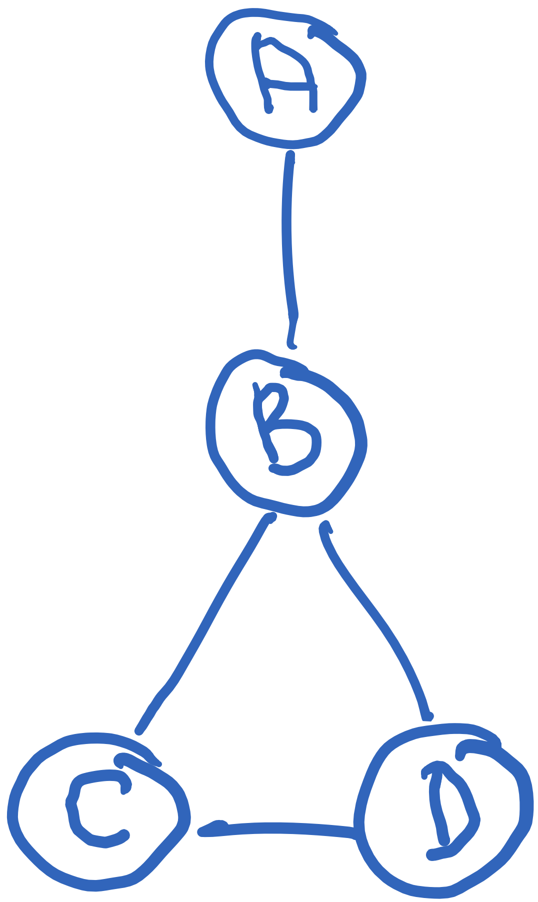
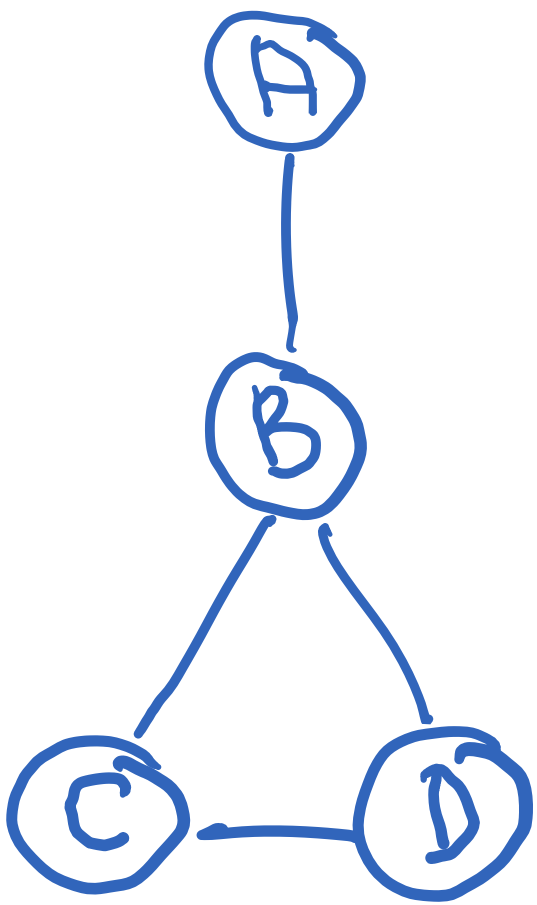
Basic Definitions

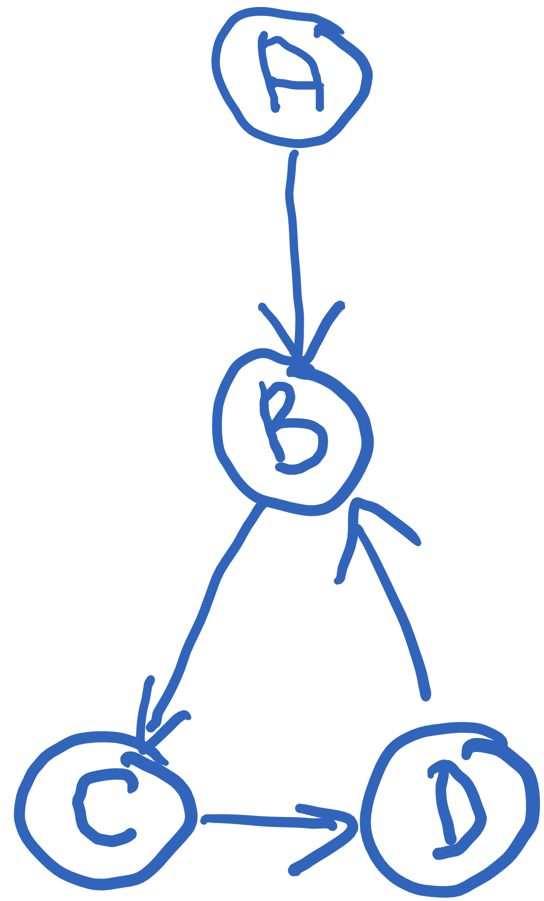
- A ***graph*** is a way of specifying relationships among a collection of items.
  - A ***graph*** consists of a set of objects called ***nodes*** (***vertices***)
  - With certain pairs of these objects connected by links called ***edges*** (***links***)
  - Two nodes are ***neighbors*** if they are connected by an ***edge***



Directed vs. Undirected



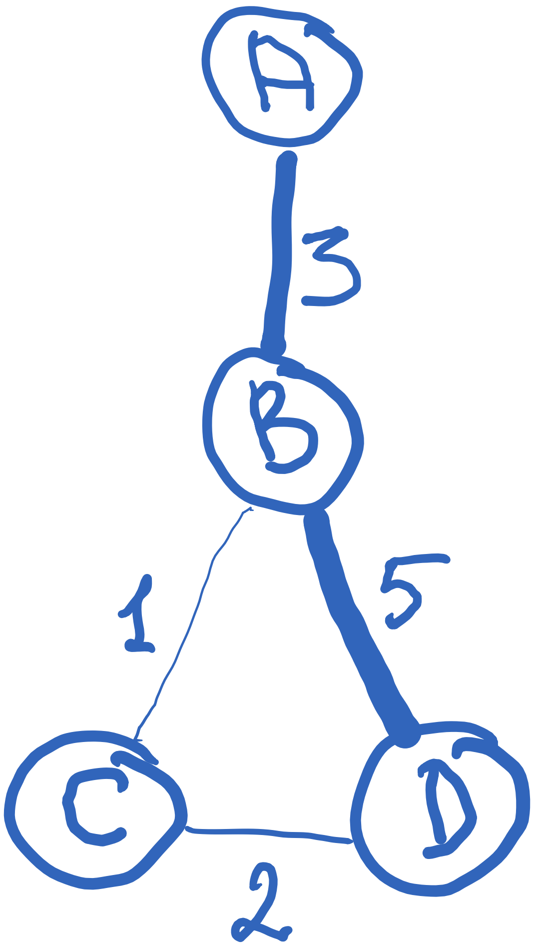
- Edges have no orientaion (e.g, Facebook)



- Edges have orientation (e.g., Twitter)

Weighted Graphs

- In a ***weighted*** graph every edge has an associated ***weight*** with it
  - What could weights represent?
  - E.g, distance, cost, frequency etc

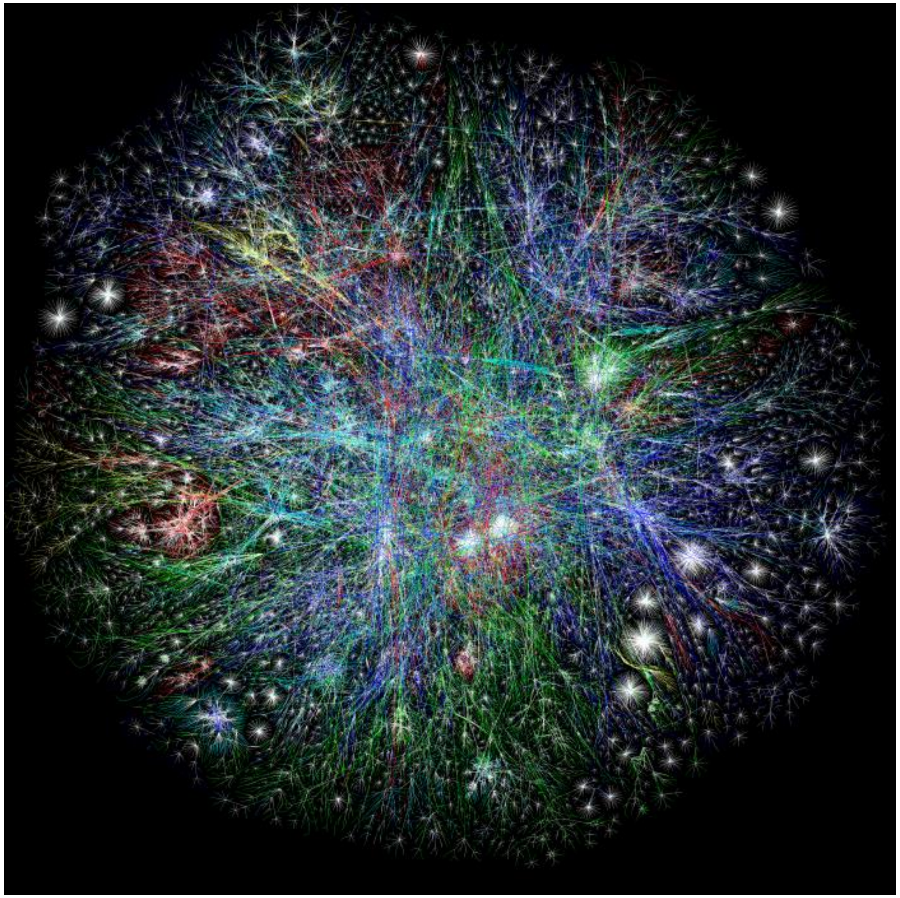


Bipartite graph

- **Bipartite graph** (or **bigraph**) is a graph whose vertices can be divided into two disjoint sets U and V and such that every edge connects a vertex in U to one in V.
  - Complete Bipartite Graph

Main Concepts

- Paths
- Cycles
- Connectivity
- (Giant) Components
- Distance



Paths

- A ***path*** in a graph is a sequence of nodes with the property that each consecutive pair in the sequence is connected by an edge
  - In a ***simple path*** nodes do not repeat
    - Sometimes can be referred to **Walks** and **Paths**

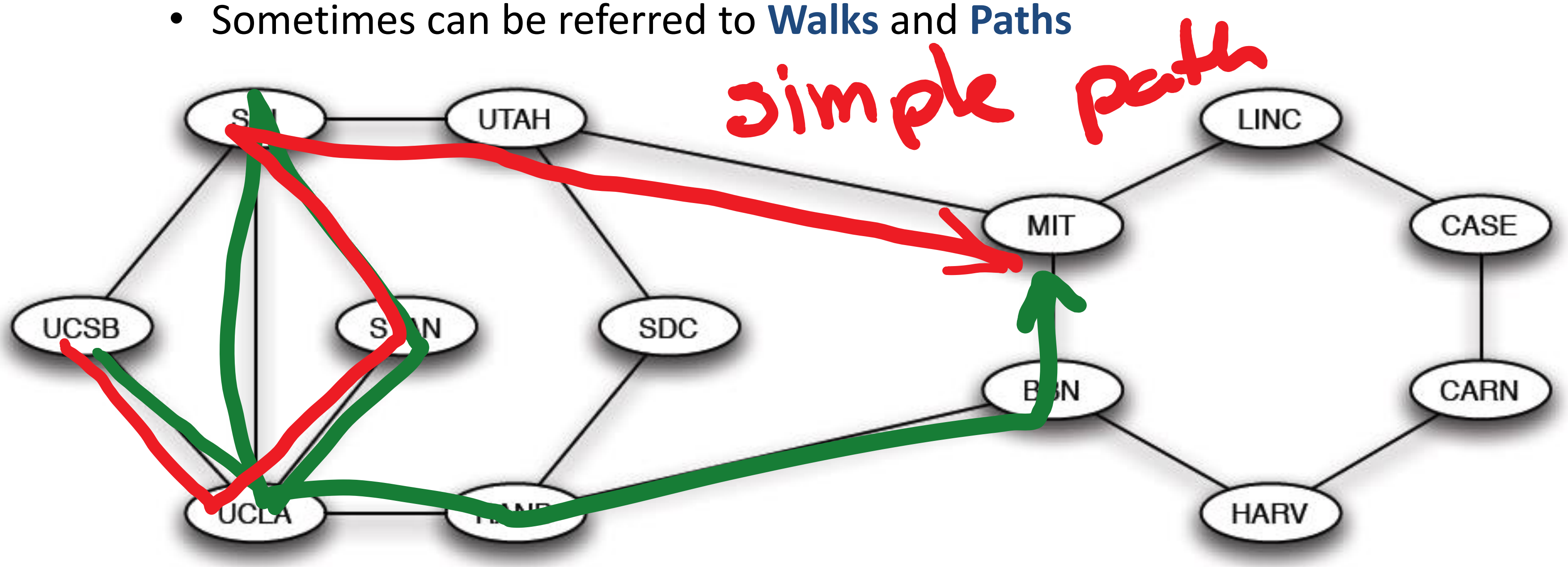
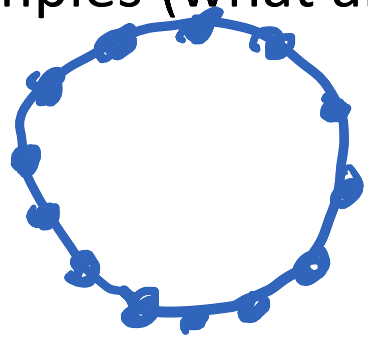
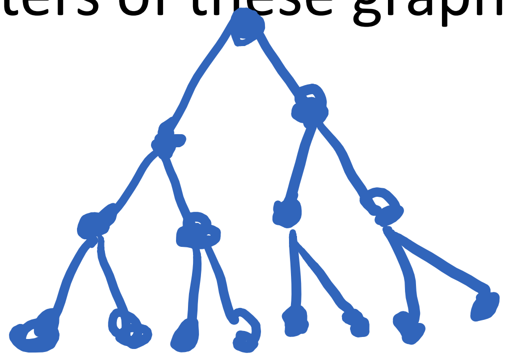


Figure 2.3: An alternate drawing of the 13-node Internet graph from December 1970.

Diameter

- Graph **diameter** can be measured as the
  - Largest shortest path
    - Any issue with that?
  - Average of shortest paths among all the nodes
    - Less sensitive to outliers
- Examples (what are the diameters of these graphs?)





- O(N) O(Log N)
- WWW: 3,1, FB: 4,74, Co-authorship graphs: 5-10
- Usually diameter is considered “small” if it is O(logN).
  - We’ll see later that it will depend on the degree of the network

Cycles

- A ***cycle*** is a closed path with at least three edges
  - All nodes are distinct except the first and the last
  - Why cycles are useful?
    - Every edge in 1970 network belongs to a cycle: design choice for making network connected even if one link failed.

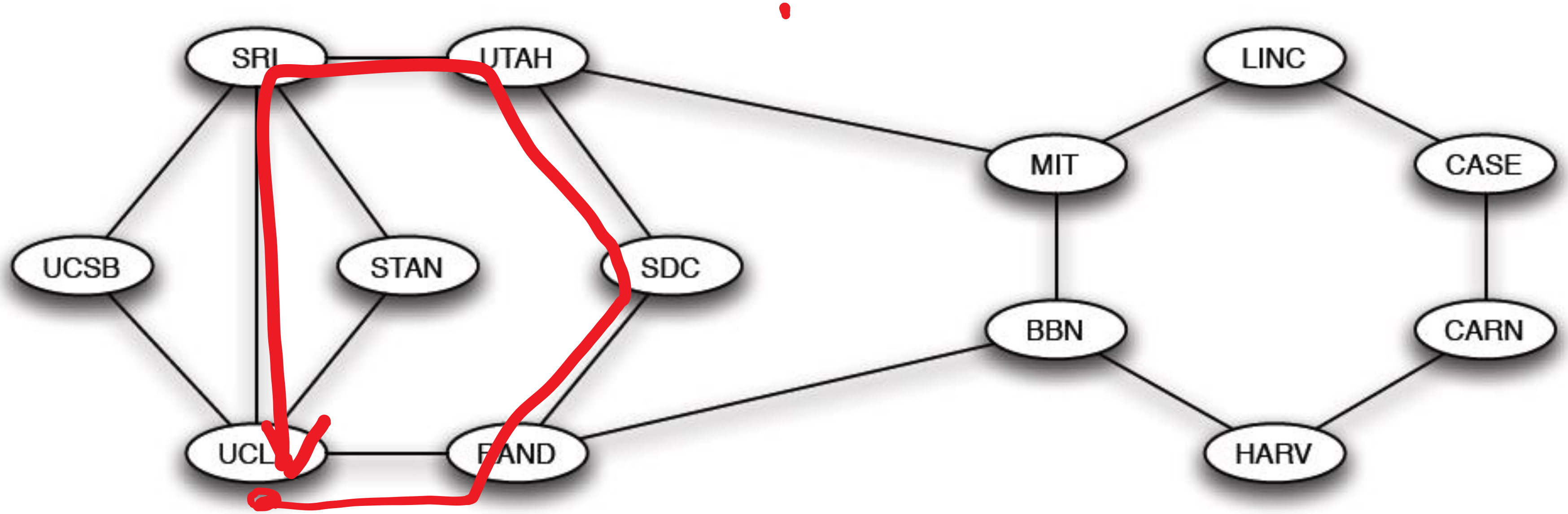
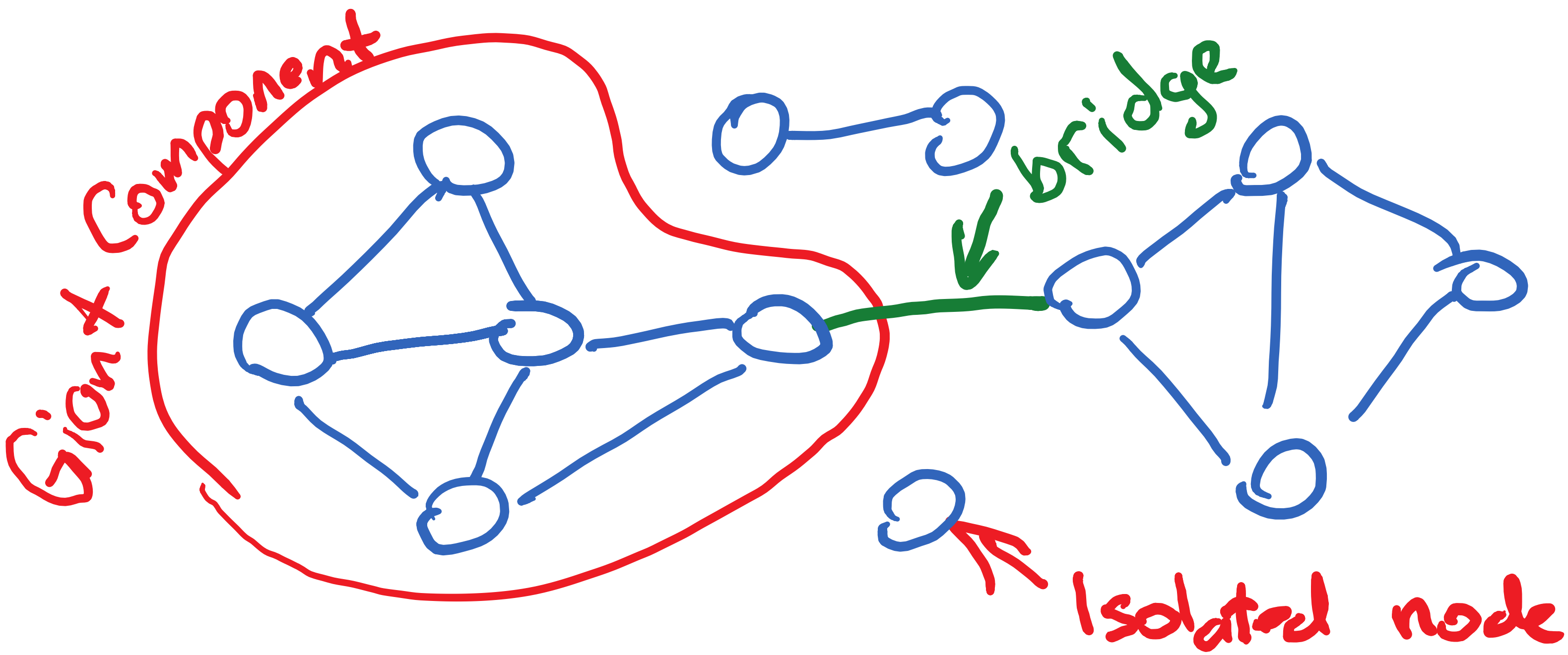


Figure 2.3: An alternate drawing of the 13-node Internet graph from December 1970.

Connectivity

- A graph is ***connected*** if for every pair of nodes there is a path between them.
- A ***disconnected*** graph is made of at least two connected sub-graphs (components)

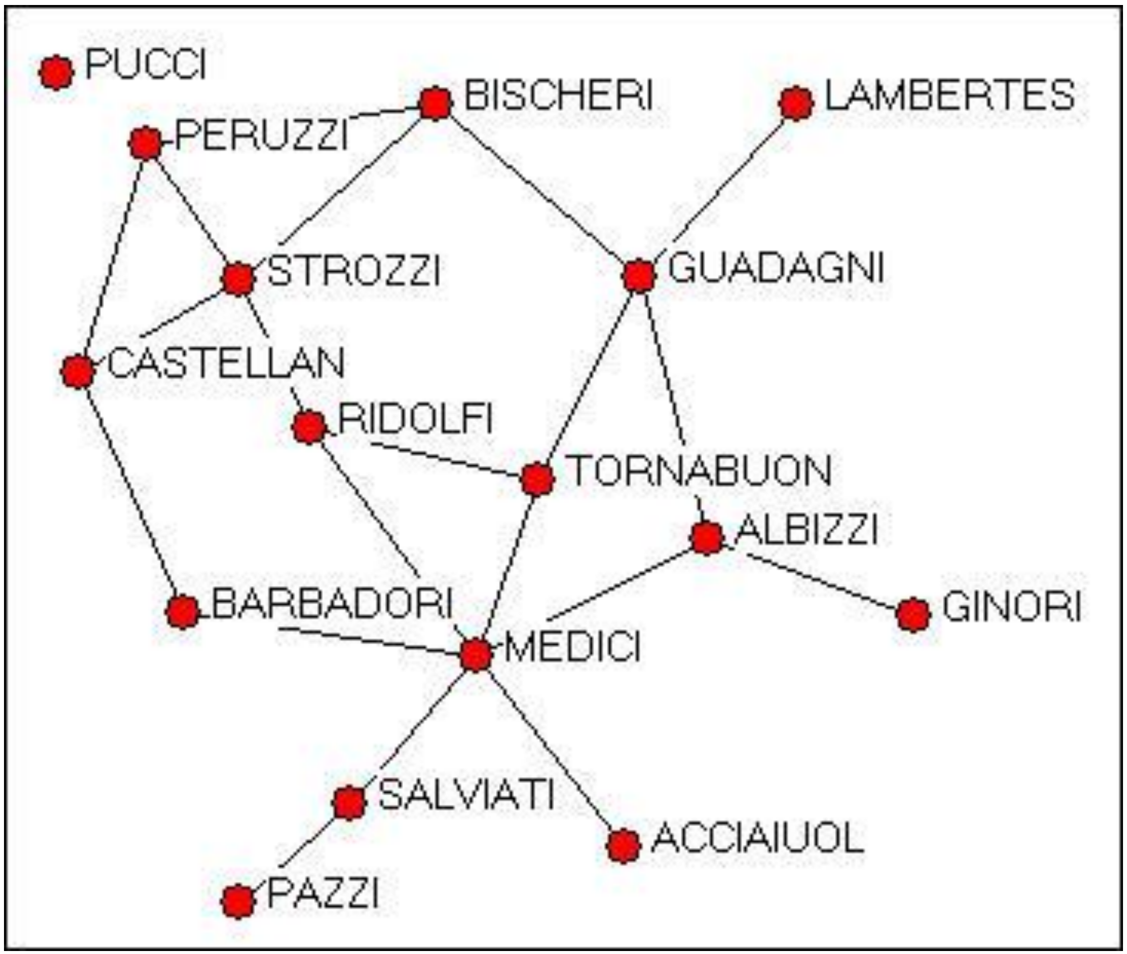


Connectivity (cont.)

- Local bridge
  - AB edge is a local bridge if A and B have no neighbors in common, but there exist another path from A to B.

- Embeddedness of the edge
  - number of mutual friends that the endpoints of the edge have in common.

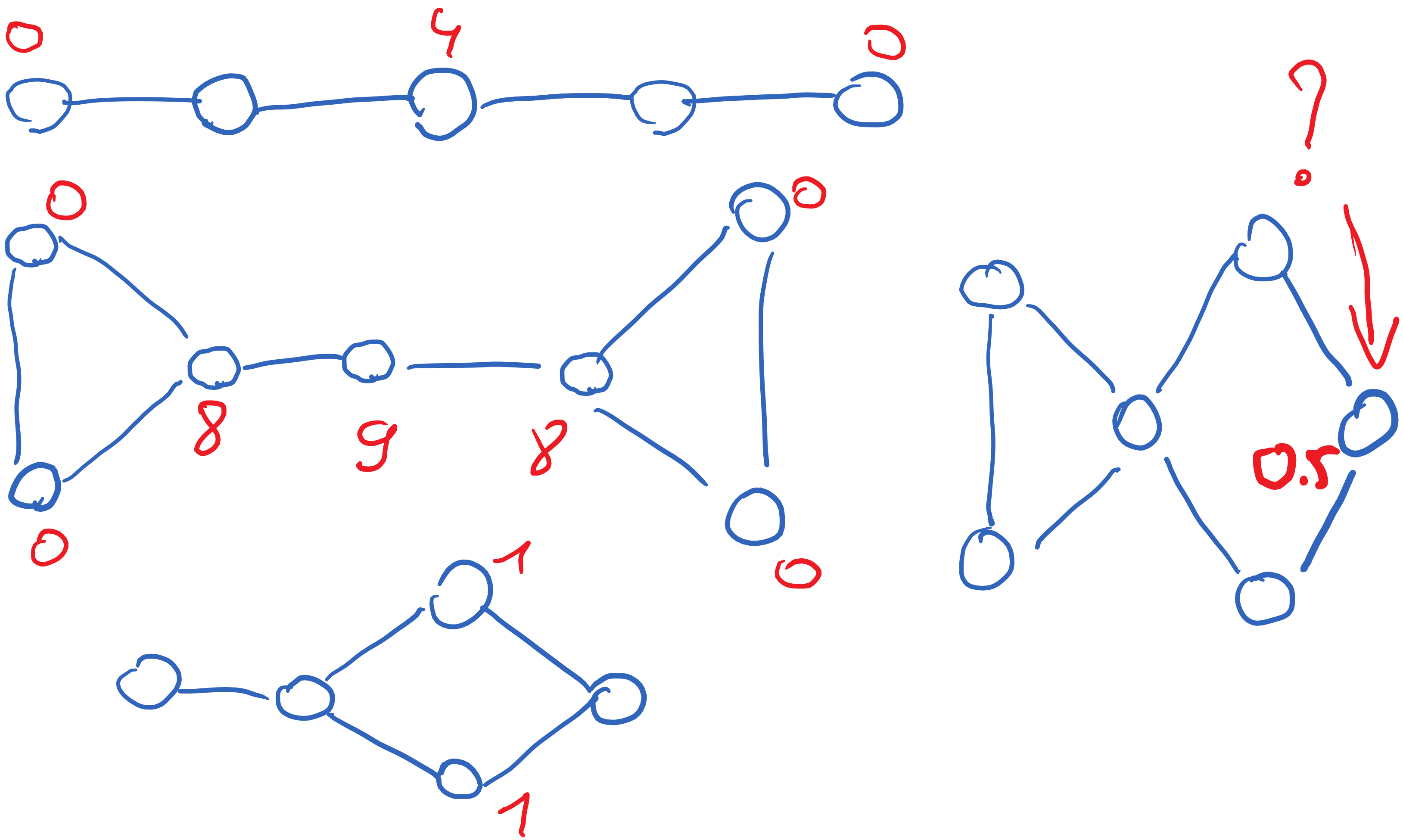
Padgett' s Florentine Families



Betweenness Centrality

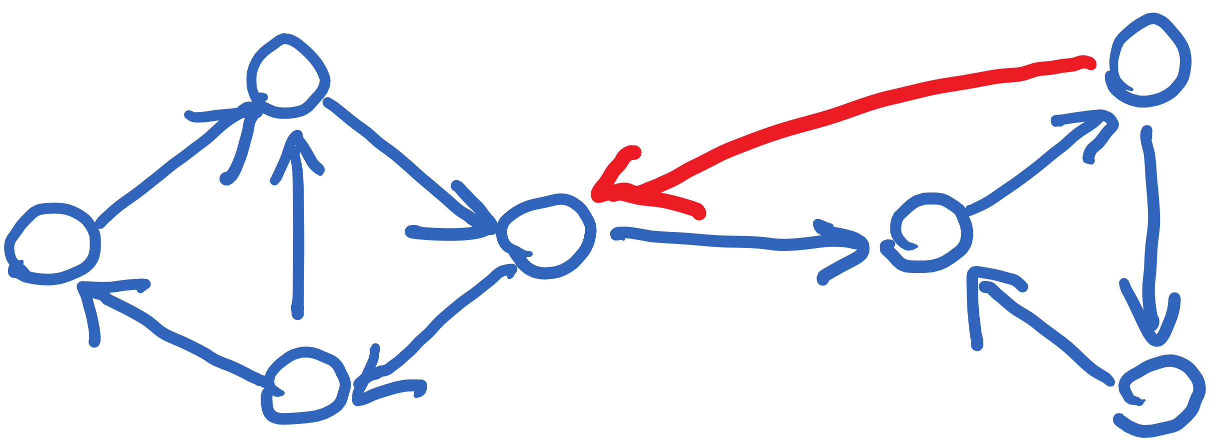
- Intuition: ***how many pairs of nodes have a shortest path through you?***
- Betweenness centrality:
  - $C_B(v) = \sum_{s \neq v \neq t \in V} \frac{\sigma_{st}(v)}{\sigma_{st}}$
  - σ_{st} number of shortest paths between s and t
  - $\sigma_{st}(v)$ number of shortest paths between s and t via v.
- Can be normalized:
  - $C'_B(v) = \frac{C_B(v)}{(n-1)(n-2)/2}$

Examples



Connectivity (directed graphs)

- A directed graph is ***strongly connected*** if for every pair of nodes there is a path between them. A ***weakly connected*** graph is connected if we disregard edge directions



Is this graph strongly connected?

Giant Component

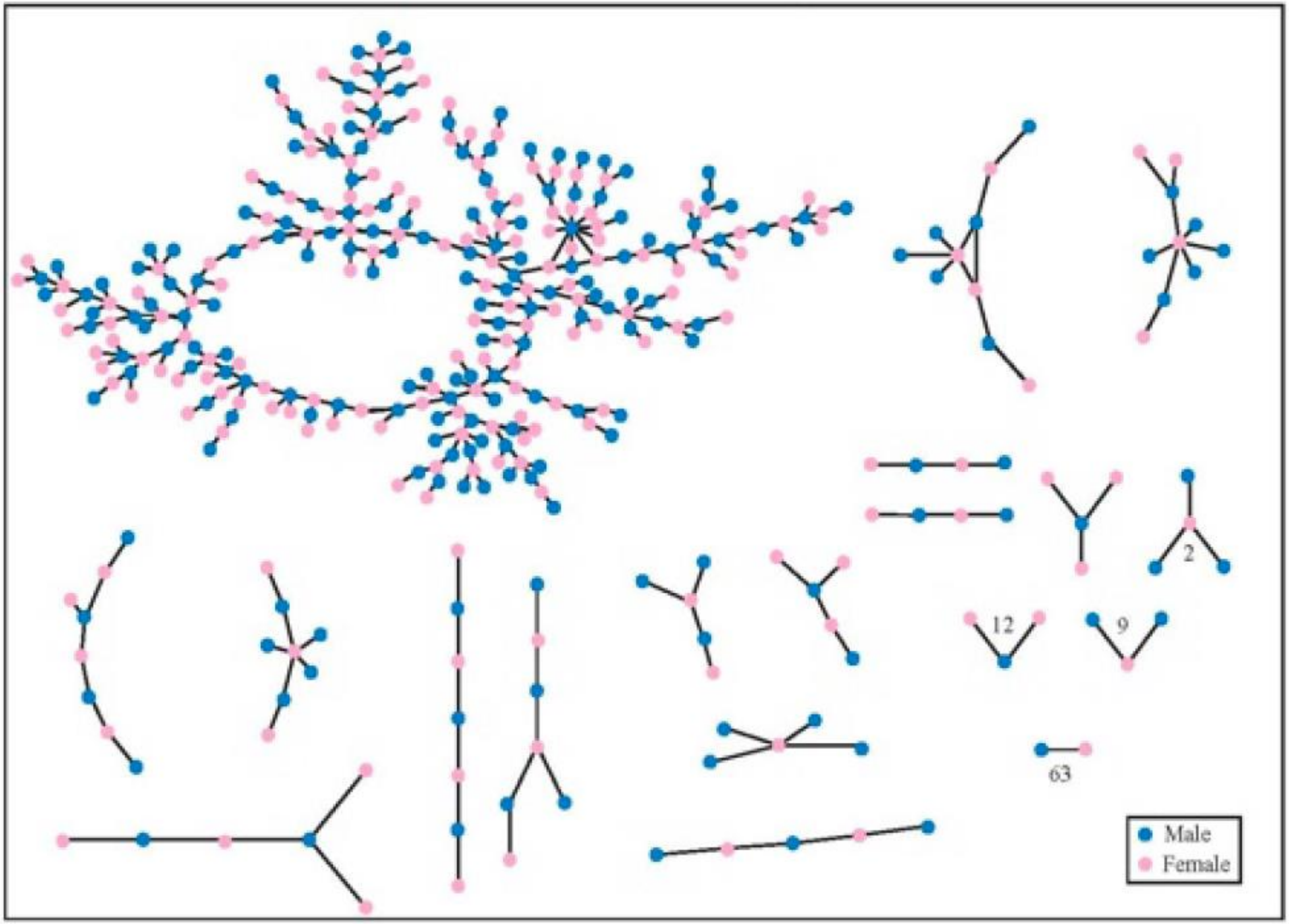
- A ***connected component*** of a graph is a subset of the nodes such that:
  - (i) every node in the subset has a path to every other;
  - and (ii) the subset is not part of some larger set with the property that every node can reach every other.
- ***Giant component***: a connected component with the largest number of nodes

Giant Components

- Real World networks often contain only a specific number of **largest** components that are **similar** in size.
  - Think what could this number be?
    - A: 1
    - B: 2
    - C: 3
    - D: 4 to 10
    - E: 11 to 99
    - F: 100 and more
- Real world networks often contain only **one giant component**

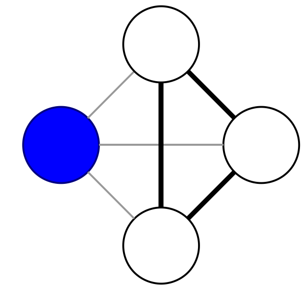
why?

Intuition on why there can’t be 2 giant components



- http://ccl.northwestern.edu/netlogo/models/GiantComponent

Clustering coefficient

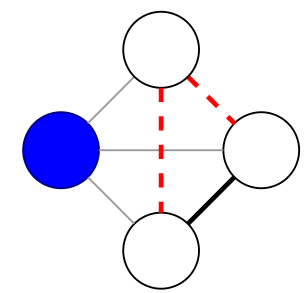


c = 1

- **Local *clustering coefficient*** *C(v)* of vertex *v* is given by the fraction of:

$$C(v) = \frac{e(v)}{deg(v)\left(deg(v)-1\right)/2}$$

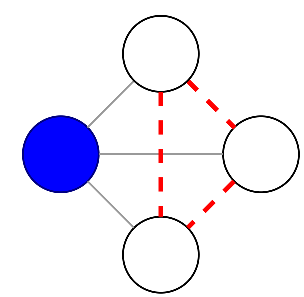
where *e(v)* denotes the links between the vertices within the neighborhood of *v*



c = 1/3

- **Network average *clustering coefficient*** $\widetilde{C}$ is given by the fraction of:

$$\widetilde{C} = \frac{1}{N}\sum_{i=1}^{N} C(i)$$



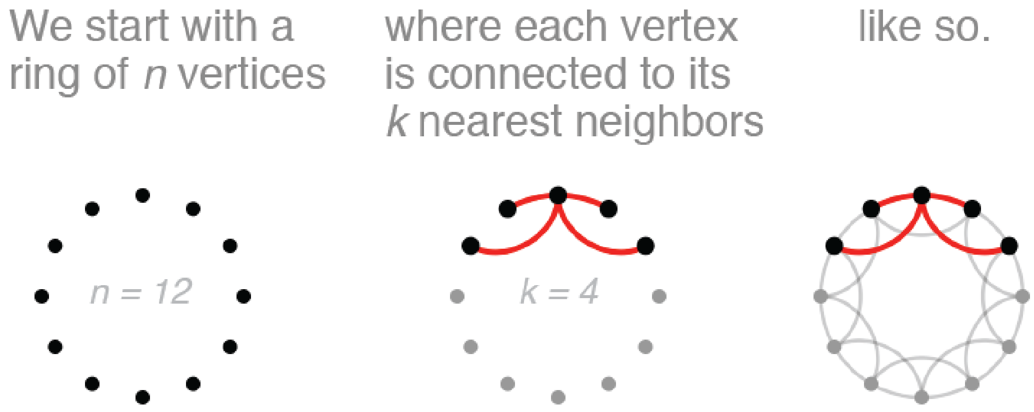
c = 0

How to interpret clustering coef.?

- Clustering coefficient denotes what is the fraction of your neighbors are neighbors themselves
- Compare to a purely random chance that the "triangles" form.
- Edge density of a network: $$\mathrm{p} = \frac{E}{0{,}5 * N(N-1)}$$
  - E is total number of edges
  - P is the probability that two nodes are connected in a random graph
- If C(G)>>p then we can claim that the **graph is clustered**

Examples

- Regular graph with degree k connected to nearest neighbors



- What's clustering coefficient when k=4?
  - Possible neighbor friendships: 6
  - Actual friendships: 3
- Clustering coef 3/6=0,5
- Compare it with random graph?

- Assume we have a graph with N=1bn, and avg degree of 100. We measure avg clustering coefficient and find it to be C(G)=0.0001.
- Can we call this graph clustered?