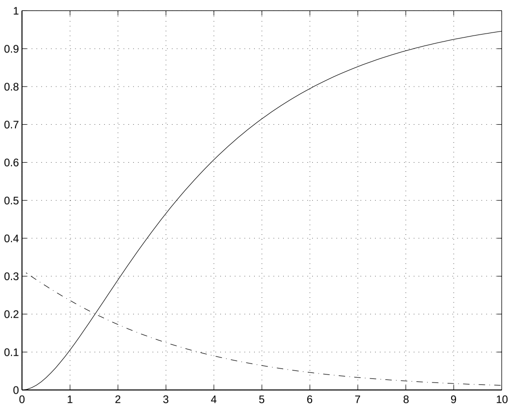
```
>> pole( Gc )
ans =
  -1.3532 + 1.1537i
  -1.3532 - 1.1537i
```

Case (iii): Repeat, this time with smaller weight on the motor angle.

```
>> L = lqr( G.a, G.b, diag([ 0 0.1 ]), 1 );
>> l_0 = 1 / ( G.c * inv( -G.a + G.b*L ) * G.b
>> Gc = ss( G.a - G.b * L, G.b * l_0, G.c, 0 );
```

Simulate the system and plot the result. The step response is now much slower.

```
>> [ y, t, x ] = step( Gc, 10 );
>> u = l_0 - x * L.';
>> plot( t, y, '-', ...
        t, u, '-.' );
>> grid
```



Compute the closed loop poles. We now get two real closed loop poles, where the pole in -0.34 causes the slow step response.

```
>> pole( Gc )
ans =
   -0.9420
   -0.3357
```

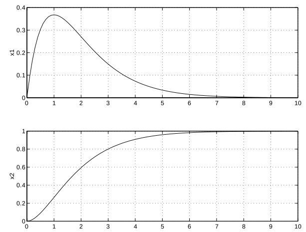
d) If we start from case (ii) and increase Q_2 the closed loop system gradually becomes slower, since we put an increasing weight on the control signal magnitude. When we reach $Q_2 = 10$ we get exactly the same result as for case (i). Since it is the "ratio" between Q_1 and Q_2 that determines the closed loop property we get the same feedback gain if we scale Q_1 and Q_2 by the same scalar.

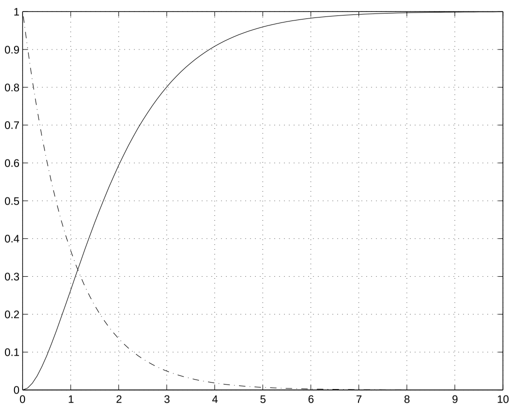
e) Compute feedback gains, adjust static gain, and compute closed loop system.

```
>> L = lqr( G.a, G.b, diag([ 1 1 ]), 1 );
>> l_0 = 1 / ( G.c * inv( -G.a + G.b*L ) * G.b );
>> Gc = ss( G.a - G.b * L, G.b * l_0, G.c, 0 );
```

Simulate the system and plot the result. Then we also plot the states, x_1 and x_2, in two different diagrams.

```
>> [ y, t, x ] = step( Gc, 10 );
>> u = l_0 - x * L.';
>> plot( t, y, '-', ...
        t, u, '-.' );
>> grid
>> figure
>> subplot( 2, 1, 1 );
>> plot( t, x(:,1) );
>> grid; ylabel( 'x1' );
>> subplot( 2, 1, 2 );
>> plot( t, x(:,2) );
>> grid; ylabel( 'x2' );
```





Increasing the weight on the angular velocity forces the motor to move slower, and then also the step response becomes slower.

9.15 Introduce the state variables

$$x_1 = q \quad x_2 = y$$

This gives the state space description

$$\dot{x} = \begin{pmatrix} -0.05 & 0 \\ 0.05 & -0.02 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

a) The system has the controllability matrix

$$\mathcal{S} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -0.05 \\ 0 & 0.05 \end{pmatrix} \qquad \det \mathcal{S} = 0.05$$

Thus, the system is controllable.

b) The control law

$$u = -Lx$$

gives the closed loop system

$$\dot{x} = (A - BL)x$$

and the poles of the closed loop system is given by the eigenvalues of $A - BL$.

$$A - BL = \begin{pmatrix} -0.05 - l_1 & -l_2 \\ 0.05 & -0.02 \end{pmatrix}$$

The characteristic equation is given by

$$\det(sI - A + BL) = s^2 + (0.07 + l_1)s + 0.001 + 0.02l_1 + 0.05l_2 = 0$$

Both poles in -0.1 implies that we shall have the equation

$$(s + 0.1)^2 = s^2 + 0.2s + 0.01 = 0$$

Identification of the coefficients gives

$$l_1 = 0.13 \quad l_2 = 0.128$$

This gives the control law

$$u = -0.13x_1 - 0.128x_2$$

c) It is desirable that the estimation error converges to zero faster than the dynamics of the system. Thus, we should place the eigenvalues of the observer to the left of the poles of the closed loop system. To avoid large amplification of the measurement noise the poles of the observer should not be placed to far into the left hand plane.

d) Only $y = x_2$ is measurable. Employ the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

The characteristic equation is

$$\det(sI - A + KC) = s^2 + (0.07 + k_2)s + 0.05k_1 + 0.05k_2 + 0.001$$

Both poles in -0.2 implies that we shall have the equation

$$s^2 + 0.4s + 0.04 = 0$$

Identification of the coefficients gives

$$k_1 = 0.45 \quad k_2 = 0.33$$

9.16 Are the specifications 1–4 fulfilled?

??. The bandwidth is $\omega_B \approx 1.1 < 5$ which is seen from the gain curve of the closed loop system.

The bandwidth requirement is not fulfilled.

??. Stability despite model errors and disturbances?

We have $Y(s) = \kappa e^{-\tau s}G(s)U(s) + E(s)$ instead of $Y(s) = G(s)U(s)$. The factor κ thus represents the gain uncertainty, while the factor $e^{-\tau s}$ represents a phase uncertainty. These uncertainties are also present (with the same magnitudes) in the loop gain $G_o = FG$.

Looking in the Nyquist curve of G_o, where amplitudes near 1 are easiest to read, one can see that there is always just one intersection with $|G_o(i\omega)| = 1$, independently of the present uncertainties in gain and phase. Thus the stability criterion based on the Bode plot applies.

The uncertain phase lag is $\omega\tau$ at the frequency ω. Thus the maximum negative phase lag occurs for $\tau^* = 0.3$ s.

Next, we must find the worst case gain crossover frequency in order to see if the worst case phase lag causes instability by reducing the phase margin below 0. Study the amplitude and phase curves for the loop gain $G_o(s)$.

Since the phase of G_o is decreasing, higher gain crossover will always be more critical since it both means a smaller phase margin to begin with, and also a bigger phase lag due to the worst case time delay.

From the gain curve of G_o it is clear that higher values of κ are more critical since those give the higher gain crossovers. By very careful inspection of the gain curve, one can see that the most critical value, $\kappa^* = 1.1$, leads to $\omega_c^* \approx 2.3$ rad/s < 3 rad/s, and $\varphi_m^* > 55°$.

Combining the worst case κ (leading to the ω_c^* and φ_m^* above) with the worst case and $\tau^* = 0.3$ s results in a total worst case phase margin of at least $55° - \omega_c^* \tau^* = 55° - 3\text{ rad/s} \cdot 0.3\text{ s} = 55° - 0.9\text{ rad} \approx 3° > 0$. Thus the system is guaranteed to be stable.

The system is stable despite the model errors.

Remark: The robustness criterion $\forall\omega : \ |Q(i\omega)| < \frac{1}{|G(i\omega)|}$ is sufficient but not necessary to show stability.

??. Both the Bode plot and the Nyquist curve of the loop gain tells us that the loop gain does not contain an integration which could remove static errors. This implies the model errors will influence the static gain. The details of this argument follow.

With $u = F_r r - F_y y$, the closed loop system is

$$G_c(s) = \frac{F_r(s)G(s)}{1 + F_y(s)G(s)}$$

The real closed loop system is

$$G_c^0(s) = \frac{F_r(s)\kappa e^{-\tau s}G(s)}{1 + F_y(s)\kappa e^{-\tau s}G(s)}$$

Since the system is stable (see **??**) the final value theorem gives the final value of the step response as

$$\lim_{t\to\infty} y^0(t) = \lim_{s\to 0} s \cdot G_c^0(s) \cdot \frac{1}{s} = \frac{F_r(0)\kappa G(0)}{1 + F_y(0)\kappa G(0)}$$

which cannot be 1 for all possible values of κ.

The gain will be different from 1 for some possible value of κ.

??. If $e(t)$ is measurement noise, then the complementary sensitivity function, $T(s)$, should be checked. If $e(t)$ is process noise, then the sensitivity function, $S(s)$, should be checked. Both $T(s)$ and $S(s)$ have peaks > 1 at exactly $\omega = 10$ rad/s, which implies the both measurement and process noise are amplified.

The (measurement) noise is amplified by the system.

9.17 a) The linearized system is given by

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u =: Ax + Bu$$

Using the state feedback law $u = -Lx = -l_1 x_1 - l_2 x_2$ gives

$$\dot{x} = A + B(-Lx) = (A - BL)x = \begin{pmatrix} l_1 & 1 + l_2 \\ -1 - l_1 & -3 - l_2 \end{pmatrix} x$$

The poles of this closed loop system are given by the eigenvalues of $A - BL$, which are the roots of the characteristic polynomial

$$\begin{aligned} P(s) &= \det(sI - (A - BL)) = \det \begin{pmatrix} s - l_1 & -1 - l_2 \\ 1 + l_1 & s + 3 + l_2 \end{pmatrix} \\ &= (s - l_1)(s + 3 + l_2) - (1 + l_1)(-1 - l_2) \\ &= s^2 + (-l_1 + l_2 + 3)s - 2l_1 + l_2 + 1 \end{aligned}$$

To place the poles in $\{-2, -4\}$, $P(s)$ must be the polynomial

$$(s + 2)(s + 4) = s^2 + 6s + 8$$

This gives the system of equations

$$\begin{aligned} -l_1 + l_2 + 3 &= 6 \\ -2l_1 + l_2 + 1 &= 8 \end{aligned}$$

which has the solution

$$l_1 = -4 \quad l_2 = -1$$

The state feedback law thus becomes $u = -Lx = 4x_1 + x_2$.

b) If only x_2 is measured, the output equation is given by

$$y = x_2 = \begin{pmatrix} 0 & 1 \end{pmatrix} x =: Cx$$

Given y (x_2) and u, x_1 can be estimated if the system is observable. The observability matrix becomes

$$\mathcal{O} = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} \qquad \det \mathcal{O} = 1$$

Hence the system is observable and x_1 can be estimated using an observer.

It is essential that the input u is known since u is required in the observer design to get an asymptotically vanishing state estimation error.

c) If u is unknown but constant, we can introduce a third state $x_3 = u$ which has the dynamics $\dot{x}_3 = 0$. Introducing $z^{\mathrm{T}} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$, the system dynamics can be rewritten as

$$\dot{z} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & -3 & 1 \\ 0 & 0 & 0 \end{pmatrix} z =: \tilde{A}z$$
$$y = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} z =: \tilde{C}z$$

The observability matrix becomes

$$\mathcal{O} = \begin{pmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \tilde{C}\tilde{A}^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -3 & 1 \\ 3 & 8 & -2 \end{pmatrix} \qquad \det \mathcal{O} = 1$$

(Tip: $\det(\mathcal{O}) \neq 0$ can be established without computing the determinant, by checking that the rows of $\mathcal{O}$ are linearly independent.) The fact that the system is observable means that x_1 (and also u) can be estimated from measurements of x_2 using an observer of the form

$$\dot{\hat{z}} = \tilde{A}\hat{z} + K(y - \tilde{C}\hat{z}) = (\tilde{A} - K\tilde{C})\hat{z} + Ky$$

where the observer gain K is selected so that the observer poles, that is, the eigenvalues of $\tilde{A} - K\tilde{C}$, are all in the left half plane.

9.18 a) The system is described by

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w$$

A P controller corresponds to $u = K(r - x_1)$, this means that the closed loop system is given by

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1-K & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ K \end{pmatrix} r + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w$$

The poles to the closed loop system are given by

$$\det \begin{pmatrix} s & -1 \\ 1+K & s \end{pmatrix} = 0$$

which leads to $s^2 + 1 + K = 0$. The poles are pure complex and thus the system doesn't have a well defined stationary error or speed of response.

b) A linear combination of r and x_2 is given by

$$u = l_0 r - l_2 x_2$$

with this controller the closed loop is

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & l_2 \end{pmatrix} x + \begin{pmatrix} 0 \\ l_0 \end{pmatrix} r + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w$$

The poles to the closed loop system are given by

$$\det \begin{pmatrix} s & -1 \\ 1 & s + l_2 \end{pmatrix} = 0$$

which means $s^2 + l_2 s + 1 = 0$. The poles can be placed with l_2 as

$$s = \frac{-l_2}{2} \pm \sqrt{\frac{l_2^2 - 4}{4}}$$

We have that $\dot{x} = 0$ at stationary which gives that $x_2 = -w$ and $x_1 = l_0 r - l_2 x_2$ if $w = 0$. If we select $l_0 = 1$ then the stationary error will be zero. If $w \neq 0$ and $l_0 = 1$ then there will be stationary error of size $l_2 w$.

c) Introduce a new state $x_3 = w$ to estimate the unknown signal. The extended system is described by

$$\dot{x} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} x$$

Create an observer to estimate the states

$$\dot{\hat{x}} = (A - KC)\hat{x} + Bu + Ky$$

the poles of the observer can be placed with

$$\det(sI - (A - KC)) = 0$$

which gives $s^3 + k_2 s^2 + (1 - k_1)s - k_3 = 0$. Place the poles for example in -2, that is, seek the polynomial $s^3 + 6s^2 + 12s + 8 = 0$. Comparison gives

$$k_1 = -11 \quad k_2 = 6 \quad k_3 = -8$$

Now, let $u = l_0 r - l_2\hat{x}_2 - l_3\hat{x}_3$. At stationary we have $\hat{x}_3 = w = -\hat{x}_2$, so with $l_3 = l_2$ we have $x_1 = l_0 r$, and with $l_0 = 1$ there will be no error.

9.19 a) Överföringsfunktionerna finnes genom:

i. $X(s)G_1(s)(U(s) - G_2(s)X(s)) = G_1(s)U(s) - G_1(s)G_2(s)X(s)$
$\Rightarrow X(s)(1 + G_1(s)G_2(s) = G_1(s)U(s)$
$\Rightarrow X(s) = \frac{G_1(s)}{1+G_1(s)G_2(s)}U(s)$
$\Rightarrow G_X(s) = \frac{G_1(s)}{1+G_1(s)G_2(s)}$

ii. $Y(s) = G - 4(s)U(s) + G_3(s)X(s) = G_4(s)U(s) + G_3(s)G_X(s)U(s)$
$= (G_4(s) + \frac{G_1(s)G_3(s)}{1+G_1(s)G_2(s)})U(s)$
$\Rightarrow G(s) = G_4(s) + \frac{G_1(s)G_3(s)}{1+G_1(s)G_2(s)}$

b) Vi kan skriva $G(s) = \frac{s+2}{s^2}$ som

$$G(s) = \frac{s+2}{s^2} = \frac{s+2}{s^2 + 0s + 0} = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}$$

Detta kan nu skrivas enkelt på t.ex. styrbar kanonisk form:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -a_1 & -a_2 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u = \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{A_s} x + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{B_s} u \\ y &= \begin{pmatrix} b_1 & b_2 \end{pmatrix} x = \underbrace{\begin{pmatrix} 1 & 2 \end{pmatrix}}_{C_s} x \end{aligned}$$

eller alternativt (det räcker med att svara med en korrekt form för att få full poäng) på observerbar kanonisk form:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -a_1 & 0 \\ -a_2 & 0 \end{pmatrix} x + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A_o} x + \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{B_o} u \\ y &= \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C_o} x \end{aligned}$$

Ett system är en minimal realisation om det är både styrbart och observerbart. Därför måste styrbarhetsmatrisen ($\mathcal{S}$) och observerbarhetsmatrisen ($\mathcal{O}$) ha full rang.

$$\begin{aligned} \det \mathcal{S} &= \det\left(\begin{bmatrix} B_s & A_s B_s \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 \neq 0 \\ \det \mathcal{O} &= \det\left(\begin{bmatrix} C_s \\ C_s A_s \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\right) = -4 \neq 0 \end{aligned}$$

Alternativt om en observerbar kanonisk representation används:

$$\begin{aligned} \det \mathcal{S} &= \det\left(\begin{bmatrix} B_o & A_o B_o \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\right) = -4 \neq 0 \\ \det \mathcal{O} &= \det\left(\begin{bmatrix} C_o \\ C_o A_o \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 \neq 0 \end{aligned}$$

I båda fallen har styrbarhetsmatrisen och observerbarhetsmatrisen full rang, eftersom determianterna är skillda från noll. Därför är systemet en minimal realisation.

c) Med tillståndsåterkopplingen $u(t) = -Lx(t) + l_0 r(t)$, blir tillståndsekvationen

$$\dot{x}(t) = Ax(t) + Bu(t) = (A - BL)x(t) + l_0 r(t)$$

Polerna ges då av egenvärden till $(A - BL)$, dvs. genom den karakteristiska ekvationen:

$$\det(sI - (A_s - B_s L)) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} l_1 & l_2 \end{bmatrix}\right)\right) =$$

$$= \det\left(\begin{bmatrix} s + l_1 & l_2 \\ -1 & s \end{bmatrix}\right) = s^2 + l_1 s + l_2 = 0$$

Önskade poler i $\{-1, -1\}$ ger följande karakteristiska ekvation:

$$(s+1)^2 = s^2 + 2s + 1$$

Genom att identifiera kofficienter erhålles:

$$l_1 = 2 \quad l_2 = 1$$

Notera: Detta kunde även inses snabbt genom att uppmärksamma att för system skrivna på styrbar kanonisk form är koefficienterna i den önskade karakteristiska ekvationen samma som parametrarna l_i i L-matrisen för återkopplingen.

Systemet från $r(t)$ är $Y(s) = C(sI - (A - BL))^{-1} B l_0 R(s)$. Den statiska förstarkningen erhålles då $s = 0$, vilket medför:

$$l_0 = \frac{1}{C_s(-A_s + B_s L)^{-1} B_s} = \frac{1}{2}$$

På liknande sätt kan L och l_0 erhållas om kanonisk observerbar form nyttjas, dvs. (A_o, B_o, C_o). Då blir

$$L = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \end{bmatrix} \quad l_0 = \frac{1}{2}$$

d) En observatör införs i systemet enligt:

$$\dot{x}(t) = Ax(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

Polerna ges nu av egenvärden till $(A - KC)$:

$$\det(sI - (A_s - KC_s)) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix}\right)\right) =$$

$$= \det\left(\begin{bmatrix} s + k_1 & 2k_2 \\ k_2 - 1 & s + 2k_2 \end{bmatrix}\right) = s^2 + (k_1 + 2k_2)s + 2k_1 = 0$$

Önskade poler i $\{-10, -10\}$ till observatören ger följande karakteristiska ekvation:

$$(s+1)^2 = s^2 + 20s + 100$$

Och genom koefficientidentifiering erhålles:

$$k_1 = 50 \quad k_2 = -15 \quad \Rightarrow \quad K = \begin{bmatrix} 50 & -15 \end{bmatrix}^T$$

På likande sätt kan K erhållas om kanonisk observerbar form nyttjas. För system skrivna på observerbar kanonisk form är koefficienterna i den önskade karakteristiska ekvationen samma som parametrarna k_i i K-matrisen för observatören, dvs.

$$K = \begin{bmatrix} 20 & 100 \end{bmatrix}^T$$

11 Implementation

11.1 Inverse Laplace transformation of

$$U(s) = KN\frac{s+b}{s+bN}E(s)$$

gives the differential equation

$$\dot{u}(t) + bNu(t) = KN\dot{e}(t) + bKNe(t) \qquad (11.1)$$

At time $t-T$ we have

$$\dot{u}(t-T) + bNu(t-T) = KN\dot{e}(t-T) + bKNe(t-T) \qquad (11.2)$$

By replacing $\dot{u}(t)$ and $\dot{e}(t)$ in (11.1) and (11.2) with $\Delta_t u(t)$ and $\Delta_t e(t)$, respectively, and then adding the equations we get

$$\begin{aligned}\Delta_t u(t) + \Delta_t u(t-T) + bNu(t) + bNu(t-T)&\\ = KN\Delta_t e(t) + KN\Delta_t e(t-T) + bKNe(t) + bKNe(t-T)&\end{aligned}$$

Tustins formula

$$\frac{1}{2}(\Delta_t u(t) + \Delta_t u(t-T)) = \frac{1}{T}(u(t) - u(t-T))$$

now gives

$$\begin{aligned}\frac{2}{T}(u(t) - u(t-T)) + bN(u(t) + u(t-T))&\\ = \frac{2}{T}KN(e(t) - e(t-T)) + bKN(e(t) + e(t-T))&\end{aligned}$$

Inserting the numerical values, $K = 2$, $T = 0.1$, $N = 10$ and $b = 0.1$, we get

$$\begin{aligned}20(u(t) - u(t-T)) + (u(t) + u(t-T))&\\ = 400(e(t) - e(t-T)) + 2(e(t) + e(t-T))&\end{aligned}$$

which gives

$$u(t) = \frac{19}{21}u(t-T) + \frac{402}{21}e(t) - \frac{398}{21}e(t-T)$$

that is

$$u(t) = 0.905u(t-T) + 19.14e(t) - 18.95e(t-T)$$

11.2 a) Consider the differential equation

$$\dot{y}(t) = u(t)$$

during the sampling interval $kT \le t < kT + T$. The input is constant during the sampling interval, $u(t) = u_k$, which gives

$$\dot{y}(t) = u_k$$

By integrating the left- and right-hand sides from $t = kT$ to $t = kT + T$ we get

$$y(kT+T) - y(kT) = \int_{kT}^{kT+T} u_k \, \mathrm{d}t = Tu_k$$

With the notation $y_{k+1} = y(kT+T)$ and $y_k = y(kT)$ this gives

$$y_{k+1} - y_k = Tu_k$$

b) The feedback

$$u_k = -Ky_k$$

gives

$$y_{k+1} = (1 - KT)y_k \quad y_0 = y^0$$

that is

$$y_k = (1 - KT)^k y^0$$

The closed loop system is asymptotically stable if

$$|y_k| \to 0, \ t \to \infty$$

This gives the condition

$$|1 - KT| < 1$$

or, equivalently, $0 < K < \frac{2}{T}$.

11.3 a) Because the prefilter is linear, the signal prior to sampling may be written

$$y(t) = y_0(t) + y_1(t)$$

where $y_1(t)$ stems from the disturbance $u_1(t) = \sin\omega_2 t$. After all transients have disappeared, we get

$$y_1(t) = A\sin(\omega_2 t + \Phi)$$

where

$$A = |G(\mathrm{i}\omega_2)| = \frac{1}{\sqrt{1+(\omega_2 T_1)^2}}$$
$$\Phi = \arg G(\mathrm{i}\omega_2) = -\arctan\omega_2 T_1$$

Let us introduce the notation $\omega_1 = \omega_\mathrm{s} - \omega_2$ where ω_s denotes the sampling frequency, $\omega_\mathrm{s} = 2\pi/T$. When $y_1(t)$ is sampled with the sampling interval T, we get

$$\begin{aligned} y_1(kT) &= A\sin(\omega_2 kT + \Phi) = A\sin((\omega_\mathrm{s}-\omega_1)kT + \Phi) \\ &= A\sin(2k\pi - \omega_1 kT + \Phi) = A\sin(-\omega_1 kT + \Phi) \\ &= -A\sin(\omega_1 kT - \Phi) = A\sin(\omega_1 kT + \pi - \Phi) \\ &= A\sin(\omega_1 kT + \varphi) \end{aligned}$$

that is

$$\begin{aligned} A &= \frac{1}{\sqrt{1+(\omega_2 T_1)^2}} \\ \omega_1 &= \frac{2\pi}{T} - \omega_2 \\ \varphi &= \pi + \arctan\omega_2 T_1 \end{aligned}$$

b) The bandwidth of the filter is obtained from the relation

$$|G(\mathrm{i}\omega_\mathrm{B})| = \frac{1}{\sqrt{1+(\omega_\mathrm{B} T_1)^2}} = \frac{1}{\sqrt{2}}$$

which gives $\omega_\mathrm{B} = 1/T_1$. The signal u_0 is in the interval $0 \le \omega < \pi/T$, and this gives the specification

$$\frac{\pi}{T} \le \frac{1}{T_1}$$

The limiting case

$$\frac{\pi}{T} = \frac{1}{T_1}$$

gives $T_1 = T/\pi$. Inserting this in the expression for A in a), we get the answer

$$A = \frac{1}{\sqrt{1+(\omega_2 T/\pi)^2}}$$

11.4 PI-regulatorn ges av

$$F(s) = K + \frac{K}{T_I s}.$$

Regulatorn är alltså

$$T_I\dot{u}(t) = KT_I\dot{e}(t) + Ke(t).$$

Euler bakåt ger

$$T_I(u(t) - u(t-1)) = KT_I(e(t) - e(t-1)) + Ke(t)$$

$$\Rightarrow u(t) = u(t-1) + \frac{KT_I + K}{T_I}e(t) - Ke(t-1).$$

Vi i dentifierar $K = T_I = 1$.

Svar: $K = T_I = 1$.

11.5 a) Vi börjar med att skriva modellen på tillståndsform, $\dot{x} = f(x,u)$. Om tillståndsvektorn väljs som

$$x = \begin{bmatrix} \theta \\ z \\ \dot{z} \end{bmatrix}$$

kan ekvationerna skrivas

$$\begin{aligned} \dot{x_1} &= Ku = f_1(x,u) \\ \dot{x_2} &= x_3 = f_2(x,u) \\ \dot{x_3} &= -\frac{m}{m+J/r^2} g\sin x_1 + \frac{m}{m+J/r^2} x_2 K^2 u^2 = f_3(x,u) \end{aligned}$$

Vid jämviktspunkten gäller $f(x_0,u_0) = 0$, vilket ger

$$\begin{aligned} x_0 &= [0 \quad z_0 \quad 0]^T \\ u_0 &= 0, \end{aligned}$$

för valfri konstant z_0, då $u_0 = \theta_0 = \dot{z}_0 = 0$ enligt uppgiften. Vi väljer $z_0 = 0$ i fortsättningen. Jakobianerna blir

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{5}{7}g\cos x_1 & \frac{5}{7}K^2u^2 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} K \\ 0 \\ \frac{10}{7}K^2x_2u \end{bmatrix}$$

I jämviktspunkten får vi

$$A = \frac{\partial f}{\partial x}(x_0, u_0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{5}{7}g & 0 & 0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}(x_0, u_0) = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix}$$

Beteckna $\triangle x = x - x_0$ och $\triangle u = u - u_0$. Det linjäriserade systemet ges då av

$$\dot{\triangle x} = A\triangle x + B\triangle u.$$

b) Om $y = \theta$ så $\dot{y} = Ku$. För $kT \leq t < (k+1)T$, har vi

$$\dot{y}(t) = Ku_k.$$

Integrerar vi över samplingsintervallet fås

$$y(kT + T) - y(kT) = \int_{kT}^{kT+T} Ku_k dt = TKu_k$$

Med $y(kT) = y_k$, fås

$$y_{k+1} = y_k + TKu_k.$$

c) Med $u_k = -K_py_k$, får vi $y_{k+1} = (1 - K_pTK)y_k$.

För asymptotisk stabilitet ($y_k \to 0$, $k \to \infty$) krävs $|1 - K_pTK| < 1$. Detta ger oss $0 < K_p < \frac{2}{KT}$.

Liten reglerteknisk ordlista

This version: August 2013

1 Engelsk-svensk

actuator	ställdon
amplitude	amplitud
attenuation	dämpning
bandwidth	bandbredd
bond graph	bindningsgraf
closed loop system	slutet system
control law	styrlag
controllability	styrbarhet
controller	regulator
convolution	faltning
correlation analysis	korrelationsanalys
credibility	trovärdighet
crossover frequency	skärfrekvens
damping	dämpning
damping ratio	relativ dämpning
describing function	beskrivande funktion
discrete event systems	händelseorienterade system
distributed parameter models	fördelade parametriska modeller
disturbance rejection	störningsundertryckning
eigenvalue	egenvärde
feedback	återkoppling
feedforward	framkoppling
flow	flöde
gain	förstärkning
gain crossover frequency	(amplitud)skärfrekvens
gain margin	amplitudmarginal
impulse response	impulssvar
initial value	begynnelsevärde
loop gain	kretsförstärkning, öppna systemet
lumped models	aggregerade modeller
magnitude	amplitud
observability	observerbarhet
observer	observatör
open-loop system	öppet system, kretsförstärkning
overfit	överanpassning
overshoot	översläng
parsimonious	sparsam
peak frequency	resonansfrekvens
peak resonance	resonanstopp
phase crossover frequency	fasskärfrekvens
phase lag	fasretarderande
phase lead	fasavancerande
phase margin	fasmarginal
ramp function	ramp
rank	rang
reset windup	integratoruppvridning
resonant frequency	resonansfrekvens
rise time	stigtid
root locus (pl. loci)	rotort
sensitivity function	känslighetsfunktion
sensor	givare
settling time	insvängningstid, lösningstid
sinusoidal	sinusformad
stability robustness	stabilitetsrobusthet
state	tillstånd
state feedback	tillståndsåterkoppling
static gain	statisk förstärkning
steady state	stationärt tillstånd
step function	steg
step repsonse	stegsvar
subspace	underrum
time delay	tidsfördröjning
transfer function	överföringsfunktion
unit step	enhetsteg
unstable	instabil
validity	giltighet
whitening filter	blekningsfilter

2 Svensk-engelsk

(amplitud)skärfrekvens	gain crossover frequency
aggregerade modeller	lumped models
amplitud	amplitude
amplitud	magnitude
amplitudmarginal	gain margin
bandbredd	bandwidth
begynnelsevärde	initial value
beskrivande funktion	describing function
bindningsgraf	bond graph
blekningsfilter	whitening filter
dämpning	damping, attenuation
egenvärde	eigenvalue
enhetsteg	unit step
faltning	convolution
fasavancerande	phase lead
fasmarginal	phase margin
fasretarderande	phase lag
fasskärfrekvens	phase crossover frequency
flöde	flow
framkoppling	feedforward
fördelade parametriska modeller	distributed parameter models
förstärkning	gain
giltighet	validity
givare	sensor
händelseorienterade system	discrete event systems
impulssvar	impulse response
instabil	unstable
insvängningstid, lösningstid	settling time
integratoruppvridning	reset windup
korrelationsanalys	correlation analysis
kretsförstärkning	loop gain, open loop system
känslighetsfunktion	sensitivity function
observatör	observer
observerbarhet	observability
ramp	ramp function
rang	rank
regulator	controller
relativ dämpning	damping ratio
resonansfrekvens	peak frequency
resonansfrekvens	resonant frequency
resonanstopp	peak resonance
rotort	root locus (pl. loci)
sinusformad	sinusoidal
slutet system	closed loop system
skärfrekvens	gain crossover frequency
sparsam	parsimonious
stabilitetsrobusthet	stability robustness
stationärt tillstånd	steady state
statisk förstärkning	static gain
steg	step function
stegsvar	step repsonse
stigtid	rise time
styrbarhet	controllability
styrlag	control law
ställdon	actuator
störningsundertryckning	disturbance rejection
tidsfördröjning	time delay
tillstånd	state
tillståndsåterkoppling	state feedback
trovärdighet	credibility
underrum	subspace
återkoppling	feedback
öppet system	open-loop system, loop gain
överanpassning	overfit
överföringsfunktion	transfer function
översläng	overshoot