KTH Teknikvetenskap

SF1624 Algebra och geometri
Exam
Friday, October 23, 2015

Time: 08:00am–1:00pm
No books/notes/calculators etc. allowed
Examiner: Tilman Bauer

This exam consists of nine problems, each worth 4 points.

Part A comprises the first three problems. The bonus points from the seminars will be automatically added to the total score of this part, which however cannot exceed 12 points.

The next three problems constitute part B, and the last three problems part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

To get full score on a problem, your solution must be well-presented and easy to follow. In particular, you should define your notation; clearly explain the logical structure of your argument in words or symbols; and motivate and explain your argument. Solutions severely lacking in these respects will achieve at most 2 points.

p.t.o.

PART A

1. The following points are given:

$$A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

in $\mathbb{R}^3$.

   (a) Find an equation for the plane H which goes through the origin 0 and through the points A and B. **(2 p)**

   (b) Find a parametric form for the line l which is orthogonal to the plane H and contains the point $Q = (1, 1, -1)$. **(1 p)**

   (c) Compute the distance between the point Q and the plane H. **(1 p)**

2. Let $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ be the standard basis of $\mathbb{R}^3$. Consider the linear map $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ which is given by

$$F(\vec{e_1}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad F(\vec{e_2}) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \text{and} \quad F(\vec{e_3}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

   (a) Compute $F(\vec{v})$, where $\vec{v} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$. **(1 p)**

   (b) Why is the image (range) of F all of $\mathbb{R}^2$? **(1 p)**

   (c) Find a basis for the image $\operatorname{Im}(F)$. **(1 p)**

   (d) Find a basis for $\operatorname{Ker}(F)$. **(1 p)**

3. For some constants a, b, the map $L\colon \mathbf{R}^4 \to \mathbf{R}^4$ is given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} ax_1 - x_3 \\ x_2 + x_4 \\ x_3 - x_4 \\ bx_1 + x_2 + x_4 \end{bmatrix}.$$

   (a) Use the determinant to determine all a, b such that L is invertible. **(2 p)**

   (b) Let $a = b = 1$. Compute the inverse map L^{-1} for this case. **(2 p)**

PART B

4. The matrix
$$A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$$
has eigenvalues 3 and 6. Compute an orthonormal basis of eigenvectors for A. **(4 p)**

5. The vector space $V \subset \mathbb{R}^4$ is spanned by the vectors
$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}.$$
   (a) Compute a basis B for V. **(2 p)**
   (b) For which number a does the vector
$$\vec{w} = \begin{bmatrix} 7 \\ -4 \\ 3 \\ a \end{bmatrix},$$
   lie in V? Compute the coordinates for $\vec{w}$ in the basis B. **(2 p)**

6. The matrix representation of the linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ in the basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$ is given by the matrix
$$D = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$
Compute $T^n\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right)$ for all integers $n > 0$. **(4 p)**

p.t.o.

PART C

7. A plane H in $\mathbb{R}^3$ contains the point $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. A light ray passes through the point $P = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, hits the plane H in the point A, is reflected, and then passes through the point $Q = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$. Find a nonzero normal vector for the plane H. **(4 p)**

8. Let a, b, c, and d be real constants with $a < b < c < d$. Show that the system of equations

$$\begin{aligned} x + y + z + w &= k_1 \\ ax + by + cz + dw &= k_2 \\ a^2x + b^2y + c^2z + d^2w &= k_3 \\ a^3x + b^3y + c^3z + d^3w &= k_4 \end{aligned}$$

with respect to x, y, z, and w has exactly one solution, however the numbers $k_1, \ldots, k_4$ are chosen. **(4 p)**

9. Let Λ be a nonzero eigenvalue of a quadratic, invertible matrix A. Show that Λ^{-1} is an eigenvalue of A^{-1}. **(4 p)**