SF2561 Finite Element Methods: Written Examination

Tuesday 2014-10-30, kl 8-13

Coordinator: Johan Hoffman

Aids: none. **Time:** 5 hours.

Answers may be given in English or in Swedish. All answers should be explained and calculations shown unless stated otherwise. A correct answer without explanation can be left without points. Do not leave integrals or systems of equations unsolved unless explicitly allowed. *Each of the 5 problems gives 10 p, resulting in a total of 50 p: 20 p for grade E, 25 p for grade D, 30 p for grade C, 35 p for grade B, and 40 p for grade A.*

Problem 1: Consider the Poisson equation:

$$\begin{aligned} -\Delta u(x) &= 1 \quad x \in \Omega \\ u(x) &= 0 \quad x \in \Gamma \end{aligned} \tag{1}$$

with $x = (x_1, x_2)$, and $\Omega \subset \mathbb{R}^2$ the unit square with corners in the points $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$, and with boundary Γ.

(a) Define a finite element mesh $\mathcal{T}_h$ over the domain Ω with at least 4 internal (non-boundary) nodes.
(b) Define a discrete finite element approximation space V_h over the mesh $\mathcal{T}_h$.
(c) Formulate a Galerkin finite element method (FEM) using the space V_h.
(d) Compute the corresponding matrix A and vector b. You do not have to solve the resulting system of equations.

Problem 2: Consider the Poisson equation (1).

(a) Derive a variational formulation of equation (1): Find $u \in V$ such that

$$a(u, v) = L(v), \qquad \text{for all } v \in V \tag{2}$$

Define the Hilbert space V and its associated norm $\|\cdot\|_V$, and define the bilinear form $a : V \times V \to \mathbb{R}$ and the linear form $L : V \to \mathbb{R}$.
(b) Prove that there exists a unique solution to the variational problem (2).

Note: *The exam continues on the next page!*

Problem 3: Consider the abstract variational problem (2).

(a) Formulate an abstract Galerkin FEM method for (2) in terms of the bilinear and linear forms, using a finite dimensional subspace $V_h \subset V$.

(b) Prove that the Galerkin FEM solution $U \in V_h$ is optimal in the energy norm, with the energy norm defined as $\|w\|_E = \sqrt{a(w,w)}$, for a function $w \in V$. That is, show that the error $\|u - U\|_E \leq \|u - v\|_E$ for any $v \in V_h$.

Problem 4: For $a(x) > 0$ and $c(x) \geq 0$, consider the problem:

$$-(a(x)u'(x))' + c(x)u(x) = f(x), \quad x \in (0,1)$$
$$u(0) = u(1) = 0$$

(a) Formulate the cG(1) method for the problem (FEM with a continuous piecewise linear approximation on a subdivision $\mathcal{T}_h$ of (0,1)).

(b) Prove the a posteriori error estimate:

$$\|u - U\|_{L_2(0,1)} \leq SC_i \|h^2 R(U)\|_{L_2(0,1)}$$

where U is the cG(1) solution, and

$$S = \max_{\xi \in L_2(0,1), \xi \neq 0} \frac{\|\varphi''\|_{L_2(0,1)}}{\|\xi\|_{L_2(0,1)}}$$

with φ the solution to the dual problem:

$$-(a(x)\varphi'(x))' + c(x)\varphi(x) = \xi(x), \quad x \in (0,1)$$
$$\varphi(0) = \varphi(1) = 0$$

(c) Prove that if $a > 0$ and $c \geq 0$ are constants, then $S \leq a^{-1}$.

(d) Formulate an algorithm for adaptive refinement of the mesh $\mathcal{T}_h$, based on the a posteriori error estimate.

Note: *The exam continues on the next page!*

Problem 5: Consider the equation:

$$-u''(x) + u(x) = 1, \quad x \in (0,1) \tag{3}$$

with boundary conditions: $u(0) = 0$ and $u'(1) = g_1$.

(a) State a variational formulation of equation (3).

(b) Formulate the cG(1) method (continuous piecewise linear approximation) for the equation (3) on a subdivision $\mathcal{T}_h$ of (0,1) with uniform mesh size h.

(c) Compute the corresponding matrix A and vector b. You do not have to solve the resulting system of equations.

(d) Show that the Neumann boundary condition $u'(1) = g_1$ is satisfied for U, the solution of the cG(1) method, in the limit when the mesh size $h \to 0$, that is:

$$\lim_{h \to 0} \frac{U(1) - U(1-h)}{h} = g_1 \tag{4}$$

Good Luck!

Johan