An example of the Method of Characteristics.

John Andersson johnan@kth.se

Example.

1. Find the solution $u(x,y) \in C^1(D)$ to

$$\begin{array}{ll} yu(x,y)\frac{\partial u(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y} = yu(x,y) & \text{in } D \\ u(s,0) = -s & \text{for } s \in (-\infty,\infty), \end{array} \tag{1}$$

   where $D = (-\infty,\infty) \times (-1,1)$.

2. Sketch the projected characteristics corresponding to the initial value problem (1). Show that, and explain why, you can not find a solution in $\tilde{D} = (-\infty,\infty) \times (-2,2)$. Which is the largest set where you can find a solution?

Solution.

1. The characteristic equations are

$$\begin{array}{ll} \frac{dx(t;s)}{dt} = y(t;s)z(t;s) & \text{and } x(0;s) = s \\ \frac{dy(t;s)}{dt} = 1 & \text{and } y(0;s) = 0 \\ \frac{dz(t;s)}{dt} = y(t;s)z(t;s) & \text{and } z(0;s) = -s. \end{array}$$

   From the second characteristic equation we immediately see that $y(t;s) = t$. The third characteristic equation becomes

$$\frac{dz(t;s)}{dt} = tz(t;s)$$

   with solution

$$z(t;s) = -se^{\frac{t^2}{2}}. \tag{2}$$

   The first equation then becomes

$$\frac{\partial x(t;s)}{\partial t} = -ste^{\frac{t^2}{2}}$$

   with solution

$$x(t;s) = -se^{\frac{t^2}{2}} + \text{function in } s.$$

   Comparing with the initial data we see that

$$x(t;s) = -se^{\frac{t^2}{2}} + 2s. \tag{3}$$

We need to calculate $u(x,y) = z(t(x,y); s(x,y))$, in particular we need to find $t(x,y)$ and $s(x,y)$. We already know that $t(x,y) = y$. To calculate $s(x,y)$ we use (3) and see

$$x = -se^{\frac{y^2}{2}} + 2s = \left(2 - e^{\frac{y^2}{2}}\right) s.$$

That is

$$s = \frac{x}{2 - e^{\frac{y^2}{2}}}.$$

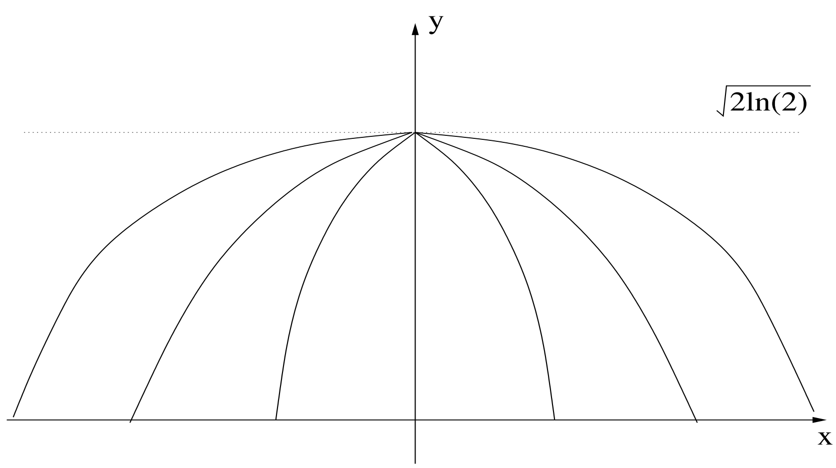
So our solution is, using (2) and the expressions for s and t,

$$u(x,y) = z(t;s) = -se^{\frac{t^2}{2}} = -\frac{xe^{\frac{y^2}{2}}}{2 - e^{\frac{y^2}{2}}}.$$

2. The projected characteristics PC_s are defined

$$CC_s = \{(x(t;s), y(t;s));\ t \in (-\alpha, \alpha)\} =$$
$$\left\{(-se^{\frac{t^2}{2}} + 2s, t);\ t \in (-\alpha, \alpha)\right\}.$$

Plotting a few characteristics we get the following diagram:



We notice that all characteristics merge at the points $x = 0$ and $y = \pm\sqrt{2\ln(2)}$. In particular,

$$x(\pm\sqrt{2\ln(2)}; s_1) = x(\pm\sqrt{2\ln(2)}; s_2)$$

for all $s_1, s_2 \in \mathbb{R}$. Since we can only calculate our solution on the projected characteristics we can not calculate the solution beyond $t = \pm\sqrt{2\ln(2)}$, that is for $y \notin (-\sqrt{2\ln(2)}, \sqrt{2\ln(2)})$.

From the calculation of $u(x, y)$ we also see that

$$\lim_{y \to \pm\sqrt{2\ln(2)}} |u(x, y)| = \infty$$

for $x \neq 0$ so it is impossible to extend the solution in any connected domain containing the lines $y = \pm\sqrt{2\ln(2)}$.