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- **Part 4**
  - Lec 12: Wireless Sensor Network Control Systems 1
Previous lecture

How to estimate phenomena from noisy measurements?
Today’s learning goals

- Which measurements are used for estimating the position of a node?
- How to estimate the position of a node?
- What is the effect of measurement errors?
Outline

- Introduction
- Specific sources of measurements
- Estimation of the position
Outline

- Introduction
- Specific sources of measurements
  - Time of arrival
  - Time difference of arrival
  - Received signal strength
  - Angle of Arrival
- Estimation of the position
  - Angle of arrival + velocity
  - Triangulation
  - Trilateration
  - Iterative and collaborative multilateration
Estimation

Centralized Estimation:
no intelligence on sensors

Distributed Estimation:
some intelligence on sensors

Completely distributed
(or decentralized) Estimation
Positioning and localization

Localization is defined as a technique to estimate the positions of nodes

- It can be categorized into
  - Centralized, where a central node estimates the position of the nodes
  - Distributed, where many nodes help each other to find their own positions
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Assuming that the nodes are synchronized the distance measurement is 

\[ d_{ij} \approx (t_2 - t_1) \cdot v \]

where \( v \) is the propagation speed of the message.

Uncertainty occurs when \( t_2 \approx t_1 \).

Problems
- Packet losses
- MAC delays
- CPU delay
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The distance measurement is

\[ d_{ij} \approx \frac{(t_2 - t_1) + (t_4 - t_3)}{2} \cdot v \]

where \( v \) is the propagation speed of the message

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Received signal strength (RSS)

Ideal propagation

\[ P_r = P_t G_t G_r \frac{\lambda^2}{(4\pi d_{ij})^2} \Rightarrow d_{ij} \approx \frac{\lambda}{4\pi} \sqrt{\frac{P_t G_t G_r \overline{PL}}{P_r}} \]

Problems: multi-path, noise, etc, affect the propagation characteristics
⇒ Channel estimation needed
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Angle of arrival

Example: Sound propagation by multiple microphones, amplitude and phase

- Odometry sensor

Problems: Errors in measurements of $\theta$ (e.g., due to magnetic fields of earth), estimation accuracy decreases with the distance.
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Estimation of the position

We now examine the specific techniques that are used to estimate the sensors’ position based on the measurements that were previously presented.
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Angle of arrival + velocity

\[ \tilde{\theta}(k) = \theta(k) + n_\theta(k) \]

- \( \tilde{\theta}(k) \) the angle measurement at discrete time \( k \) given by, e.g., odometry sensor
- \( \theta(k) \) the real angle value
- \( n_\theta(k) \) the additive noise

\[ \tilde{v}(k) = v(k) + n_v(k) \]

- \( \tilde{v}(k) \) the velocity measurement at discrete time \( k \) given by, e.g., accelerometer
- \( v(k) \) the real velocity value
- \( n_v(k) \) the additive noise
Angle of arrival + velocity

Node moving with velocity $v$

Let $X_r (k) = \begin{bmatrix} x(\k) \\ y(\k) \end{bmatrix}$ be the true position of the node that we want to estimate.

Then

\[
\hat{x}(\k+1) = \hat{x}(\k) + \tilde{v}(\k) \cdot T \cdot \cos \tilde{\theta}(\k)
\]

\[
\hat{y}(\k+1) = \hat{y}(\k) + \tilde{v}(\k) \cdot T \cdot \sin \tilde{\theta}(\k)
\]

where $T$ is the sampling time.

A problem is that the resulting estimator is **biased**

\[
\mathbb{E}\{ \hat{x}(\k+1) \} = \mathbb{E}\{ \hat{x}(\k) \} + \mathbb{E}\{ \tilde{v}(\k) \} \cdot T \cdot \mathbb{E}\{ \cos \tilde{\theta}(\k) \} =
\]

\[
= \mathbb{E}\{ \hat{x}(\k) \} + v(\k) \cdot T \cdot \cos \theta(\k) \cdot e^{-\frac{\sigma^2_{\tilde{\theta}}}{2}} \neq x(\k+1)
\]

We need to estimate the bias as well.
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Triangulation

- Anchors: nodes with fixed position used for determining the unknown position of a node

Let \( X_r = \begin{bmatrix} x_r \\ y_r \end{bmatrix} \) be the true position of the node that we want to estimate.

The angle measurements are

\[
\tilde{\theta}_i = \theta_i (X_r) + n_i \quad i = 1, 2, 3
\]

where \( \theta_i (X_r) = \arctan \frac{x_r - x_i}{y_r - y_i} \)
Triangulation

Setting

- \( \mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \) as the vector of noises with \( \mathbb{E} \{ \mathbf{n} \cdot \mathbf{n}^T \} = \mathbf{R} = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix} \)

- \( \mathbf{\theta}(\mathbf{X}_r) = \begin{bmatrix} \theta_1(\mathbf{X}_r) \\ \theta_2(\mathbf{X}_r) \\ \theta_3(\mathbf{X}_r) \end{bmatrix} \) as the vector of true node position and

- \( \mathbf{Y} = \begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\theta}_3 \end{bmatrix} \) as the vector of angle measurements

we can estimate \( \mathbf{X}_r \) from

\[
\mathbf{Y} = \mathbf{\theta}(\mathbf{X}_r) + \mathbf{n}
\]

where \( \mathbf{\theta}(\mathbf{X}_r) \) is a non-linear function
Triangulation

Error covariance (cf. slide 19 in Lecture 8)

\[
C\left(\hat{X}_r\right) = \left[\theta\left(\hat{X}_r\right) - Y\right]^T \cdot R^{-1} \cdot \left[\theta\left(\hat{X}_r\right) - Y\right] = \sum_{i=1}^{3} \frac{\left(\theta_i\left(\hat{X}_r\right) - \tilde{\theta}_i\right)^2}{\sigma_i^2}
\]

Goal: Choose \(\hat{X}_r\) that keeps oscillations of \(\theta\left(\hat{X}_r\right)\) around \(Y\)

Therefore

\[
\min_{\hat{X}_r} \quad C\left(\hat{X}_r\right) \equiv \begin{cases} 
\frac{dC\left(\hat{X}_r\right)}{dx_r} = 0 \\
\frac{dC\left(\hat{X}_r\right)}{dy_r} = 0 
\end{cases}
\]
Triangulation

We can use iterative methods for solving the above system of non-linear equations, e.g.,

### Newton-Gauss method

Consider the non-linear equation

\[ g(x) = 0 \quad g : \mathbb{R}^N \to \mathbb{R}^N \quad x \subset \mathbb{R}^N \]

Iterative method

\[ x_{t+1} = x_t - \delta_t \cdot g(x_t) \]

where

\[ \delta_t = \nabla^{-1} g(x_t) \]

\[ \lim_{t \to \infty} x_t \to x^* : g(x^*) = 0 \]

\( \nabla \) is the matrix of all first-order partial derivatives (Jacobian)

In this specific case, we use the iterative method to solve

\[ g(x) = \nabla_x C(\hat{X}_r) = 0 \]
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Trilateration

Let \( X_r = \begin{bmatrix} x_r \\ y_r \end{bmatrix} \) be the true position of the node that we want to estimate.

The distance measurements are

\[
\tilde{d}_i = d_i + n_i \quad i = 1, \ldots, N
\]

From the trigonometry

\[
\begin{align*}
(x_1 - x_r)^2 + (y_1 - y_r)^2 &= \tilde{d}_1^2 \\
&
\vdots \\
(x_N - x_r)^2 + (y_N - y_r)^2 &= \tilde{d}_N^2
\end{align*}
\]
After substracting these $N$ equations we arrive at the following system of equations

$$A \cdot X_r = Y$$

where

$$A \in \mathbb{R}^{(N-1) \times 2}, \quad X_r \in \mathbb{R}^2, \quad Y \in \mathbb{R}^{(N-1) \times 1}$$

$$A = 2 \cdot \begin{bmatrix}
(x_N - x_1) & (y_N - y_1) \\
\vdots & \vdots \\
(x_N - x_{N-1}) & (y_N - y_{N-1})
\end{bmatrix}$$

$$Y = \begin{bmatrix}
\tilde{d}_1^2 - \tilde{d}_N^2 - x_1^2 - y_1^2 + x_N^2 + y_N^2 \\
\vdots \\
\tilde{d}_{N-1}^2 - \tilde{d}_N^2 - x_{N-1}^2 - y_{N-1}^2 + x_N^2 + y_N^2
\end{bmatrix}$$
Trilateration

From the previous lecture we considered a measurements model $Y = HX + n$, but we now have

$$Y = AX_r$$

$A$ is not a square matrix and there is no explicit noise (all is included in $Y$). $X_r$ is constant.

We can apply the Linear Minimum Mean Squared Estimator (LMMSE) with

$$\hat{X}_r = L \cdot Y$$

where

$$L = \left(A^T A\right)^{-1} A^T$$
Trilateration

We can redefine \( \hat{X}_r \) as a \textbf{least square} solution.

This is equivalent to define the following cost function

\[
C(X_r) = (AX_r - Y)^T (AX_r - Y)
\]

and search for \( X_r \) that minimizes \( C(X_r) \)

\[
\frac{dC(X_r)}{dX_r} = 2A^T (A\hat{X}_r - Y) = 0 \Rightarrow A^T A\hat{X}_r = A^T Y \Rightarrow \hat{X}_r = (A^T A)^{-1} A^T Y
\]
Trilateration

Note that $\hat{X}_r$ is a random variable due to $Y$ being noisy

Taking the expected values,

$$E \{ \hat{X}_r \} = (A^T A)^{-1} A^T E \{ Y \}$$

$$E \{ Y \} = \begin{bmatrix}
E \{ \tilde{d}_1^2 \} - E \{ \tilde{d}_N^2 \} - x_1^2 - y_1^2 + x_N^2 + y_N^2 \\
\vdots \\
E \{ \tilde{d}_{N-1}^2 \} - E \{ \tilde{d}_N^2 \} - x_{N-1}^2 - y_{N-1}^2 + x_N^2 + y_N^2 
\end{bmatrix}$$

$\tilde{d}_i = d_i + n_i \Rightarrow \tilde{d}_i^2 = d_i^2 + n_i^2 + 2d_i n_i \Rightarrow E \{ \tilde{d}_i^2 \} = d_i^2 + E \{ n_i^2 \} = d_i^2 + \sigma_i^2$

where $\sigma_i^2 = \sigma_0^2 e^{k \sigma d_i}$

The result is that $\hat{X}_r$ is a biased estimator since

$$E \{ \hat{X}_r \} = (A^T A)^{-1} A^T E \{ Y \} \neq X_r$$
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Iterative and collaborative multilateration

- In lateration techniques, at least three anchor nodes are required for estimating position.
- Iterative and collaborative multilateration are an extension of lateration that do not require three neighbouring anchors.
- In iterative multilateration:
  - Any node can become an anchor after estimating its position and sending anchor messages on the network.
  - In this way, all nodes can estimate their positions after several iterations.
- For example, in the figure below:
  - Gray node estimates its position by three black anchors.
  - Blue node estimates by gray and two black anchor nodes.
  - White node is localized by blue and two black anchor nodes.
Iterative and collaborative multilateration

- It is possible that nodes cannot have three anchors even after several iterations.
- In that case collaborative multilateration is used in which:
  - A graph of participating nodes is constructed.
  - Participating nodes are the ones that are either anchors or have at least three participating neighbours.
  - This gives set of over-constrained quadratic equations relating distance among nodes and their neighbours.
  - These equation are solved to estimate positions.
Summary

- We have studied the basic of localization for sensor networks
- Localizing the nodes consists in applying estimation techniques
Next lecture

- Application of estimation and detection to synchronization