8.11 cG(1): Find $U \in V_h$ such that

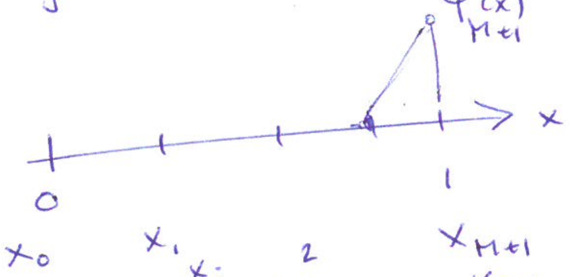
$$\begin{cases} -(au')' = f \text{ in } (0,1) \\ u(0) = 0 \\ a(1)u'(1) = g_1 \end{cases} \qquad \int_0^1 aU'v'\,dx = \int_0^1 fv\,dx + g_1 v(1) \quad \forall v \in V_h$$

$a = f = 1,\ g_1 = 1$, uniform partition $h_i = h$

$$\Rightarrow \int_0^1 U'v'\,dx = \int_0^1 v\,dx + v(1)$$

$$U = \sum_{j=1}^{M+1} \xi_j \varphi_j$$

$$\Rightarrow \sum_{j=1}^{M+1} \xi_j \int_0^1 \varphi_j' \varphi_i'\,dx = \int_0^1 \varphi_i\,dx + \varphi_i(1)$$



$$a_{ii} = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(\frac{-1}{h}\right)^2 dx = \frac{2}{h} \qquad i = 1, \dots, M$$

$$a_{M+1\,M+1} = \int_{x_M}^{x_{M+1}} \left(\frac{1}{h}\right)^2 dx = \frac{1}{h}$$

$$a_{i-1\,i} = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)\left(\frac{-1}{h}\right) dx = \frac{-1}{h}$$

$$a_{M\,M+1} = \int_{x_M}^{x_{M+1}} \left(\frac{1}{h}\right)\left(\frac{-1}{h}\right) dx = -\frac{1}{h}$$

$$b_i = \int_{x_{i-1}}^{x_{i+1}} \varphi_i\,dx = h \qquad i = 1, \dots, M$$

$$b_{M+1} = \int_{x_M}^{x_{M+1}} \varphi_{M+1}\,dx + 1 = \frac{h}{2} + 1$$

8.11 Discrete eqn. at $x=1$:

$$-\frac{1}{h}\xi_M + \frac{1}{h}\xi_{M+1} = \frac{h}{2} + 1$$

$$\cancel{\xi_{M+1} = \xi_M + \frac{h^2}{2} + h}$$

$$\frac{\xi_{M+1} - \xi_M}{h} = \frac{h}{2} + 1$$

$$\frac{U(1) - U(1-h)}{h} = \frac{h}{2} + 1$$

$$h \to 0 \Rightarrow U'(1) = 1$$

8.9

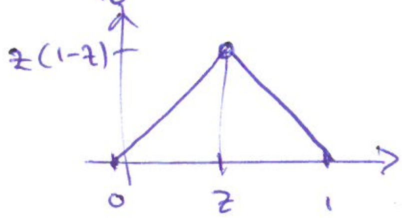
$$\begin{cases} -u'' = f & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

(1) Weak formulation : $\int_0^1 u'v'\,dx = \int_0^1 fv\,dx \quad \forall v \in V$

(2) cG(1) : find $U \in V_h$: $\int_0^1 U'v'\,dx = \int_0^1 fv\,dx \quad \forall v \in V_h \subset V$

$e = u - U$

(1) − (2) : $\int_0^1 e'v'\,dx = 0 \quad \forall v \in V_h$



$$g_z(x) = \begin{cases} (1-z)x & 0 \le x \le z \\ z(1-x) & z \le x \le 1 \end{cases}$$

$$g_z'(x) = \begin{cases} 1-z & 0 \le x \le z \\ -z & z \le x \le 1 \end{cases}$$

$$\int_0^1 g_z'(x)e'(x)\,dx = \int_0^z (1-z)e'\,dx + \int_z^1 (-z)e'\,dx$$

$$= (1-z)(e(z) - e(0)) - z(e(1) - e(z))$$

$$= e(z) \qquad [e(0) = e(1) = 0]$$

$$e(x_j) = \int_0^1 g_{x_j}'(x)e'(x)\,dx = 0 \qquad g_{x_j} \in V_h \text{ for } j = 1, \dots, M$$

8.10

Prove that $\begin{cases} -g_z'' = \delta_z & \text{in } (0,1) \\ g_z(0) = g_z(1) = 0 \end{cases}$.

δ_z delta function at $x = z$.

$$\int_0^1 -g_z'' v \, dx = \int_0^1 g_z' v' \, dx - [g_z' v]_0^1$$

$$= \int_0^z (1-z) v' \, dx + \int_z^1 (-z) v' \, dx - \left((-z) v(1) - (1-z) v(0) \right)$$

$$= (1-z)(v(z) - v(0)) - z(v(1) - v(z)) + z v(1) + (1-z) v(0)$$

$$= v(z) + v(0)\left(\cancel{-(1-z)} + (1-z)\right) + v(1)\left(\cancel{z} - z\right) = v(z)$$

$$\Rightarrow -g_z'' = \delta_z \quad \square$$

$$\begin{cases} -u'' = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

$$\int_0^1 g_z(x) f(x) \, dx = \int_0^z (1-z) x f(x) \, dx + \int_z^1 z(1-x) f(x) \, dx$$

$$= (1-z) \int_0^z x f(x) \, dx + z \int_z^1 (1-x) f(x) \, dx$$

$$= (1-z) \int_0^z x(-u'') \, dx + z \int_z^1 (1-x)(-u'') \, dx$$

$$= (1-z) \left[\int_0^z u' \, dx - [x u']_0^z \right] + z \left[\int_z^1 (-1) u' \, dx - [(1-x) u']_z^1 \right]$$

$$= (1-z) \left[(u(z) - u(0)) - z u'(z) \right] + z \left[(u(z) - u(1)) + (1-z) u'(z) \right]$$

$$= (1-z) u(z) - \cancel{z(1-z) u'(z)} + z u(z) + \cancel{z(1-z) u'(z)}$$

$$= u(z) \quad \square$$

15.16

FEM: Find $U \in V_h$ s.t. $(\nabla U, \nabla v) = (f, v) \quad \forall v \in V_h$

$\{\varphi_1, \dots, \varphi_M\}$ nodal basis for V_h

$$\Rightarrow \quad v \in V_h : \quad v(x) = \xi_1 \varphi_1(x) + \dots + \xi_M \varphi_M(x)$$

$$= v(N_1)\varphi_1(x) + \dots + v(N_M)\varphi_M(x)$$

$\{N_1, \dots, N_M\}$ enumeration of internal nodes $\mathcal{N}_h$

If $(\nabla U, \nabla \varphi_i) = (f, \varphi_i)$ for all $i = 1, \dots, M$

then for any $v \in V_h$

$$(\nabla U, \nabla(v(N_1)\varphi_1(x) + \dots + v(N_M)\varphi_M(x))) - ((f, \dots)$$

$$= v(N_1)(\nabla U, \nabla \varphi_1) + \dots + v(N_M)(\nabla U, \nabla \varphi_M) - (v(N_1)(f, \varphi_1) + \dots$$

$$\Rightarrow v(N_1)\big(\underbrace{(\nabla U, \nabla \varphi_1) - (f, \varphi_1)}_{=0}\big) + \dots = 0$$