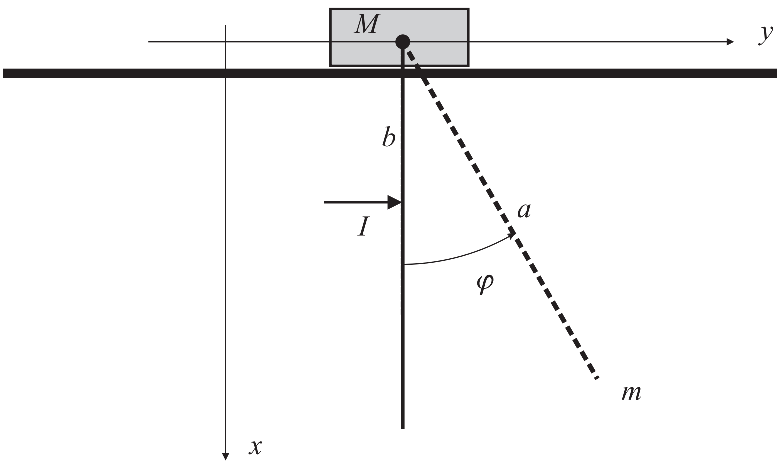
Rigid Body Dynamics, SG2150

Solutions to Exam, 2013 10 25

Calculational problems

Problem 1: A block of mass M can slide freely along a straight horizontal track. A rod of mass m and length a is hinged with one end at the center of mass of the block so that it can rotate freely about a horizontal axis perpendicular to the track. At the equilibrium position, with the rod straight down and the system at rest, the rod is impacted with an impulse I perpendicular to the rod and parallel to the track at a distance b below the hinge. Find the values of b that a) makes the angular velocity of the rod $\dot{\varphi}$ zero after the impact and b) makes the translational velocity $\dot{y}$ zero after impact.



Solution 1: The kinetic energy becomes

$$T = \frac{1}{2}(M+m)\dot{y}^2 + \frac{1}{2}m\frac{a^2}{3}\dot{\varphi}^2 + \frac{1}{2}ma\,\dot{y}\dot{\varphi}\cos\varphi.$$

The work of the force $F(t)$ whose time integral is I can be written (at $\varphi = 0$, the value at impact)

$$\delta W = \delta W_y + \delta W_\varphi = Q_y \mathrm{d}y + Q_\varphi \mathrm{d}\varphi = F\mathrm{d}y + Fb\mathrm{d}\varphi$$

so $Q_y = F$ and $Q_\varphi = bF$. The generalized impulses are thus $I_y = I$ and $I_\varphi = bI$.

Lagrange's impact equations ($p_a \equiv \partial L/\partial \dot{q}_a = \partial T/\partial \dot{q}_a$) are

$$\begin{aligned} p_y(t+\tau) - p_y(t) &= I_y \\ p_\varphi(t+\tau) - p_\varphi(t) &= I_\varphi \end{aligned}$$

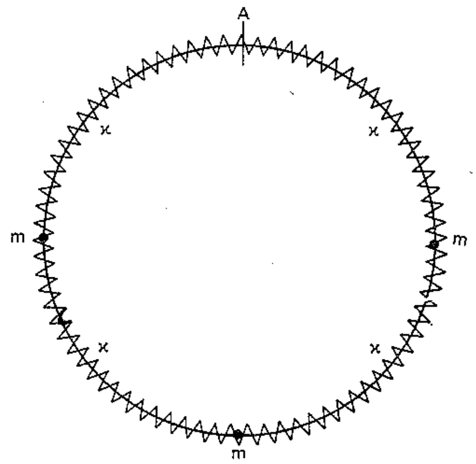
and these now give

$$\begin{aligned} (M+m)\dot{y} + \frac{1}{2}ma\dot{\varphi} &= I \\ m\frac{a^2}{3}\dot{\varphi} + \frac{1}{2}ma\dot{y} &= bI \end{aligned}$$

If we put $\dot{\varphi} = 0$ here we find that b must be $b = (m/2)a/(m+M)$. One notes that this is the center of mass position. If we instead put $\dot{y} = 0$ we find $b = 2a/3$, so:

Answer: a) For $b = (m/2)a/(m+M)$ there is no angular velocity after impact.
b) For $b = 2a/3$ there is no translational velocity of the block after impact.

Problem 2: Find the angular frequencies for the system shown: three particles, all of mass m, move freely along a smooth horizontal fixed ring. Four equal springs of stiffness κ along the smooth ring connect the particles and a fixed point A on the ring. The figure shows the equilibrium positions. (Hint: the secular equation factors into a linear and a quadratic; no need to solve a cubic.)



Solution 2: The kinetic energy of the three particles is,

$$T = \tfrac{1}{2}m(\dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^2)$$

where $u_a, a = 1, 2, 3$, are the deviations of the particles from the equilibrium positions. The corresponding potential energy of the four springs is

$$V = \tfrac{1}{2}\kappa[u_1^2 + (u_2 - u_1)^2 + (u_3 - u_2)^2 + u_3^2] =$$

$$\tfrac{1}{2}\kappa(2u_1^2 + 2u_2^2 + 2u_3^2 - 2u_1u_2 - 2u_2u_3)$$

From this we can read off the M- and K-matrices:

$$\mathbf{M} = m\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{K} = \kappa\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

If we put $\Omega^2 = \kappa/m$ and $x = \omega^2$ we get the secular equation

$$\begin{vmatrix} -x + 2\Omega^2 & -\Omega^2 & 0 \\ -\Omega^2 & -x + 2\Omega^2 & -\Omega^2 \\ 0 & -\Omega^2 & -x + 2\Omega^2 \end{vmatrix} = 0$$

which gives

$$(-x + 2\Omega^2)^3 - 2(-x + 2\Omega^2)\Omega^4 = 0$$

$$(-x + 2\Omega^2)[(-x + 2\Omega^2)^2 - 2\Omega^4] = 0$$

The roots are $x_{1,2} = (2 \pm \sqrt{2})\Omega^2, \;\; x_3 = 2\Omega^2$, so the angular frequencies become

Answer: $\omega_{1,2} = \sqrt{2 \pm \sqrt{2}}\sqrt{\kappa/m}$ and $\omega_3 = \sqrt{2}\sqrt{\kappa/m}$.

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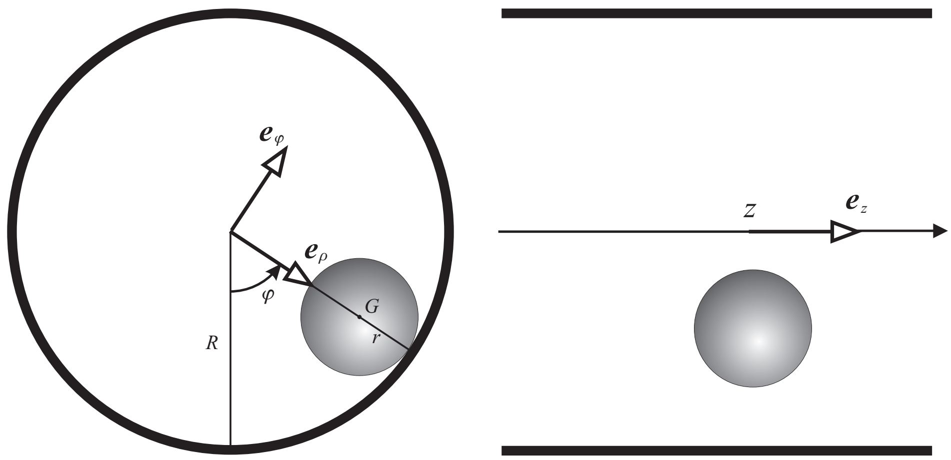
Problem 3: A ball of mass m and radius r rolls inside a fixed horizontal cylindrical pipe of radius R. Use cylindrical coordinates ρ, φ, z for the center of mass of the ball, with z-axis along the axis of the cylinder, as indicated in the figure below. Also use cylindrical components of the angular velocity vector, $\boldsymbol{\omega} = \omega_\rho\, \boldsymbol{e}_\rho + \omega_\varphi\, \boldsymbol{e}_\varphi + \omega_z\, \boldsymbol{e}_z$. Find the equations of motion, in particular, the equation for the φ motion, which decouples from the others, and the (two) relations between $\dot{\varphi}$, ω_ρ and ω_φ, and their time derivatives, $\dot{\omega}_\rho$ and $\dot{\omega}_\varphi$.



Solution 3: Kinematics: Center of mass position and velocity:

$$\boldsymbol{r}_G = (R-r)\,\boldsymbol{e}_\rho + z\,\boldsymbol{e}_z, \quad \boldsymbol{v}_G = (R-r)\dot{\varphi}\,\boldsymbol{e}_\varphi + \dot{z}\,\boldsymbol{e}_z \tag{1}$$

Center of mass acceleration:

$$\dot{\boldsymbol{v}}_G = -(R-r)\dot{\varphi}^2\,\boldsymbol{e}_\rho + (R-r)\ddot{\varphi}\,\boldsymbol{e}_\varphi + \ddot{z}\,\boldsymbol{e}_z \tag{2}$$

Angular velocity is $\boldsymbol{\omega} = \omega_\rho\,\boldsymbol{e}_\rho + \omega_\varphi\,\boldsymbol{e}_\varphi + \omega_z\,\boldsymbol{e}_z$ so using $\dot{\boldsymbol{e}}_\rho = \dot{\varphi}\boldsymbol{e}_\varphi$, $\dot{\boldsymbol{e}}_\varphi = -\dot{\varphi}\boldsymbol{e}_\rho$ we find for the angular acceleration

$$\dot{\boldsymbol{\omega}} = (\dot{\omega}_\rho - \omega_\varphi\dot{\varphi})\,\boldsymbol{e}_\rho + (\dot{\omega}_\varphi + \omega_\rho\dot{\varphi})\,\boldsymbol{e}_\varphi + \dot{\omega}_z\,\boldsymbol{e}_z. \tag{3}$$

The rolling condition requires that the velocity of the contact point of the ball with the pipe wall is zero and this gives

$$\mathbf{0} = \boldsymbol{v}_G + \boldsymbol{\omega}\times r\,\boldsymbol{e}_\rho = [(R-r)\dot{\varphi}\,\boldsymbol{e}_\varphi + \dot{z}\,\boldsymbol{e}_z] + r[\omega_z\,\boldsymbol{e}_\varphi - \omega_\varphi\,\boldsymbol{e}_z] \tag{4}$$

so the two components give the rolling constraints,

$$(R-r)\dot{\varphi} + r\omega_z = 0 \tag{5}$$
$$\dot{z} - r\omega_\varphi = 0. \tag{6}$$

Now to dynamics. The equations of motion are

$$m\dot{\boldsymbol{v}}_G = \boldsymbol{F} \tag{7}$$
$$J_G\dot{\boldsymbol{\omega}} = \boldsymbol{M}_G \tag{8}$$

The force is the sum of gravity, normal force and friction force

$$\boldsymbol{F} = mg(\cos\varphi\,\boldsymbol{e}_\rho - \sin\varphi\,\boldsymbol{e}_\varphi) - N\,\boldsymbol{e}_\rho + (f_\varphi\,\boldsymbol{e}_\varphi + f_z\,\boldsymbol{e}_z). \tag{9}$$

The moment of force with respect to center of mass G is then

$$\boldsymbol{M}_G = r\,\boldsymbol{e}_\rho \times (f_\varphi\,\boldsymbol{e}_\varphi + f_z\,\boldsymbol{e}_z) = -rf_z\,\boldsymbol{e}_\varphi + rf_\varphi\,\boldsymbol{e}_z. \tag{10}$$

Equation (7) with (2) and (9) then gives the three component equations

$$-m(R-r)\dot{\varphi}^2 = mg\cos\varphi - N \quad (11)$$
$$m(R-r)\ddot{\varphi} = -mg\sin\varphi + f_\varphi \quad (12)$$
$$m\ddot{z} = f_z \quad (13)$$

Equation (8) with (3) and (10) further gives the three component equations

$$J_G(\dot{\omega}_\rho - \omega_\varphi\dot{\varphi}) = 0 \quad (14)$$
$$J_G(\dot{\omega}_\varphi + \omega_\rho\dot{\varphi}) = -rf_z \quad (15)$$
$$J_G\dot{\omega}_z = rf_\varphi \quad (16)$$

Here $J_G = (2/5)mr^2$ for the solid ball.

We can use equation (16) and the time derivative of equation (5), i.e. $(R-r)\ddot{\varphi} - r\dot{\omega}_z = 0$, to find f_φ,

$$f_\varphi = (J_G/r)\dot{\omega}_z = (2/5)mr[-(R-r)\ddot{\varphi}/r] \quad (17)$$

Insertion of this in (12) then gives

$$m(R-r)\ddot{\varphi} = -mg\sin\varphi - (2/5)m(R-r)\ddot{\varphi}. \quad (18)$$

The decoupled equation for the φ-motion is then the **Answer:**

$$\ddot{\varphi} = -\frac{5g}{7(R-r)}\sin\varphi. \quad (19)$$

For the relations between angular velocities we note that we already have equation (14). To get the other one we combine equations (15) and (13) with the time derivative of equation (6), which is $\ddot{z} = r\dot{\omega}_\varphi$. This gives us the **Answer:**

$$\dot{\omega}_\rho - \omega_\varphi\dot{\varphi} = 0 \quad (20)$$
$$7\dot{\omega}_\varphi + 2\omega_\rho\dot{\varphi} = 0 \quad (21)$$

This concludes the solution of this problem.

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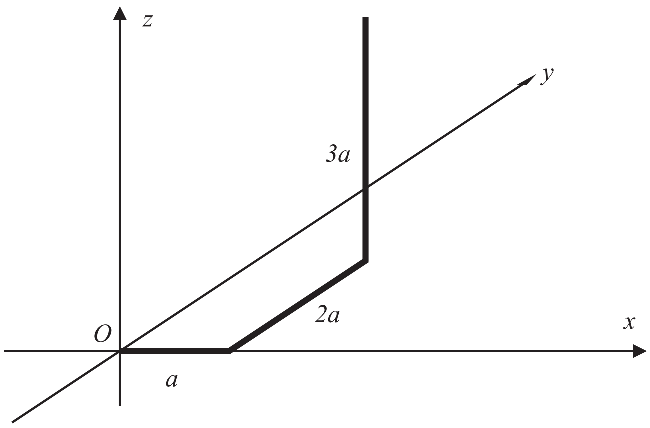
Idea problems:

Problem 4: Three rods of lengths $a, 2a$, and $3a$ and mass $m, 2m$, and $3m$ respectively, are welded together at the end points at right angles, see figure below. Find the inertia matrix $\mathbf{J}_O$ of the body in the indicated coordinate system, with origin at one end of the rod of length a.



Solution 4:
We need the inertia matrices for rod 1, plus the inertia matrices for rods 2 and 3 with respect to their centers of mass, plus the two Steiner contributions from the centers of mass of rods 2 and 3. The first three give

$$\mathbf{J}_0 = \frac{1}{3}ma^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{12}(2m)(2a)^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{12}(3m)(3a)^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The center of mass of rod 2 is $\boldsymbol{r}_2 = a\,\boldsymbol{e}_x + a\,\boldsymbol{e}_y$ and the center of mass of rod 3 is at $\boldsymbol{r}_3 = a\,\boldsymbol{e}_x + 2a\,\boldsymbol{e}_y + (3/2)a\,\boldsymbol{e}_z$. The formula for a Steiner contribution is,

$$\mathbf{J}_i = m_i \begin{pmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & x_i^2 + z_i^2 & -y_i z_i \\ -x_i z_i & y_i z_i & x_i^2 + y_i^2 \end{pmatrix}$$

We must thus add

$$\mathbf{J}_S = 2ma^2 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + 3ma^2 \begin{pmatrix} 25/4 & -2 & -3/2 \\ -3/2 & 13/4 & -3 \\ -3/2 & -3 & 5 \end{pmatrix} = ma^2 \begin{pmatrix} 83/4 & -8 & -9/2 \\ -8 & 47/4 & -9 \\ -9/2 & -9 & 19 \end{pmatrix}.$$

We thus get **Answer:**

$$\mathbf{J}_O = ma^2 \begin{pmatrix} 35/12 & 0 & 0 \\ 0 & 31/12 & 0 \\ 0 & 0 & 1 \end{pmatrix} + ma^2 \begin{pmatrix} 83/4 & -8 & -9/2 \\ -8 & 47/4 & -9 \\ -9/2 & -9 & 19 \end{pmatrix} = ma^2 \begin{pmatrix} 71/3 & -8 & -9/2 \\ -8 & 43/3 & -9 \\ -9/2 & -9 & 20 \end{pmatrix}.$$

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Problem 5: Derive Euler's dynamic equations for the components of the angular velocity vector of a rigid body in the body fixed principal axes system. Solve them for the special case of a free symmetric top.

Solution 5:
See Section 4.2, pages 64-65, in Essén's *Dynamics of Bodies.* For the free symmetric top solution see Example 5.2 (page 79) in the same text.

Problem 6: Derive the Euler-Lagrange equation from the variational principle of least action, i.e. assuming that the action $S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q})\, \mathrm{d}t$ has a minimum for the real path $q(t)$ between given fixed end points $q_1 = q(t_1)$, $q_2 = q(t_2)$.
Solution 6:
This is shown in Section 18.1, pages 24-25, in Essén's *The Theory of Lagrange's Method* (2013).

Each problem gives maximum 3 points, so that the total maximum is 18. Grading: 1-3, F; 4-5, FX; 6, E; 7-9, D; 10-12, C; 13-15, B; 16-18; A.
Allowed equipment: Handbooks of mathematics and physics. One A4 size page with your own compilation of formulas.

HE and AN, 2013 10 25