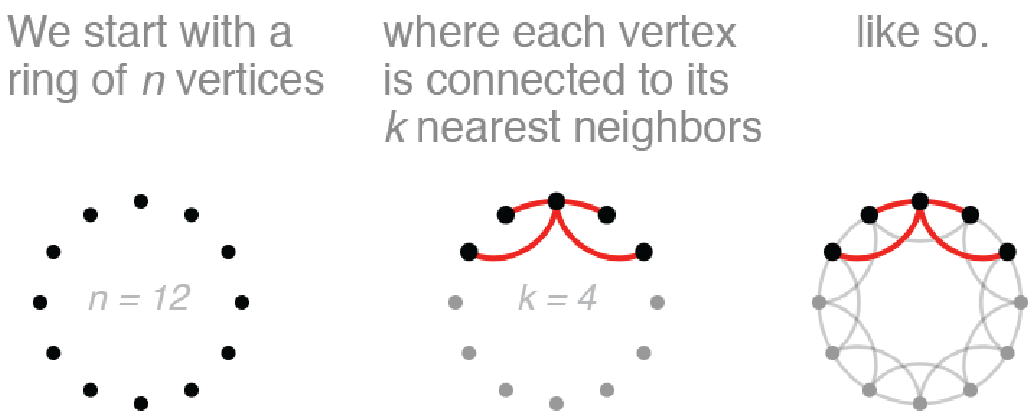
Watts-Strogatz Model

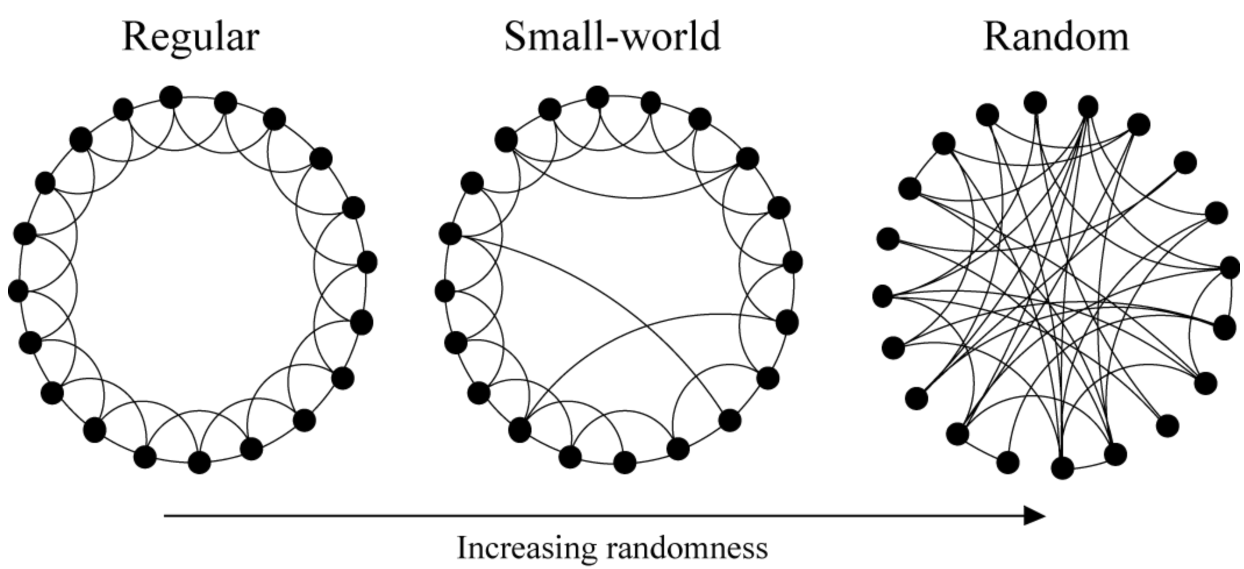
- Regular graph with degree k connected to nearest neighbors

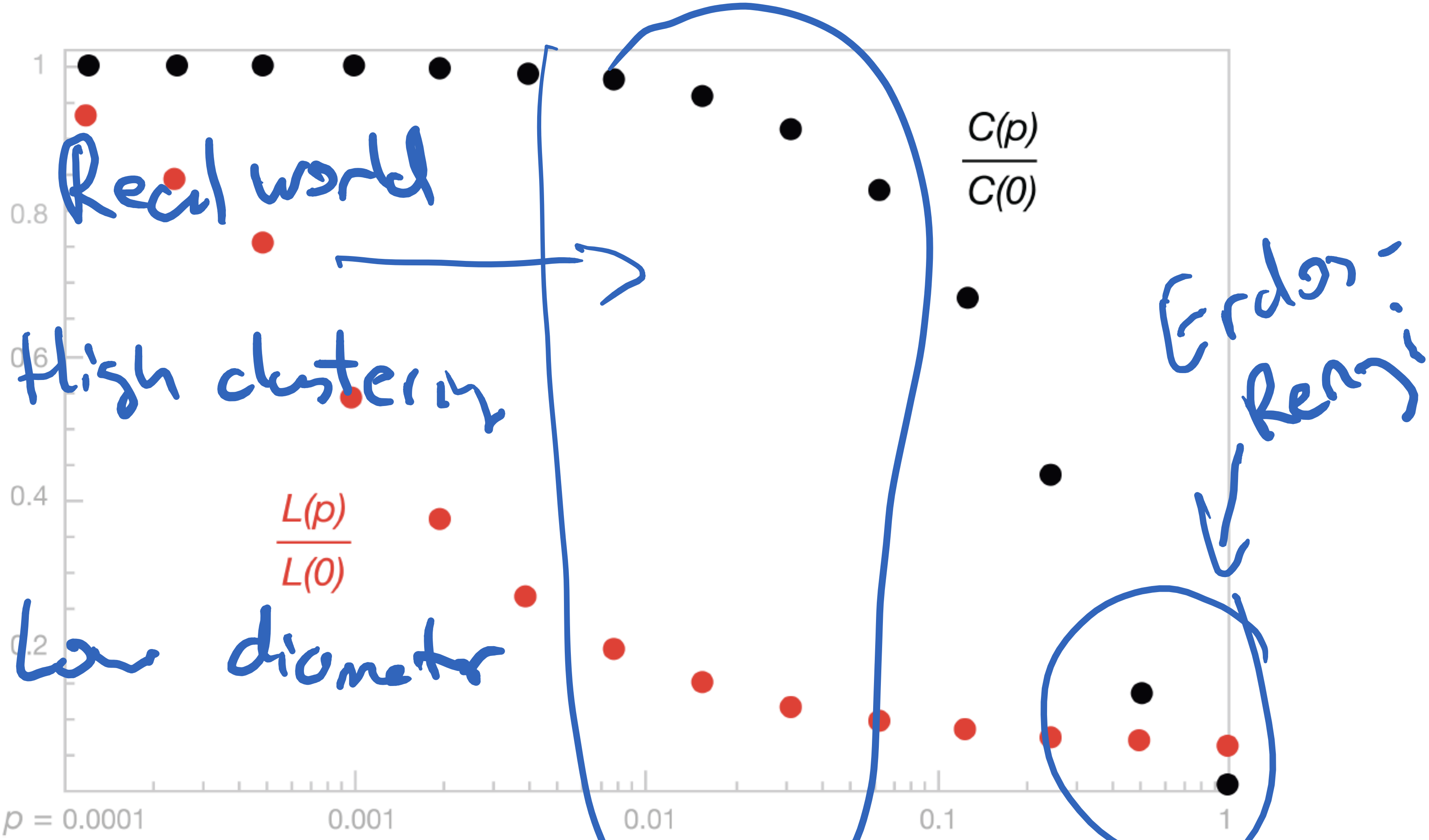


- Can be also a grid, torus, or any other "geographical" structure which has high clusterisation and high diameter
- ***With probability p rewire each edge in the network to a random node.***
  - ***Q: What happens when p=1?***

Watts-Strogatz Model (cont.)

- When p=1 we have ~Erdos-Renyi network
- There is a range of p values where the network exhibits properties of both: random and regular graphs:
  - High clusterisation;
  - Short path length.





The data shown in the figure are averages over 20 random realizations of the rewiring process, and have been normalized by the values L(0), C(0) for a regular lattice. All the graphs have n = 1000 vertices and an average degree of k = 10 edges per vertex. We note that a logarithmic horizontal scale has been used to resolve the rapid drop in L(p), corresponding to the onset of the small-world phenomenon. During this drop, C(p) remains almost constant at its value for the regular lattice, indicating that the transition to a small world is almost undetectable at the local level.

Demo Small Worlds

- http://ccl.northwestern.edu/netlogo/models/SmallWorlds

Small Worlds and Real Networks

- More realistic than Erdos-Renyi
  - Low path length
  - High clusterisation
- **What other properties of real world networks are missing?**
- Degree distribution
  - Watts-Strogatz model might be good for certain applications (e.g., P2P networks)
  - But can not model real world networks where degree distributions are usually power law.

Preferential attachment Model

- Start with Two connected nodes
  - Add a new node v
  - Create a link between v and one of the existing nodes with **probability proportional to the degree** of the that node
    - P(u,v) = d(u)/Total_network_degree
- **Rich get richer** phenomenon!
- Exhibits power-law distributions
- Can be extended to m links (Barabasi-Albert model)
  - Start with **connected network of m_0** nodes
  - Each new node connects to m nodes ($m \leq m_0$) with aforedescribed pref. attachment principle.

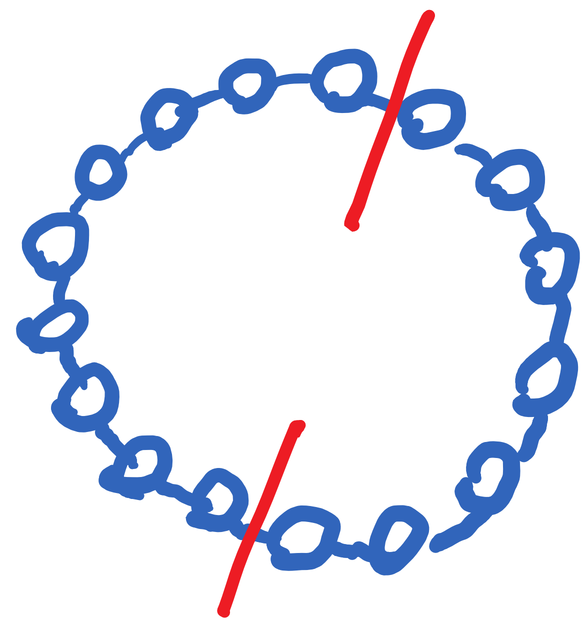
- http://ccl.northwestern.edu/netlogo/models/PreferentialAttachment

Recap: Network properties

- Each network is unique "***microscopically***", but in large scale networks one can observe ***macroscopic*** properties:
  - **Diameter** (six-degrees of separation);
  - **Clustering coefficient** (triangles, friends-of-friends are also friends);
  - **Degree distribution** (are there many "hubs" in the network?);

Expanders

- ***Expanders are graphs with very strong connectivity properties***.
  - sparse yet very well-connected
- Example
  - N nodes, E edges N=E
  - ***Does this graph have good connectivity properties?***
  - 2 edges fail -> isolates up to N/2 nodes



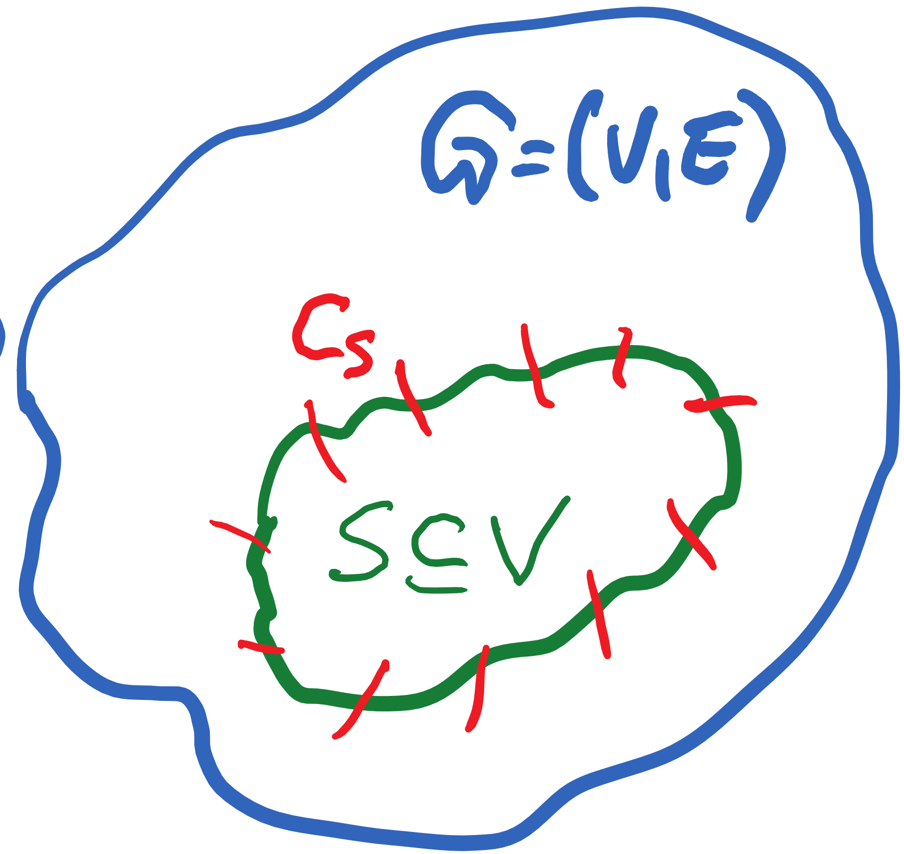
Can one isolate large number of nodes by removing small number of edges?

Expansion

- **Expansion *α*:**

$$\alpha = \min_{S \subseteq V} \frac{C_s}{\min(n_s, n-n_s)}$$

- How robust are your graphs?
- ***To isolate*** ***k*** ***nodes one needs to remove at least*** ***α*k*** ***edges***



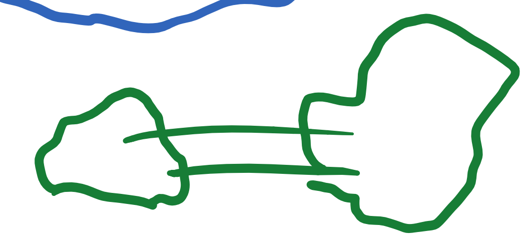
$|V| = n$ $\quad |S| = n_s$

Examples

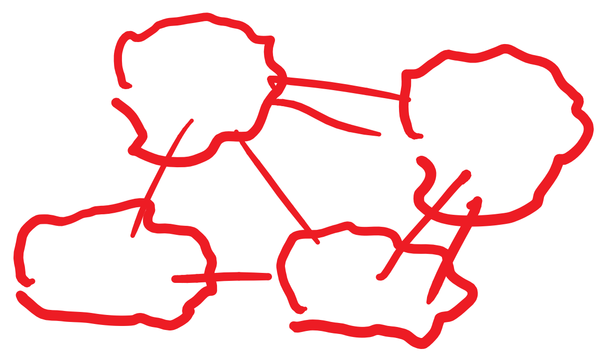
- Which networks do you think have good expansion (random graph, tree, grid)?

High expansion

Low expansion



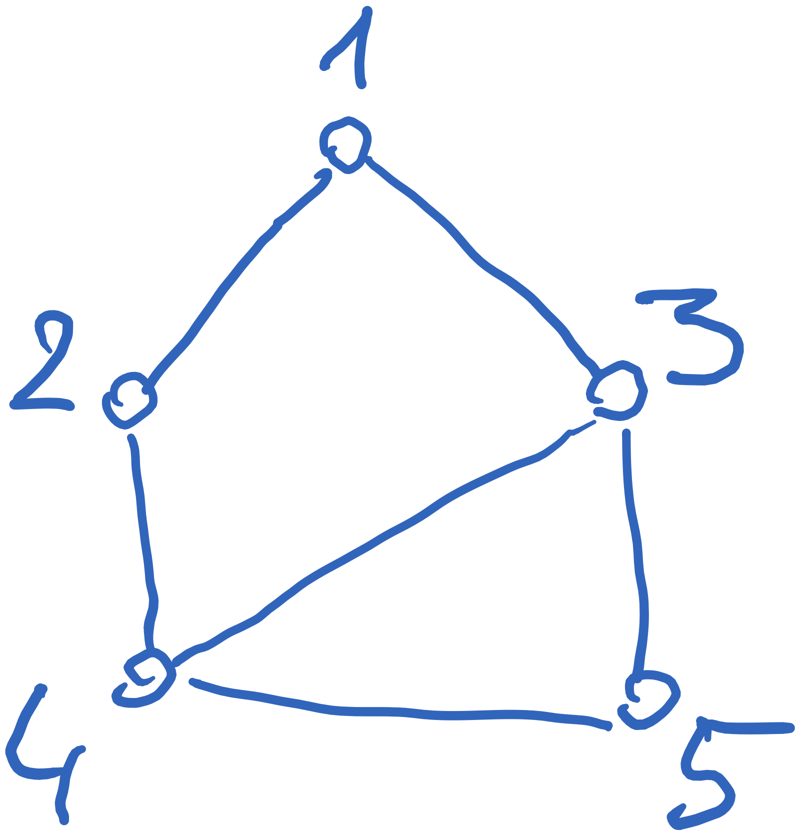
Social networks



Properties of Expander Graphs

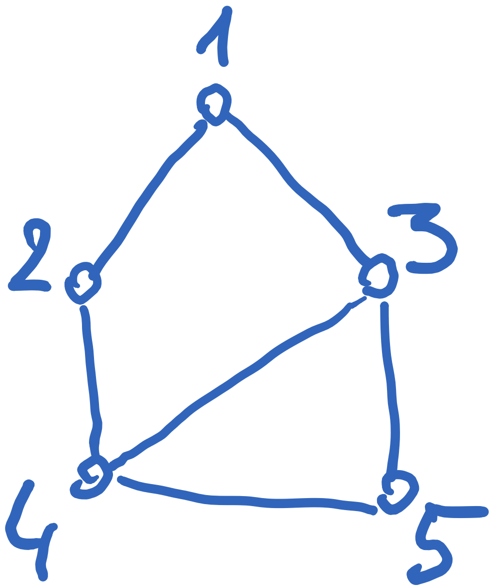
- A graph **is an expander** if the number of edges originating from every subset of vertices is larger than the number of vertices at least by a constant factor (more than 1).
    - Sparse yet very well-connected (no small cuts, no bottlenecks)
    - Small second eigenvalue λ_2(we will talk about it later)
    - Rapid convergence of random walk
- For all practical reasons, random walk of TTL = O(logN) on an expander graph with fixed node degree gives an uniform random node from the population

Random Walks



- Let G(V,E) be connected graph.
- Consider random walk on G from node v.
  - We move to a neighboring node with probability 1/d(v)
  - The sequence of random walks is Markov chain

Random Walks



	Node 1	Node 2	Node 3	Node 4	Node 5
Time 0	1	0	0	0	0
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					

- The initial node can be fixed, but also can be drawn from some initial distribution P_0

Convergence

- For any ***connected non-bipartite*** bidirectional graph, and any starting point, the random walk converges
  - Converges to **unique** stationary distribution
  - **Power Iteration**

t	1	0	0	0	0
1	0.00	0.50	0.50	0.00	0.00
2	0.42	0.00	0.00	0.42	0.17
3	0.00	0.35	0.43	0.08	0.14
4	0.32	0.03	0.10	0.39	0.17
5	0.05	0.29	0.37	0.13	0.16
6	0.27	0.07	0.15	0.35	0.17
7	0.08	0.25	0.33	0.17	0.17
8	0.24	0.10	0.18	0.32	0.17
9	0.11	0.22	0.31	0.19	0.17
10	0.22	0.12	0.20	0.30	0.17
11	0.13	0.21	0.29	0.21	0.17
12	0.20	0.13	0.22	0.28	0.17
13	0.14	0.19	0.28	0.22	0.17
14	0.19	0.14	0.23	0.27	0.17
15	0.15	0.19	0.27	0.23	0.17
16	0.18	0.15	0.23	0.27	0.17
17	0.15	0.18	0.26	0.24	0.17
18	0.18	0.16	0.24	0.26	0.17
19	0.16	0.18	0.26	0.24	0.17
20	0.17	0.16	0.24	0.26	0.17
21	0.16	0.17	0.26	0.24	0.17
22	0.17	0.16	0.24	0.26	0.17
23	0.16	0.17	0.25	0.25	0.17
24	0.17	0.16	0.25	0.25	0.17
25	0.16	0.17	0.25	0.25	0.17
26	0.17	0.16	0.25	0.25	0.17
27	0.16	0.17	0.25	0.25	0.17
28	0.17	0.16	0.25	0.25	0.17
29	0.17	0.17	0.25	0.25	0.17
30	0.17	0.17	0.25	0.25	0.17

Stationary Distribution

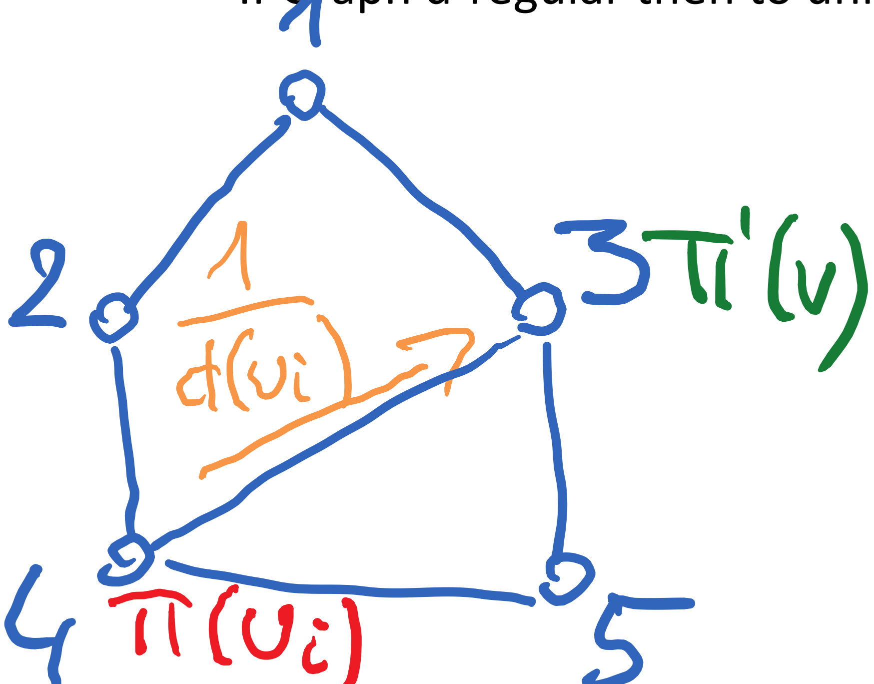
- Which distribution does the random walk converge in our graph?

π	0.17	0.17	0.25	0.25	0.17
π'	0.17	0.17	0.25	0.25	0.17

- Random walk converges to the stationary distribution:

$$\boldsymbol{\pi(v) = d(v)/2m}$$

  - d(v) = degree of v, i.e. # of neighbors.
  - m: |E|, i.e. # of edges.
  - If Graph d-regular then to uniform distribution



$$\pi'(v) = \sum_{u:\,(u,v)\in E} \pi(u) \frac{1}{d(u)}$$

$$= \sum_{u:\,(u,v)\in E} \frac{d(u)}{2m} \cdot \frac{1}{d(u)}$$

$$= \sum_{u:\,(u,v)\in E} \frac{1}{2m}$$

$$= \frac{d(v)}{2m}$$

$$= \pi(v)$$

Implications

- The stationary distribution

  $$\pi(v) = d(v)/2m$$

  is **proportional to the degree of v**.
  - What's the intuition?
  - The more neighbors you have, the more chance you'll be visited.
    - We'll talk about it later

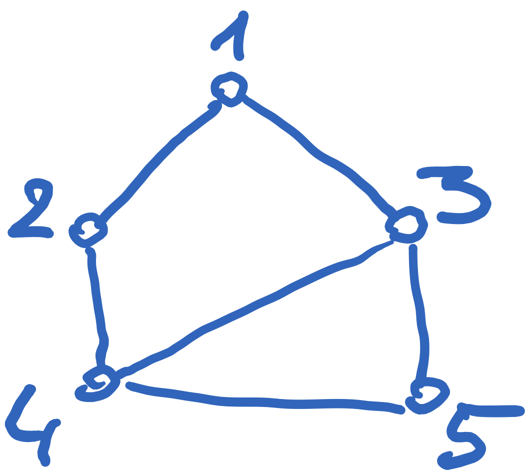
Definitions

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Adjacency Matrix

$$D = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

Diagonal matrix with $D_{i,i} = 1/d(i)$

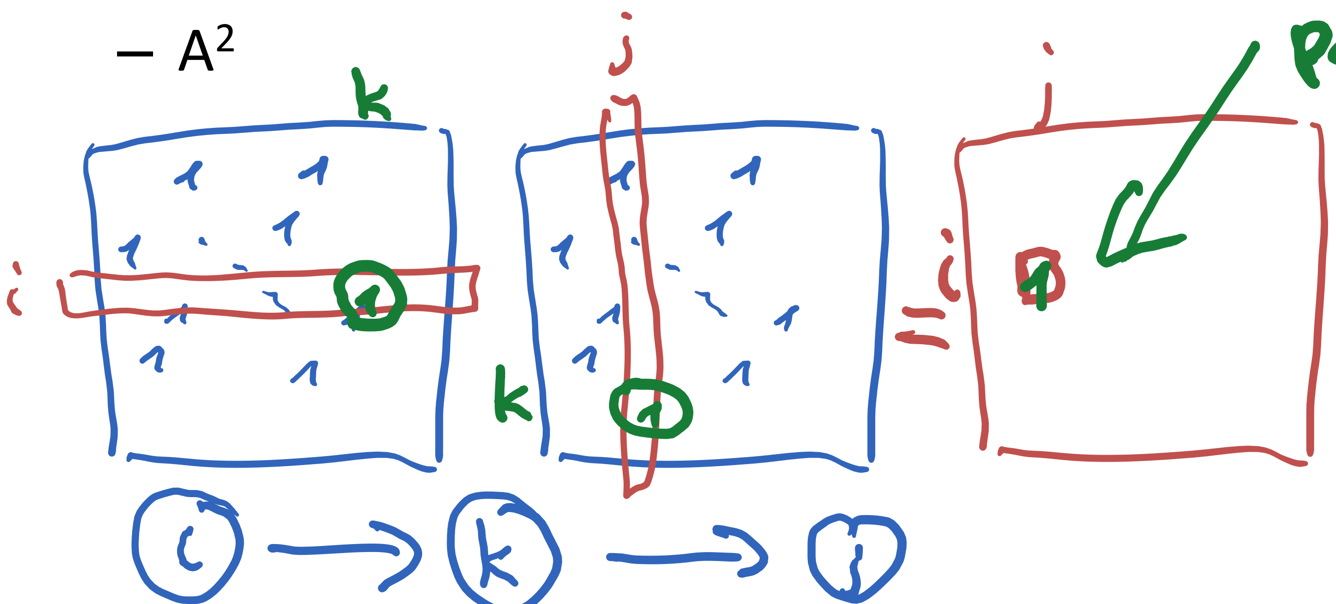


$$M= \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

Transition (random walk) Matrix
M=DA

Adjacency Matrix

- A is *n x n* adjacency matrix of G=(V,E)
  - A_{ij} is 1 if there is a link between i and j nodes and 0 otherwise
- Gives us all 1-hop paths.
- How to count # of 2-hop paths?
  - A^2

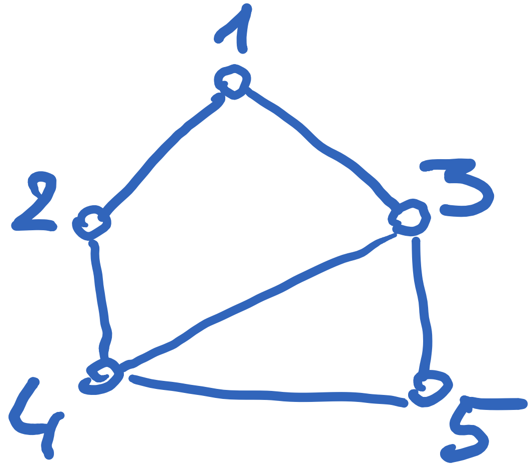


# of 2-hop paths from i to j

Matrix manipulations

- A^2 gives us # of 2-hop paths
- A^3 gives us ?
  - # of 3-hop paths, etc.
  - not simple paths! Can refer as walks.
- What about taking a vector v=(1 0 0 0 0) that represents a message at the first node and multiplying it by A?

$$vA = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$



- vA=(0 1 1 0 0)
  - indicates how many walks of length 1 from node 1 end up in node i.
- vA^2 = (2 0 0 2 1)
  - Indicates how many walks of length 2 from node 1 end up in node i.
- vA^3 = (0 4 5 1 2)
  - Indicates how many walks of length 3 from node 1 end up in node i.

Matrix manipulations (cont.)

- What about multiplying by a Random Walk Matrix?

$$M=\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

Transition (random walk) Matrix
M=DA

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}=\begin{pmatrix} 0.42 & 0 & 0 & 0.42 & 0.17 \end{pmatrix}$$

What about Random Walk Matrix M? (cont.)

- Recap: Random walk on a graph G: we start at a node v_0 and at the t-th step we are at a node v_t. We move to a neighbor of v_t with probability $1/d(v_t)$.
  - The sequence of random nodes (v_t :t=0,1,2…) is a Markov chain
- We start from the initial state of the system, e.g. P_0: [1 0 0 0 0];
  - Can also be drawn from some initial distribution
- **$P_t = P_0 M^t$**
  - Or can be written as $P_t = (M^T)^t P_0$ if we represent P as a column vector

Again the same Example

$P_t = P_0 M^t$

$$(1 \quad 0 \quad 0 \quad 0 \quad 0)\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix} = (0 \quad 1/2 \quad 1/2 \quad 0 \quad 0)$$

$$(0 \quad 1/2 \quad 1/2 \quad 0 \quad 0)\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix} = (0.42 \quad 0 \quad 0 \quad 0.42 \quad 0.17)$$

- When $P_{t+1} = P_t = \pi$, we have reached stationary distribution, i.e. **$\pi M = \pi$**
- Recall: that v is **eigenvector** of matrix M and λ its eigenvalue if **$vM = \lambda v$**
  - so ***π is eigenvector of M with eigenvalue λ=1***