

9.11 X and Y stochastic with $m_y = 4$ $\sigma_y = 1$ and $E[XY] = 1$
 $m_x = 0$ $\sigma_x = 2$

- a) Determine a that minimizes $E\{(X - aY)^2\}$ and the corresponding minimum value.

orthogonality principle

error, hence X parameter to estimate
 a "filter" Y data

$$(E\{(X - aY)Y\} = 0) \quad E\{XY\} - aE\{Y^2\} = 0$$

$$1 - a(\sigma_y^2 + m_y^2) = 0 \quad 1 - a(1 + 16) = 0 \quad 1 - a(17) = 0 \quad a = \frac{1}{17}$$

$$\theta = E\{(X - aY)^2\} \Big|_{a = \frac{1}{17}} = E\{(X - \frac{1}{17}Y)^2\} = E\{X^2 - 2\frac{1}{17}XY + \frac{1}{(17)^2}Y^2\} =$$

$$= (\sigma_x^2 + m_x^2) - 2\frac{1}{17}E[XY] + \frac{1}{(17)^2}(\sigma_y^2 + m_y^2) = 4 - \frac{2}{17} + \frac{1}{(17)^2}(17) =$$

$$= 4 - \frac{2}{17} + \frac{1}{17} = \frac{68 - 2 + 1}{17} = \frac{67}{17}$$

- b) Constants a_0 and a_1 that minimize $E\{(X - (a_0 + a_1Y))^2\}$

orthogonality principle:

$$\underline{a} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \quad \underline{d} = \begin{pmatrix} 1 \\ Y \end{pmatrix}$$

$$E\{(\underbrace{X - \underline{a}^T \underline{d}}_{\text{error}}) \underbrace{\underline{d}^T}_{\text{data}}\} = \underline{0}^T$$

$$E\{X \underline{d}^T\} - E\{\underline{a}^T \underline{d} \underline{d}^T\}$$

$$E\{X(1, Y)\} - \underline{a}^T E\left\{\begin{pmatrix} 1 \\ Y \end{pmatrix} (1 \ Y)\right\} =$$

$$= (m_x, E[XY]) - \underline{a}^T \begin{pmatrix} 1 & m_y \\ m_y & \sigma_y^2 + m_y^2 \end{pmatrix} = 0 \Rightarrow \text{Take transpose}$$

$$\begin{pmatrix} m_x \\ E[XY] \end{pmatrix} - \begin{pmatrix} 1 & m_y \\ m_y & \sigma_y^2 + m_y^2 \end{pmatrix} \underline{a} = 0$$

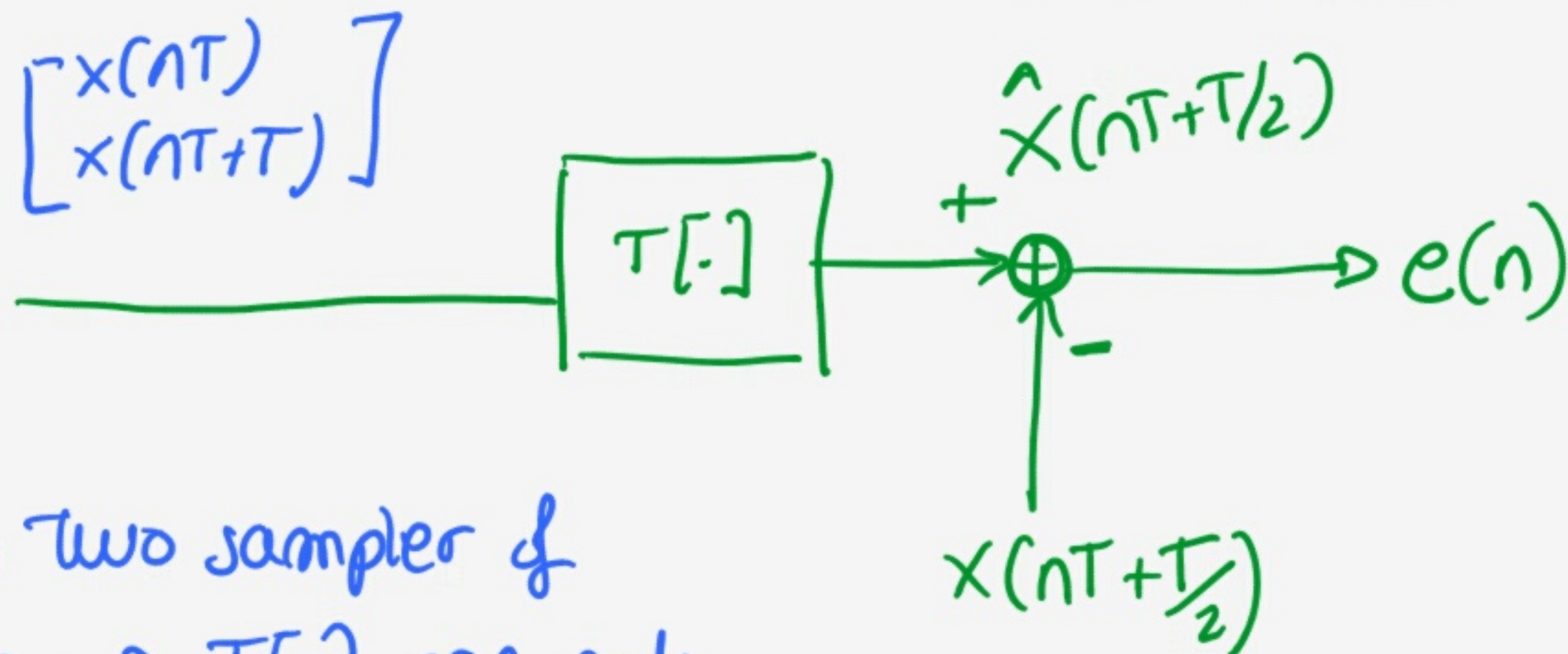
$$\underline{a} = \begin{pmatrix} 1 & m_y \\ m_y & \sigma_y^2 + m_y^2 \end{pmatrix}^{-1} \begin{pmatrix} m_x \\ E[XY] \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 4 & 17 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 17 & 4 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 4 & 17 \end{pmatrix}^{-1} = \frac{\begin{pmatrix} 17 & -4 \\ -4 & 1 \end{pmatrix}}{17 - 16} = \begin{pmatrix} 17 & -4 \\ -4 & 1 \end{pmatrix} \quad \text{Hence,} \quad \underline{a} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} \Rightarrow \begin{matrix} a_0 = -4 \\ a_1 = 1 \end{matrix}$$

9.7.

$x(t)$ Realisation of $X(t)$ with ACF $r_x(\tau)$
sampled signal $x(nT)$.

Try to estimate $x(nT + T/2)$ determining a linear combination of $x(nT)$ and $x(nT + T)$ which minimizes MMSE between interpolated value and the true value.



only two samples of data $\Rightarrow T[\cdot]$ can only have 2 coefficients!

$$E\{e(n) [x(nT), x(nT+T)]\} = \underline{0}^T$$

$$\begin{aligned} e(n) &= \hat{x}(nT+T/2) - x(nT+T/2) = \\ &= [h_1, h_2] \begin{bmatrix} x(nT) \\ x(nT+T) \end{bmatrix} - x(nT+T/2) \end{aligned}$$

$$E\{([h_1, h_2] \begin{bmatrix} x(nT) \\ x(nT+T) \end{bmatrix} - x(nT+T/2)) [x(nT), x(nT+T)]\} = \underline{0}$$

$$E\{(\underline{h}^T \underline{x} - x(nT+T/2)) \underline{x}^T\} = 0$$

$$\underline{h}^T E\{\underline{x} \underline{x}^T\} - E\{x(nT+T/2) \underline{x}^T\} = 0$$

$$\underline{h}^T E\left\{ \begin{bmatrix} x(nT) \\ x(nT+T) \end{bmatrix} [x(nT) \ x(nT+T)] \right\} - E\left\{ x(nT+T/2) [x(nT) \ x(nT+T)] \right\} = 0$$

$$= 0$$

$$\underline{h}^T \begin{pmatrix} R_x(0) & R_x(T) \\ R_x(T) & R_x(0) \end{pmatrix} - \begin{pmatrix} R_x(T/2) \\ R_x(T/2) \end{pmatrix}^T = 0 \Rightarrow \text{take transpose}$$

$$\begin{pmatrix} R_x(0) & R_x(T) \\ R_x(T) & R_x(0) \end{pmatrix} \underline{h} - \begin{pmatrix} R_x(T/2) \\ R_x(T/2) \end{pmatrix} = 0$$

$$\underline{h} = \begin{pmatrix} R_x(0) & R_x(T) \\ R_x(T) & R_x(0) \end{pmatrix}^{-1} \begin{pmatrix} R_x(T/2) \\ R_x(T/2) \end{pmatrix}$$

$$\underline{h} = \begin{pmatrix} R_x(0) & R_x(T) \\ R_x(T) & R_x(0) \end{pmatrix}^{-1} \begin{pmatrix} R_x(T/2) \\ R_x(T/2) \end{pmatrix}$$

$$\underline{h} = \frac{\begin{pmatrix} R_x(0) & -R_x(T) \\ -R_x(T) & R_x(0) \end{pmatrix}}{R_x(0)^2 - R_x(T)^2} \begin{pmatrix} R_x(T/2) \\ R_x(T/2) \end{pmatrix} \quad h_1 = h_2 \text{ (symmetry of matrix and vector being all same)}$$

$$h_1 = h_2 = \frac{R_x(0)R_x(T/2) - R_x(T)R_x(T/2)}{(R_x(0) + R_x(T))(\cancel{R_x(0) - R_x(T)})} = \frac{R_x(T/2)}{R_x(0) + R_x(T)}$$

• a straight line interpolation can be expressed as:

$$\hat{x}(nT + T/2) = \frac{1}{2} (x(nT) + x(nT + T))$$

• mmse interpolation

$$\hat{x}_{\text{MMSE}}(nT + T/2) = \frac{R_x(T/2)}{R_x(0) + R_x(T)} (x(nT) + x(nT + T))$$

9.19 Let $x(n)$ and $y(n)$ be two stochastic processes. $x(n)$ given by

$$\hat{x}(n) = w(n) * y(n) = \sum_{k=-\infty}^{\infty} h(k) y(n-k)$$

Estimator to be a Wiener filter should be chosen such that the MSE $E\{(x(n) - \hat{x}(n))^2\}$ is minimized.

a) Show that the orthogonality principle gives the following necessary condition on the Wiener filter.

$$E\{(x(n) - \hat{x}(n))^2\} = E\{(x(n) - \sum_{k=-\infty}^{\infty} h(k) y(n-k))^2\}$$

$$\nabla_{\underline{h}} E\{(x(n) - \sum_{k=-\infty}^{\infty} h(k) y(n-k))^2\} = \underline{0}$$

given $h(a)$ we have that $\forall a$:

$$\frac{d}{dh(a)} E \left\{ \left(x(n) - \sum_{k=-\infty}^{\infty} h(k) y(n-k) \right)^2 \right\} = 0$$

By using the chain rule and linearity of the expectation:

$$-2 E \left\{ \left(x(n) - \sum_{k=-\infty}^{\infty} h(k) y(n-k) \right) y(n-a) \right\} = 0$$

$\forall a \quad \forall n$

defining $l = n - a$ we have that:

$$E \left\{ \left(x(n) - \sum_{k=-\infty}^{\infty} h(k) y(n-k) \right) y(l) \right\} = 0 \quad \forall n, \forall l$$

b) Show that the frequency function $H(\nu) = \mathcal{F}\{h(k)\} = \sum_{k=-\infty}^{\infty} h(k) e^{-j2\pi\nu k}$

of the Wiener filter is $H(\nu) = \frac{R_{xy}(\nu)}{R_y(\nu)}$

$$E \left\{ \left(x(n) - \sum_{k=-\infty}^{\infty} h(k) y(n-k) \right) y(l) \right\} = 0$$

$$E \{ x(n) y(l) \} - \sum_{k=-\infty}^{\infty} h(k) E \{ y(n-k) y(l) \} = 0 \quad \forall n, \forall l$$

$$R_{xy}(n-l) - \sum_{k=-\infty}^{\infty} h(k) R_y(n-k-l) = 0 \quad n-l =$$

$$R_{xy}(n-l) - \sum_{k=-\infty}^{\infty} h(k) R_y(n-k-l) = R_{xy}(n-l) - h(n) * R_y(n-l)$$

$$\{n-l=m\} \rightarrow R_{xy}(m) - \sum_{k=-\infty}^{\infty} h(k) R_y(m-k) = R_{xy}(m) - h(m) * R_y(m) \\ = \emptyset$$

We take Fourier transform on both sides $\Rightarrow R_{xy}(\nu) - H(\nu) R_y(\nu) = \emptyset$

$$H(\nu) = \frac{R_{xy}(\nu)}{R_y(\nu)} \quad \rfloor$$