

7.1. ACF $r_y(k)$ for ARMA(1,1) expressed in the parameters σ^2, a_1, b_1 . $E[X(n)] = \sigma^2$

$$Y(n) + a_1 Y(n-1) = X(n) + b_1 X(n-1)$$

Can be seen as a linear system

$$Y(\omega) + a_1 e^{-j\omega} Y(\omega) = X(\omega) + b_1 e^{-j\omega} X(\omega) \quad | \quad \omega = 2\pi\nu$$

$$Y(\omega)(1 + a_1 e^{-j\omega}) = X(\omega)(1 + b_1 e^{-j\omega})$$

$$H(\omega) = \frac{1 + b_1 e^{-j\omega}}{1 + a_1 e^{-j\omega}}$$

$$|H(\omega)|^2 = \frac{1 + b_1^2 + 2b_1 \cos(2\pi\nu)}{1 + a_1^2 + 2a_1 \cos(2\pi\nu)} = \frac{A}{1 + a_1^2 + 2a_1 \cos(2\pi\nu)} + B$$

$$B(1 + a_1^2 + 2a_1 \cos(2\pi\nu)) + A = 1 + b_1^2 + 2b_1 \cos(2\pi\nu)$$

$$B(1 + a_1^2) + A = 1 + b_1^2$$

$$2a_1 B \cos(2\pi\nu) = 2b_1 \cos(2\pi\nu) \Rightarrow B = \frac{b_1}{a_1}$$

$$\rightarrow A = 1 + b_1^2 - B(1 + a_1^2) = 1 + b_1^2 - \frac{b_1}{a_1}(1 + a_1^2)$$

$$= \frac{b_1}{a_1} + \frac{1 + b_1^2 - \frac{b_1}{a_1}(1 + a_1^2)}{1 + a_1^2 + 2a_1 \cos(2\pi\nu)} = \frac{b_1}{a_1} +$$

$$+ \underbrace{\frac{1 + b_1^2 - \frac{b_1}{a_1}(1 + a_1^2)}{1 - a_1^2}}_d \cdot \frac{1 - a_1^2}{1 + a_1^2 + 2a_1 \cos(2\pi\nu)} =$$

$$= \frac{b_1}{a_1} + d \cdot \frac{1 - a_1^2}{1 + a_1^2 + 2a_1 \cos(2\pi\nu)}$$

$$d = \sigma_x^2 \cdot d$$

$$R_y(\nu) = |H(\nu)|^2 \underbrace{R_x(\nu)}_{\sigma_x^2} = R_y(\nu) = \sigma_x^2 \frac{b_1}{a_1} + c \frac{1 - a_1^2}{1 + a_1^2 + 2a_1 \cos(2\pi\nu)}$$

$$r_y(k) = \sigma_x^2 \delta(k) \frac{b_1}{a_1} + c (-a_1)^{|k|}$$

7.5. $X(n)$ sinusoidal $S(n)$ and noise $V(n)$ $X(n) = S(n) + V(n)$

ACF for $X(n)$ is

$$r_x(k) = \sigma_s^2 \cos(2\pi \nu_s k) + \sigma_v^2 \delta(k)$$

a_1, a_2 for AR(2) adapted to $X(n)$

$$Y(n) = -a_1 Y(n-1) - a_2 Y(n-2) + X(n)$$

$$Y^2(n) = -a_1 Y(n-1)Y(n) - a_2 Y(n-2)Y(n) + X(n)Y(n)$$

$$E[Y^2(n)] = -a_1 E[Y(n-1)Y(n)] - a_2 E[Y(n-2)Y(n)] + E[X(n)Y(n)]$$

$$r_y(0) = -a_1 r_y(1) - a_2 r_y(2) + E[X(n)(-a_1 Y(n-1) - a_2 Y(n-2) + X(n))]$$

$$r_y(0) = -a_1 r_y(1) - a_2 r_y(2) + \sigma_n^2$$

independent of!

$$Y(n)Y(n-1) = -a_1 Y^2(n-1) - a_2 Y(n-1)Y(n-2) + X(n)Y(n-1)$$

$$E[Y(n)Y(n-1)] = -a_1 E[Y^2(n-1)] - a_2 E[Y(n-1)Y(n-2)] + E[X(n)Y(n-1)]$$

$$r_y(1) = -a_1 r_y(0) - a_2 r_y(1)$$

indep
and $X(n)$
 $\emptyset$ mean

$$Y(n)Y(n-2) = -a_1 Y(n-1)Y(n-2) - a_2 Y^2(n-2) + X(n)Y(n-2)$$

$$E[Y(n)Y(n-2)] = -a_1 E[Y(n-1)Y(n-2)] - a_2 E[Y^2(n-2)] + E[X(n)Y(n-2)]$$

$$r_y(2) = -a_1 r_y(1) - a_2 r_y(0)$$

indep and
 $X(n)$ is $\emptyset$
mean

We have the equations

$$\begin{pmatrix} r_y(0) & r_y(1) & r_y(2) \\ r_y(1) & r_y(0) & r_y(1) \\ r_y(2) & r_y(1) & r_y(0) \end{pmatrix} \begin{pmatrix} 1 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \sigma^2 \\ 0 \\ 0 \end{pmatrix} \quad a_1? \quad a_2? \quad \sigma^2?$$

The process $X(n)$ has ACF: $r_x(k) = \sigma_s^2 \cos(2\pi \nu_s k) + \sigma_v^2 \delta(k)$

$$r_x(0) = \sigma_s^2 \cos(0) + \sigma_v^2 = \sigma_s^2 + \sigma_v^2$$

$$r_x(1) = \sigma_s^2 \cos(2\pi \nu_s)$$

$$r_x(2) = \sigma_s^2 \cos(4\pi \nu_s)$$

plug in the matrix
and solve the system,
numerical solution
in the solutions

$$\frac{\nu}{\nu_s} \rightarrow 0 \quad \nu \rightarrow 0 \quad \begin{matrix} a_1 = -2\cos(2\pi\nu_s) \\ a_2 = 1 \end{matrix} \left. \begin{matrix} \text{solution to} \\ \text{finding limits at the} \\ \text{end} \end{matrix} \right\}$$

What happens to the poles of the AR model?

$$Y(n) = -a_1 Y(n-1) - a_2 Y(n-2) + X(n)$$

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) + X(z)$$

$$Y(z) (1 + a_1 z^{-1} + a_2 z^{-2}) = X(z)$$

$$H(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{z^2}{z^2 + a_1 z + a_2}$$

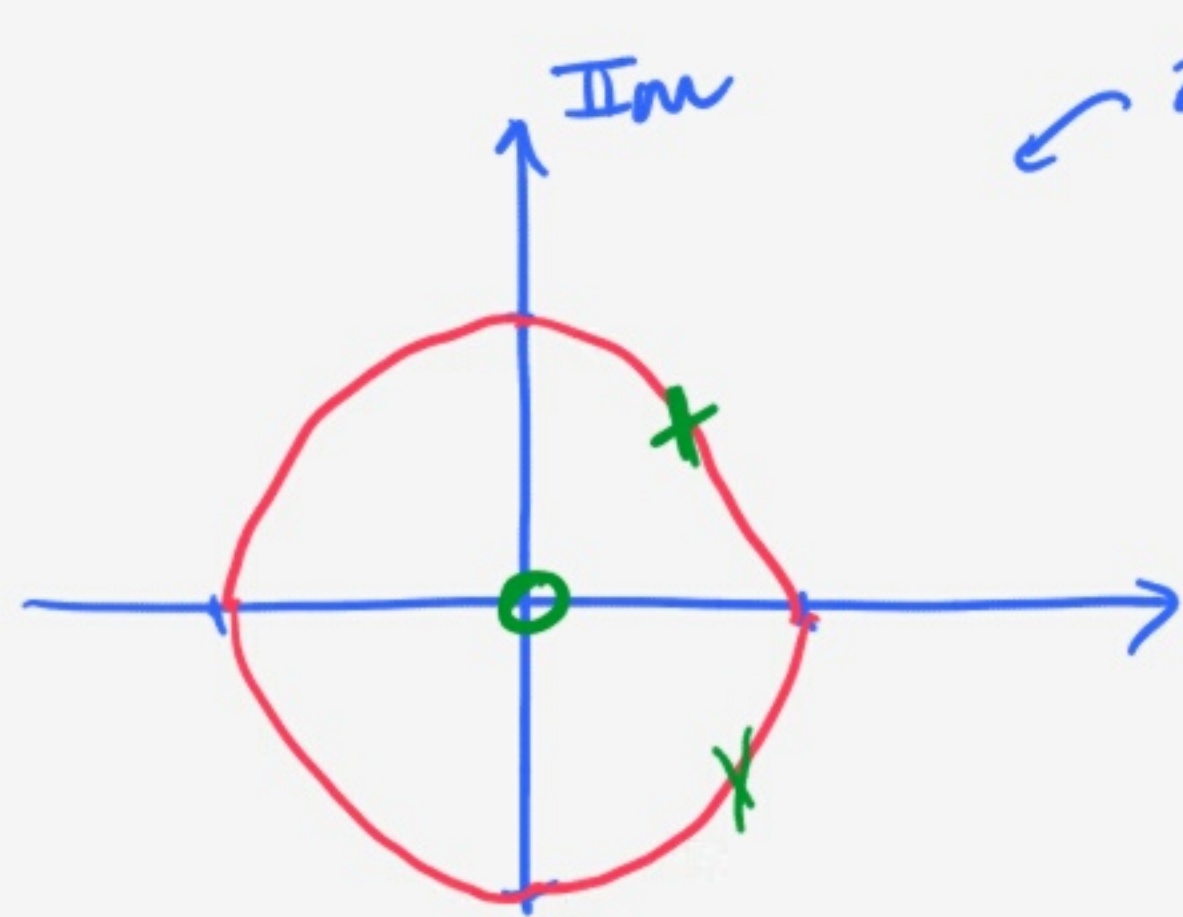
$$\text{poles } z^2 + a_1 z + a_2 = 0 \quad z = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$\frac{\nu}{\nu_s} \rightarrow 0 \quad \begin{matrix} a_1 = -2\cos(2\pi\nu_s) \\ a_2 = 1 \end{matrix}$$

$$z = \frac{2\cos(2\pi\nu_s) \pm \sqrt{4\cos^2(2\pi\nu_s) - 4}}{2} =$$

$$z = \cos(2\pi\nu_s) \pm \sqrt{-(1 - \cos^2(2\pi\nu_s))} =$$

$$= \cos(2\pi\nu_s) \pm (1-j) \sin(2\pi\nu_s) = \cos(2\pi\nu_s) \pm j \sin(2\pi\nu_s)$$



z-plane

unitary circle

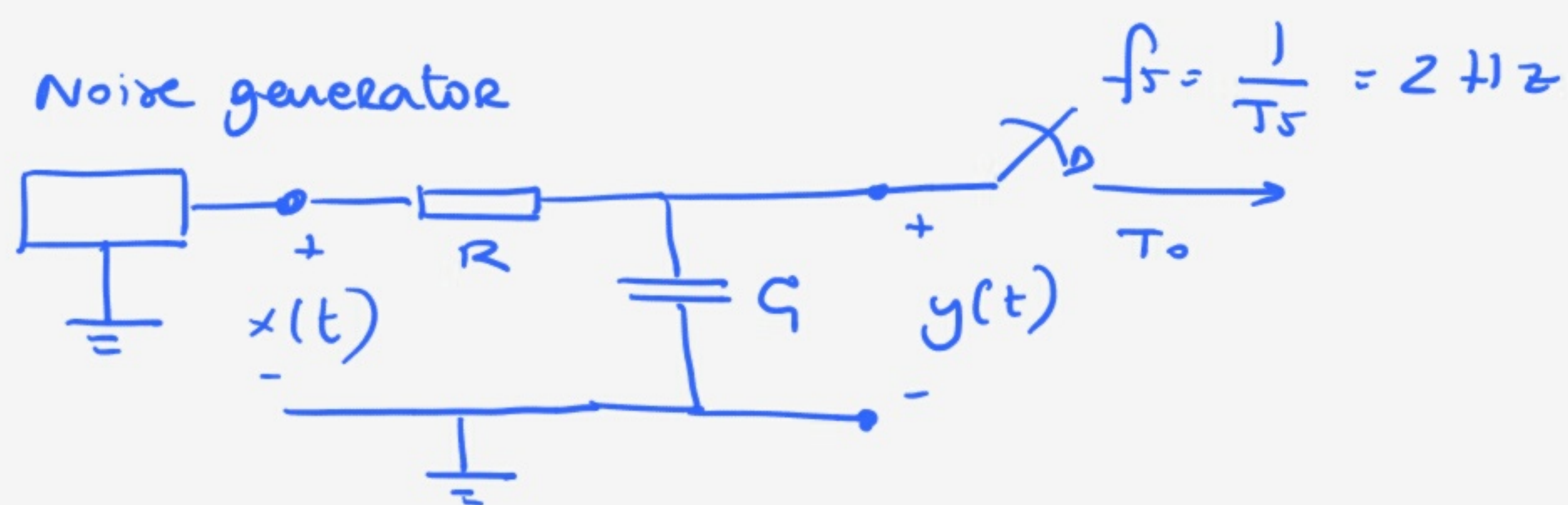
○ zero (double in this case)

x pole

a pole is a point where $H(z)$ goes to infinity! Hence we will only have

a system that outputs a sinusoid of frequency ν_s ! $\Rightarrow$ we have an oscillator at frequency ν_s .

2.7.



$$y(n) + a y(n-1) = x(n) \quad E[x^2(n)] = \sigma^2$$

$$\left. \frac{dy(t)}{dt} \right|_{t=kT_s} \approx \frac{y(kT_s) - y((k-1)T_s)}{T_s} \quad (\text{1st order approximation of derivative})$$

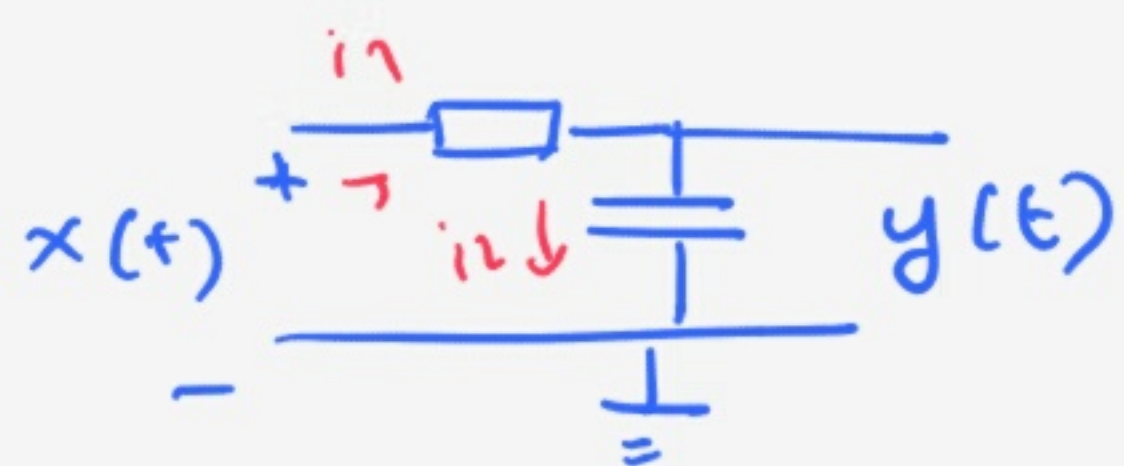
$$\hat{a} = \left[\frac{1}{N} \sum_{n=0}^{N-1} y(n)^2 \right]^{-1} \frac{1}{N} \sum_{n=0}^{N-1} y(n)y(n-1) \approx -0.5$$

$\omega_c = 1/RC$?

$$y(n) - 0.5 y(n-1) = x(n)$$

We try to identify the coefficients of the RC filter with the AR model

An RC circuit



$$i_C(t) = C \frac{dV_C(t)}{dt}$$

Current over a capacitor : $i_2(t) = C \frac{dy(t)}{dt}$

$$i_1(t) = i_2(t) \quad \frac{x(t) - y(t)}{R} = C \frac{dy(t)}{dt}$$

$$\left. \frac{dy(t)}{dt} \right|_{t=kT_s} \approx \frac{y(kT_s) - y((k-1)T_s)}{T_s}$$

$$\frac{x(k) - y(k)}{R} = \frac{C}{T_s} (y(k) - y(k-1)), \quad x(k) - y(k) = \frac{RC}{T_s} (y(k) - y(k-1))$$

$$x(k) = y(k) \left(1 + \frac{RC}{T_s} \right) - \frac{RC}{T_s} y(k-1)$$

$$y(n) - \frac{\frac{RC}{T_s}}{\left(1 + \frac{RC}{T_s}\right)} y(n-1) = \frac{x(n)}{\left(1 + \frac{RC}{T_s}\right)} ;$$

$$y(n) - \frac{RC}{T_s + RC} y(n-1) = \frac{T_s x(n)}{T_s + RC}$$

$$a = \frac{-RC}{T_s + RC} = \left\{ T_s = \frac{1}{2} \right\} = \frac{-RC}{\frac{1}{2} + RC} = -0.5$$

$$\frac{RC}{0.5 + RC} = 0.5 \quad RC = 0.25 + 0.5RC$$

$$0.5RC = 0.25 \Rightarrow RC = \frac{1}{2}$$

$$a_1 = - \frac{\sigma_s^2 \cos(2\pi\nu_s) (\sigma_s^2 + \sigma_v^2 - \sigma_s^2 \cos(4\pi\nu_s))}{(\sigma_s^2 + \sigma_v^2)^2 - \sigma_s^4 \cos^2(2\pi\nu_s)}$$

$$\lim_{\sigma_v \rightarrow 0} a_1 = - \frac{\cancel{\sigma_s^4} \cos(2\pi\nu_s) (1 - \cos(4\pi\nu_s))}{\cancel{\sigma_s^4} (1 - \cos^2(2\pi\nu_s))} = \frac{\cos(4\pi\nu_s) = \cos^2(2\pi\nu_s) - \sin^2(2\pi\nu_s)}{-\cos(2\pi\nu_s) (1 - \cos^2(2\pi\nu_s) + \sin^2(2\pi\nu_s))} = \frac{1 - \cos^2(2\pi\nu_s) = \sin^2(2\pi\nu_s)}{-\cos(2\pi\nu_s) (2\sin^2(2\pi\nu_s))} = -2\cos(2\pi\nu_s)$$

$$a_2 = - \left(\frac{(\sigma_s^2 + \sigma_v^2) \sigma_s^2 \cos(4\pi\nu_s) - \sigma_s^4 \cos^2(2\pi\nu_s)}{(\sigma_s^2 + \sigma_v^2)^2 - \sigma_s^4 \cos^2(2\pi\nu_s)} \right)$$

$$\lim_{\sigma_v \rightarrow 0} a_2 = - \frac{\cancel{\sigma_s^4} (\cos(4\pi\nu_s) - \cos^2(2\pi\nu_s))}{\cancel{\sigma_s^4} (1 - \cos^2(2\pi\nu_s))} =$$

$$= \frac{-2\cos^2(2\pi\nu_s) + 1 + \cos^2(2\pi\nu_s)}{1 - \cos^2(2\pi\nu_s)} = 1$$

$$1 = \cos^2(x) + \sin^2(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$1 + \cos(2x) = 2\cos^2(x)$$

$$\cos(2x) = 2\cos^2(x) - 1$$