Theory of PDE, Examples Sheet 2

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1. Consider the following PDE

$$\frac{\partial u(x,y)}{\partial x} = 0. \tag{1}$$

a) Find the characteristics and projected characteristics corresponding to (1) when the initial data is given by

$$u(0,y) = h(y). \tag{2}$$

Find the solution, u.

b) Find the projected characteristics corresponding to (1) when the initial data is given by

$$u(x,0) = \tilde{h}(x). \tag{3}$$

Show that the initial line $\{(f(s), g(s)) = (s,0);\ s \in (-\infty, \infty)\}$ is a projected characteristic.

Prove that we have no solutions to (1) with initial conditions (3) unless $\tilde{h}(x) = c$.

c) Assume that $\tilde{h} = c =$constant in part b). Show that any solution to (1) with initial conditions (2) is also a solution to (1) with initial conditions (3) as long as $h(0) = c$. Conclude that b) has infinitely many solutions.

2. Consider Burger's equations

$$\begin{array}{ll} uu_x + u_y = 0 & \text{in } \mathbb{R}^2 \\ u(x,0) = h(x) & \text{for } x \in (-\infty, \infty). \end{array}$$

Assume furthermore that $h(1) = 4$ and $h(2) = 1$.

Find the point (x^0, y^0) where the projected characteristics PC_1 and PC_2 intersect. Show that $\lim_{(x,y)\to(x^0,y^0)} u(x,y)$ does not exist and thus that $u \notin C(\mathbb{R}^2)$.

3. Continuation of 6. on sheet 1.

a) Still assuming that $h > 0$, plot a typical characteristic projection from part b) of question 6 on sheet 1.

b) Assume furthermore that $h'(s) > -1$ for all $s \in \mathbb{R}$ and show that the projected characteristics do not intersect each other. Conclude that the solution exist for all $y > 0$.

c) Show that $\lim_{y\to\infty} u(x,y) = 0$ for each x.

4. Let $F(t, y(t)) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a function satisfying for all $s \in \mathbb{R}$ and some L:

$$|F(s,p) - F(s,q)| \leq L|p-q|.$$

Furthermore let $y(t) : (-\epsilon, \epsilon) \to \mathbb{R}^n$ be a solution to the equation

$$y(t) = y_0 + \int_0^t F(s, y(s))ds.$$

a) Show that $y(t) \in C((-\epsilon, \epsilon))$.
(Hint: What is $|y(t_1) - y(t_2)|$.)

b) Show that $y(t) \in C^1((-\epsilon, \epsilon))$.

5. A function $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the condition $|F(p) - F(q)| \leq L|p-q|$ for some constant L is called *Lipschitz* with constant L.
Show that $F(p)$ is Lipschitz in the convex set $\Omega \subset \mathbb{R}^n$ when

a) $F(p) = 4|p|$.

b) $F(p) \in C^1(\overline{\Omega})$. (Hint: The mean value Theorem together with the fact that a function that is continuous in a closed set is bounded might be useful.)

6. Show that the following ODE has a solution close to the point $(s,t) = (0,0)$

$$\begin{array}{ll} \frac{dx(t;s)}{dt} = \tan(e^{-x}) & \text{and } x(0;s) = s \\ \frac{dy(t;s)}{dt} = \arctan(1+x^2) & \text{and } y(0;s) = 0. \end{array}$$

7. Consider the PDE

$$\begin{array}{ll} 3\frac{\partial u(x)}{\partial x} + 2\frac{\partial u(x,y)}{\partial y} = 0 & \text{in } \Omega \\ u(x,0) = \frac{1}{1+|x^2|} & \text{on the line } y = 0. \end{array}$$

Find an equation for the curve $\mathcal{B}$ where the solution blows-up.

8. In the first paragraph of the proof that $p \in C^1$ in the proof of Theorem 2 there is a statement:

...we will show that there exists a function $w(t; s_0)$ such that

$$|p(t;s) - p(t;s_0) - (s_0)p_0'(s-s_0)w(t;s_0)| = o(|s-s_0|)$$

for each t. Then it follows that $\frac{\partial p(t;s)}{\partial s}$ is continuous.

In this exercise we will show that this is indeed the case. Let $f(x)$ be any continuous function and assume that there exists a continuous function $\sigma(\delta) > 0$ such that $\lim_{\delta \to 0^+} \sigma(\delta) = 0$[1] and that there for each x exists a number $d(x)$ such that

$$|f(x) - f(y) - d(x)(x-y)| \leq \sigma(|x-y|)|x-y|. \tag{4}$$

1. Show that this implies that f is differentiable and that $f'(x) = d(x)$.

2. Show that $d(x)$ is continuous and f is therefore in C^1.

   HINT: *Assume the contrary that there exists a sequence $x^j \to x^0$ such that $d(x^j) \not\to d(x^0)$. Draw the picture of the situation using the assumption (4).*

3. COnvince yourself that Theorem 2 is true.

9: Assume that f is a function such that $f \in C^1(\mathbb{R}^2)$, $\nabla f(x,y) = (a(x,y), b(x,y))$ and $|\nabla f| \neq 0$. Prove that any solution u to the partial differential equation

$$a(x,y)\frac{\partial u(x,y)}{\partial x} + b(x,y)\frac{\partial u(x,y)}{\partial y} = 0$$

can be written as $u(x,y) = G(f(x,y))$ for some function $G : \mathbb{R} \mapsto \mathbb{R}$.

[1]Such a function is called a modulus of continuity.