Theory of PDE, Examples Sheet 1

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1. Find the solution to the following first order PDE

$$\begin{array}{ll} 3\frac{\partial u(x,y)}{\partial x} + 2\frac{\partial u(x,y)}{\partial y} = x^2 & \text{in } \mathbb{R}^{\nvDash} \\ u(s,0) = \sin(s) & \text{for all } s \in \mathbb{R}. \end{array}$$

2. Suppose that $u(x,y) = x\cos(y)$ show that

$$\frac{du(s,s^2)}{ds} = \frac{\partial u}{\partial x}(s,s^2) + 2s\frac{\partial u}{\partial y}(s,s^2) = \cos(s^2) - 2s^2\sin(s^2).$$

3. Sketch the curves $(x(s), y(s))$ in $\mathbb{R}^2$ when
a) $x(s) = s$ and $y(s) = \sin(s)$ for $s \in (-2,5)$,
b) $x(s) = s|s|$ and $y(s) = s^3$ for $s \in (-5,5)$,
c) $x(s) = s^2$ and $y(s) = s^3$ for $s \in (-5,5)$.

4.
a) Solve the following initial value problem

$$\begin{array}{ll} \frac{\partial u}{\partial x} + 3x^2\frac{\partial u}{\partial y} = 1, & \\ u(x,0) = h(x) & h \in C^1(\mathbb{R}),\ \ u \in C^1(\mathbb{R}^2). \end{array}$$

b) Show that, and explain why, the problem

$$\begin{array}{ll} \frac{\partial u}{\partial x} + 3x^2\frac{\partial u}{\partial y} = 1, & \\ u(s,s^3) = 1 & h \in C^1(\mathbb{R}),\ \ u \in C^1(\mathbb{R}^2), \end{array}$$

has no solution.
c) Show that, and explain why, the problem

$$\begin{array}{ll} \frac{\partial u}{\partial x} + 3x^2\frac{\partial u}{\partial y} = 1, & \\ u(s,s^3) = s-1 & h \in C^1(\mathbb{R}),\ \ u \in C^1(\mathbb{R}^2), \end{array}$$

Infinitely many solutions.

5. Solve the following initial value problems and verify your solutions:
a) $u_x + u_y = u^2$ and $u(x,0) = h(x)$,
b) $u_y = \frac{u_x}{(1-u)(1-y)}$ and $u(x,0) = x$,
c)*(Difficult)* $xu_x + yu_y + u_z = u$ and $u(x,y,0) = h(x,y)$.
Here $u_x = \frac{\partial u}{\partial x}$ etc..
(HINT: In part c) you encounter a first order PDE in $\mathbb{R}^3$ for the first time. That is $u = u(x,y,z)$. The characteristic equations in $\mathbb{R}^3$ for the equation

$$a(x,y,z)\frac{\partial u}{\partial x} + b(x,y,z,u)\frac{\partial u}{\partial x} + c(x,y,z,u)\frac{\partial u}{\partial x} = d(x,y,z,u)$$

with initial data $u(f(s_1,s_2),g(s_1,s_2),h(s_1,s_2))=l(s_1,s_2)$ are given by:

$$\begin{array}{ll}\frac{dx(t;s_1,s_2)}{dt}=a(x,y,z,w) & x(0,s_1,s_2)=f(s_1,s_2)\\ \frac{dy(t;s_1,s_2)}{dt}=b(x,y,z,w) & y(0,s_1,s_2)=g(s_1,s_2)\\ \frac{dz(t;s_1,s_2)}{dt}=c(x,y,z,w) & z(0,s_1,s_2)=h(s_1,s_2)\\ \frac{dw(t;s_1,s_2)}{dt}=d(x,y,z,w) & x(0,s_1,s_2)=l(s_1,s_2),\end{array}$$

where $w(t;s_1,s_2)=u\big(x(t;s_1,s_2),y(t;s_1,s_2),z(t;s_1,s_2)\big)$.)

6.

a) Solve the equation

$$\begin{array}{ll}uu_x+u_y=-u & \text{in } D=\{(x,y);x\in\mathbb{R},\ y>0\}\\ u(x,0)=h(x).\end{array} \tag{1}$$

(NOTE: The solution can only be given in implicit form.)

b) Show that if $h(x)>0$ for all $x\in\mathbb{R}$ then, for each $s\in\mathbb{R}$, the curve

$$y_s(x)=\log\left(\frac{h(s)}{h(s)+s-x}\right),\quad s\le x\le s+h(x)$$

is a characteristic projection for the equation (1).