

8.1.1 $X(n) = a_0 \sin(2\pi \nu_0 n + \Phi) + w(n)$ $\Phi \in [0, 2\pi)$ uniform

$w(n)$ and Φ are independent.

a_0, ν_0 ?

$R_w(k) = \sigma^2 \delta(k)$

$$\hat{P} = \frac{1}{N} \sum_{n=0}^{N-1} x^2(n) - \sigma^2$$

(a) $E[\hat{P}]$

$$E[\hat{P}] = \frac{1}{N} \sum_{n=0}^{N-1} E[x^2(n)] - \sigma^2$$

$$E[x^2(n)] = E[a_0^2 \sin^2(2\pi \nu_0 n + \Phi) + w^2(n)] =$$

Φ and $w(n)$ independent
and $w(n)$ has 0 mean

$$= a_0^2 E[\sin^2(2\pi \nu_0 n + \Phi)] + \sigma^2 = \frac{a_0^2}{2\pi} \int_0^{2\pi} \sin^2(2\pi \nu_0 n + \Phi) d\Phi + \sigma^2$$

$$= \left\{ \begin{array}{l} -\cos(2\alpha) = -\cos^2(\alpha) + \sin^2(\alpha) \\ 1 = \cos^2(\alpha) + \sin^2(\alpha) \end{array} \right\} = \frac{a_0^2}{2\pi} \int_0^{2\pi} \left(\frac{1 - \cos(4\pi \nu_0 n + 2\Phi)}{2} \right) d\Phi$$

$$+ \sigma^2 = \frac{a_0^2}{4\pi} \left[\frac{1}{2} \Phi - \frac{1}{4\pi} \sin(4\pi \nu_0 n + 2\Phi) \right]_0^{2\pi} + \sigma^2 =$$

$$= \frac{a_0^2}{4\pi} \left[\frac{1}{2} 2\pi - \frac{1}{4\pi} \sin(4\pi \nu_0 n + 4\pi) + \frac{1}{4\pi} \sin(4\pi \nu_0 n + 0) \right] + \sigma^2 =$$

$$= \frac{a_0^2}{4\pi} \frac{1}{2} 2\pi + \sigma^2 = \frac{a_0^2}{2} + \sigma^2$$

$$E[\hat{P}] = \frac{1}{N} \sum_{n=0}^{N-1} \left(\frac{a_0^2}{2} + \sigma^2 \right) - \sigma^2 = \frac{N}{N} \left(\frac{a_0^2}{2} + \sigma^2 \right) - \sigma^2$$

$$= \frac{a_0^2}{2}$$

unbiased

The power of sinusoid is $\frac{a_0^2}{2}$ hence, its

(b) Φ deterministic but unknown

$E[\check{P}]$ for the corresponding estimator of this modified estimator

$$\begin{aligned} E[\check{P}] &= \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{E[x^2(n)]}_{\text{nothing stochastic in } x(n) \text{ except for the noise!}} - \sigma^2 \\ &= \frac{1}{N} \left[\sum_{n=0}^{N-1} (a_0^2 \sin^2(2\pi\nu_0 n + \Phi) + \sigma^2) \right] - \sigma^2 \\ &= \frac{1}{N} \sum_{n=0}^{N-1} a_0^2 \cdot \sin^2(2\pi\nu_0 n + \Phi) = \frac{a_0^2}{N} \sum_{n=0}^{N-1} \sin^2(2\pi\nu_0 n + \Phi) \\ &= \frac{a_0^2}{N} \sum_{n=0}^{N-1} \frac{1}{2} (1 - \cos(4\pi\nu_0 n + 2\Phi)) = \frac{a_0^2}{2N} \sum_{n=0}^{N-1} 1 + \\ &\quad - \frac{a_0^2}{2N} \sum_{n=0}^{N-1} \cos(4\pi\nu_0 n + 2\Phi) = \frac{a_0^2}{2} - \underbrace{\frac{a_0^2}{2N} \sum_{n=0}^{N-1} \cos(4\pi\nu_0 n + 2\Phi)} \end{aligned}$$

→ unless special case is taken

generally different from ρ

(by knowing before hand the value of Φ)

the estimator is biased.

c) $E[\check{P}]$? $N \rightarrow \infty$

$$\begin{aligned} E[\check{P}] &= \frac{a_0^2}{2} - \frac{a_0^2}{2N} \sum_{n=0}^{N-1} \cos(4\pi\nu_0 n + 2\Phi) \\ \lim_{N \rightarrow \infty} \frac{a_0^2}{2} - \frac{a_0^2}{2N} \sum_{n=0}^{N-1} \cos(4\pi\nu_0 n + 2\Phi) &= \frac{a_0^2}{2} \end{aligned}$$

unbiased!

(average of a sinusoid is 0)

8.2] $X(t)$ be WSS with unknown mean m and with ACF:

$$r_x(\tau) = e^{-|\tau|} + m^2$$

$$\hat{m}_1 = \int_0^1 X(t) dt \quad \text{Bias, variance?}$$

$$\hat{m}_2 = \frac{1}{2} \sum_{t=0}^1 X(t) \quad \text{Bias, variance?}$$

$$\begin{aligned} \underline{\hat{m}_1} \\ \bullet E[\hat{m}_1] &= E\left[\int_0^1 X(t) dt\right] = \int_0^1 E[X(t)] dt = \int_0^1 m dt = \\ & m t \Big|_0^1 = \underline{m} \end{aligned}$$

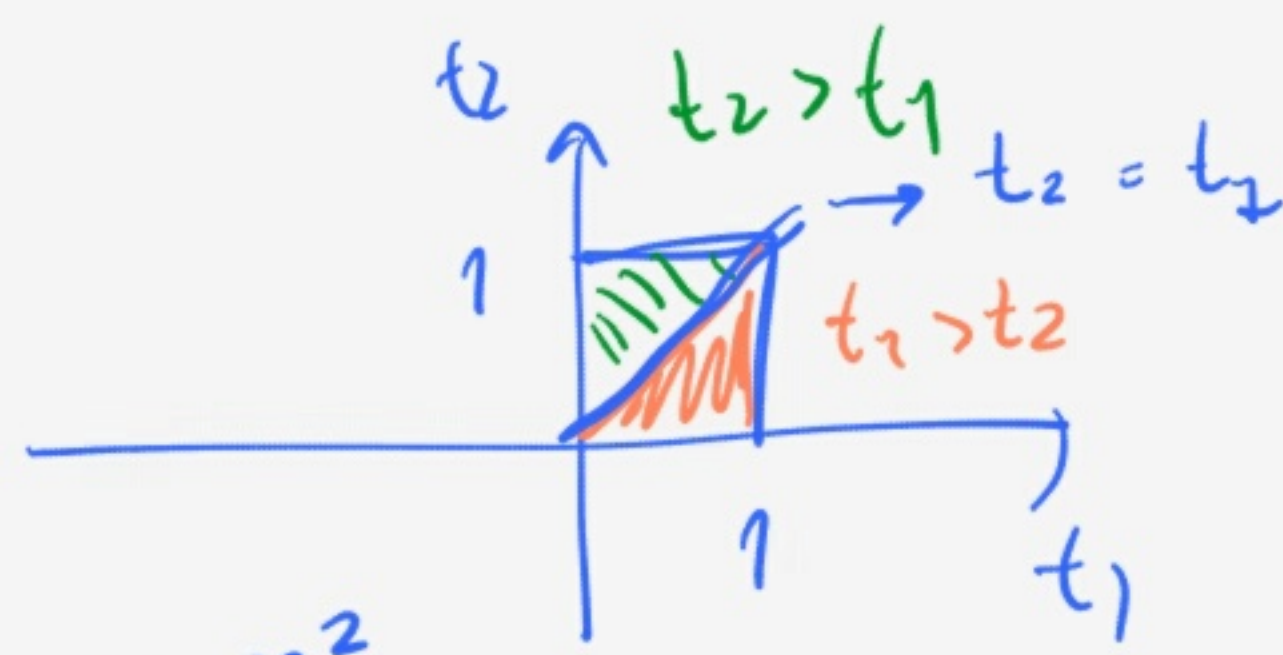
$$\bullet E[(\hat{m}_1 - m)^2] = E[\hat{m}_1^2] - m^2$$

$$E[\hat{m}_1^2] = E\left[\int_0^1 \int_0^1 X(t_1) X(t_2) dt_1 dt_2\right] =$$

$$= \int_0^1 \int_0^1 E[X(t_1) X(t_2)] dt_1 dt_2 = \int_0^1 \int_0^1 R_x(t_1, t_2) dt_1 dt_2$$

$$= \text{WSS} = \int_0^1 \int_0^1 (e^{-|t_2 - t_1|} + m^2) dt_2 dt_1 =$$

$$= \underbrace{\int_0^1 dt_1 \int_0^{t_1} e^{-|t_2 - t_1|} dt_2}_{t_1 > t_2} + \underbrace{\int_0^1 dt_2 \int_0^{t_2} e^{-|t_2 - t_1|} dt_1}_{t_2 > t_1} + m^2 =$$



$$= \int_0^1 e^{-t_1} dt_1 \int_0^{t_1} e^{t_2} dt_2 + \int_0^1 e^{-t_2} dt_2 \int_0^{t_2} e^{t_1} dt_1 + m^2 =$$

$$= 2 \int_0^1 e^{-t_1} dt_1 \int_0^{t_1} e^{t_2} dt_2 + m^2 = 2 \int_0^1 e^{-t_1} (e^{t_1} - 1) dt_1 + m^2 =$$

$$= 2 \int_0^1 (1 - e^{-t_1}) dt_1 + m^2 = 2(t_1 + e^{-t_1}) \Big|_0^1 + m^2 =$$

$$= 2(1 + e^{-1}) - 2(0 + e^0) + m^2 = 2e^{-1} + m^2$$

$$E[(\hat{m}_1 - m)^2] = 2e^{-1} + m^2 - m^2 = 2e^{-1}$$

$\hat{m}_2$:

$$E[\hat{m}_2] = E\left[\frac{1}{2} \sum_{t=0}^1 X(t)\right] = \frac{1}{2} \sum_{t=0}^1 E[X(t)] = \frac{1}{2} \sum_{t=0}^1 m = \frac{2}{2} m$$

unbiased.

$$E[(\hat{m}_2 - m)^2] = E[\hat{m}_2^2] - m^2$$

$$E[\hat{m}_2^2] = \frac{1}{4} E\left[\sum_{t_1=0}^1 X(t_1) \sum_{t_2=0}^1 X(t_2)\right] = \frac{1}{4} \sum_{t_1=0}^1 \sum_{t_2=0}^1 E[X(t_1)X(t_2)] =$$

$$= \frac{1}{4} \sum_{t_1=0}^1 \sum_{t_2=0}^1 R_X(t_1, t_2) = \left\{ \text{possible combinations are } (0,0), (0,1), (1,0), (1,1) \right\} =$$

$$= \frac{1}{4} (2R_X(0) + 2R_X(1)) = \frac{1}{4} (2e^0 + 2e^{-1}) =$$

$$= \frac{1}{2} (1 + e^{-1})$$

9.5 | $S(t)$ WSS with ACF $r_S(\tau)$

$$z = \int_0^T S(t) dt \quad \text{estimated using } \hat{z} = aS(0) + bS(T)$$

a and b so that $\hat{z}$ is an MMSE estimate

$$\min E[(\hat{z} - z)^2] \Rightarrow \begin{aligned} E[(\hat{z} - z)S(0)] &= 0 \\ \uparrow \\ E[(\hat{z} - z)S(T)] &= 0 \end{aligned}$$

These orthogonal
to data!

$$E[\hat{z}S(0)] - E[zS(0)] = E[aS(0) + bS(T)S(0)]$$

$$- E\left[\int_0^T S(t)S(0)dt\right] = aR_S(0) + bR_S(T) - \int_0^T R_S(t)dt = 0$$

$$E[\hat{z}S(T)] - E[zS(T)] = aR_S(T) + bR_S(0) - \int_0^T R_S(T-t)dt = 0$$

Note! $\int_0^T R_S(T-t)dt = \int_0^T R_S(t)dt$

Proof: $T-t=a \rightarrow \int_T^0 R_S(a)(-da) = \int_0^T R_S(a)da = \int_0^T R_S(t)dt \checkmark$

$$aR_S(0) + bR_S(T) = \int_0^T R_S(t)dt \quad \left\{ \text{solve for a and b:} \right.$$

$$aR_S(T) + bR_S(0) = \int_0^T R_S(t)dt \quad \left. \begin{aligned} a &= b = \frac{\int_0^T r_S(t)dt}{r_S(0) + r_S(T)} \end{aligned} \right\}$$