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- **Part 4**
  - Lec 12: Wireless Sensor Network Control Systems 1
Previous lecture

How to synchronize nodes?
Today’s learning goals

- How the process state dynamics over time are mathematically modeled?

- How such state dynamics can be controlled by closing the loop process→controller→process?

- How to discretize the continuous time model of the dynamics?

- What is the concept of state stability of closed loop control systems?
Outline

- Wireless Sensor Network Control Systems (WSNCS)
- State space description of a control system
- Stability and asymptotic stability of a control system
Outline

- **Wireless Sensor Network Control Systems (WSNCS)**

- State space description of a control system
  - Linear model
    - Continuous time description
    - Discretization of state space model
  - Non-linear model

- Stability and asymptotic stability of a control system
Closed-loop system

\[ x(t) \]
\[ x(kh) \] sampled state

Plant/Process

\[ y(t) \]
\[ y(kh) \] sampled

communication line

WSN
IEEE 802.15.4
Wireless HART

actuation line

\[ u(t) \]
\[ u(kh) \] sampled

Controller
Wireless Sensor Network Control Systems

\( k \): discrete time
\( h \): sampling interval

- \( u(kh) \): control decision
- \( x(kh) \): state of the process/plant
- \( y(kh) \): output of the state (measured by sensors)

- The **GOAL** of the controller is to bring the state \( x(kh) \) in a desired region by taking measurements \( y(kh) \) and a control decision \( u(kh) \)
- Delay and packet loss probability affect the way the measurements \( y(t) \) are received in the controller

This lecture gives the basic control theory background for WSNCS. The effect of the network on the controller is studied next lecture
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- Stability and asymptotic stability of a control system
Continuous time description

Let $x(t)$ be the state of the system (temperature, position, ...)
We assume that the physical process is described by the time-invariant state space model

**Linear model**

\[
\frac{dx(t)}{dt} \triangleq \dot{x}(t) = Ax(t) + Bu(t) \quad \text{state model} \tag{1}
\]

\[y(t) = Cx(t) + Du(t) \quad \text{measurement model}\]

where $A, B, C, D$ are assumed to be known matrices
Continuous time description

Assuming that $x(kh)$ is known, then the solution of the differential equation (1) is

$$x(t) = e^{A(t-kh)} \cdot x(kh) + \int_{kh}^{t} e^{A(t-\tau)} Bu(\tau) d\tau \quad t > kh \quad (2)$$

The control decision $u(t)$ can be properly chosen to bring the system state $x(t)$ in a desired region.
Example: "The step response"

Suppose \( x(0) = 0 \) and \( x(t) \in \mathbb{R} \)

The step response is defined as the evolution of the state \( x(t) \), i.e., the solution of (1), for an input \( u(t) = \begin{cases} 
0 & t \leq 0 \\
1 & t > 0 
\end{cases} \)

The state may evolve to a stabilized condition after possible oscillations
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Discretization of state space model

Assume $u(t)$ constant in the interval $kh \leq t \leq kh + h$

Then, (2) becomes

$$x(t) = e^{A(t-kh)} \cdot x(kh) + \int_{kh}^{t} e^{A(t-\tau)} B u(\tau) \, d\tau = e^{A(t-kh)} \cdot x(kh) + \int_{0}^{t-kh} e^{A\tau} B u(kh) \, d\tau = \phi_{t} x(kh) + \Gamma_{t} u(kh)$$

Let $t = kh + h$

$$x(kh + h) = \phi x(kh) + \Gamma u(kh) \quad (3)$$

where $\phi = e^{Ah}$ and $\Gamma = \int_{0}^{h} e^{A\tau} B \, d\tau$
Discretization of state space model

Recursively from (3),

\[ x(kh + 2h) = \phi x(kh + h) + \Gamma u(kh + h) \]

Therefore, the solution of (3), given \( x(0) \) and \( u(kh) \) \( \forall k \), is

\[ x(kh) = \phi^k x(0) + \sum_{j=0}^{k-1} \phi^{k-1-j} \Gamma u(jh) \]

Note that the matrix exponential \( e^{Ah} \) can be expressed equivalently as a power series

\[ e^{Ah} = I + Ah + \frac{A^2h^2}{2} + \ldots \]
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  - Stability and asymptotic stability of a control system
Non-linear model of the state

Observation

Control decision is chosen as a function of the state

\[ u(t) = f(x(t)) \]

Therefore, consider a state that evolves according to a non-linear law

\[ \dot{x}(t) = a(x(t)) \]
\[ y(t) = c(x(t)) \]

where \( a \) and \( c \) are be non-linear functions in general

What is the solution of that system?
Non-linear model of the state

Non-linear differential equation

\[ x(t + kh) = x(t) + \int_{t}^{t+kh} a(x(\tau)) \, d\tau \]

- In general, the integral difficult to solve.
- However, by deriving opportune upper bounds of \( a(x(t)) \), it is possible to show important properties of the system (e.g., stability).
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- Stability and asymptotic stability of a control system
Stability

Let us consider the discrete-time differential equation

\[ x(kh + h) = g(x(kh)) \] \hspace{1cm} (4)

where \( g \) can be linear or non-linear

**Definition**

A specific solution of (4), \( x^*(kh) \), is called stable, if \( \forall \varepsilon > 0 \ \exists \delta(\varepsilon) : \forall \) other solution \( x(kh) \)

\[ \|x(0) - x^*(0)\| \leq \delta \Rightarrow \|x(kh) - x^*(kh)\| \leq \varepsilon \ \forall k \]
Asymptotic stability

We consider the same equation as on the previous slide:

\[ x(kh + h) = g(x(kh)) \]  (5)

where \( g \) can be linear or non-linear.

**Definition**

A specific solution \( x^*(k) \) of (5) is called **asymptotically stable** if it is stable and if there is a \( \delta > 0 \) such that for every other solution \( x(k) \) it holds that:

\[
\|x(0) - x^*(0)\| \leq \delta \Rightarrow \|x(k) - x^*(k)\| \to 0 \quad \text{as} \quad k \to \infty
\]
Example 1

Assume that $x^*(kh) = 0$ is a solution of (5). The figure shows the typical behaviour of other solutions in case $x^*(kh)$ is stable or asymptotically stable.
The linear case

Consider a linear difference equation

\[ x(kh + h) = A \cdot x(kh) \]  \hspace{1cm} (6)

where \( A \) is a known matrix \( \in \mathbb{R}^{n \times n} \).

**Definition**

A linear difference equation of the form (6) is (asymptotically) stable if the constant solution \( x^*(k) = 0 \) is (asymptotically) stable.

How do we choose matrix \( A \) in order to have

1. stability?
2. asymptotic stability?
The linear case

The answer is given by the following theorem:

**Theorem (Stability of linear difference equations)**

Let $\rho(A) = \max\{|\lambda|, \lambda \text{ is an eigenvalue of } A\}$.

(i) $x(kh + h) = A \cdot x(kh)$ is stable if and only $\rho(A) \leq 1$.

(ii) $x(kh + h) = A \cdot x(kh)$ is asymptotically stable if and only $\rho(A) < 1$. 
The linear case: intuition

1. \( \|x(0) - 0\| \leq \delta \Rightarrow \|x(kh)\| \leq \varepsilon \)?

\[
\|x(kh)\| = \|A \cdot x((k - 1)h)\| = \ldots = \|A^k \cdot x(0)\| \leq \|A^k\| \cdot \|x(0)\| \leq \|A\|^k \cdot \delta
\]

To achieve stability, choose matrix \(A\) that does not grow with \(k\)

This is when the maximum absolute eigenvalue of \(A\), \(\rho(A) \leq 1\)

2. \(\lim_{k \to \infty} \|x(kh)\| \leq \lim_{k \to \infty} \|A\|^k \cdot \delta \to 0\)

In this case, \(\rho(A) < 1\)

- In scalar case, \(A\) is constant and \(\rho(A) = A\)
- If the eigenvalues are larger than 1 \(\Rightarrow\) instability
Summary

- We have seen the basic aspects of control systems
  - Mathematical description of the state evolution
  - Discretization
  - Stability
Next lecture

- WSNCS, robustness to packet delays and losses