KTH Teknikvetenskap

SF1625 Calculus in one variable
Solutions to exam 2015-10-27

Del A

1. Let the function f be given by $f(x) = 1 + x + \dfrac{4}{(x-2)^2}$.

   A. Determine the domain of definition of f.
   B. Find the intervals where f is increasing and decreasing, respectively.
   C. Find all local extreme values of f.
   D. Find all asymptotes to the graph $y = f(x)$.
   E. Sketch, using the above, the graph $y = f(x)$.

Solution. A. The domain of definition is all $x \neq 2$.

B. We differentiate and obtain

$$f'(x) = 1 - \frac{8}{(x-2)^3},$$

that exists for all $x \neq 2$. We see that $f'(x) = 0 \iff x = 4$. We study the sign of the derivative:

If $x < 2$ then $f'(x)$ is positive.

If $2 < x < 4$ then $f'(x)$ negative.

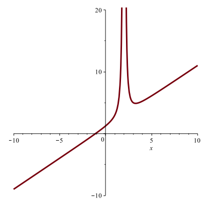
If $x > 4$ then $f'(x)$ positive.

It follows that f is strictly increasing in the interval $x < 2$, strictly decreasing in the interval $2 < x < 4$ and strictly increasing in the interval $x > 4$,

C. It follows from the above that f has exactly one extremal point, namely a local minimum at $x = 4$.

D. Since $\lim_{x\to 2} f(x) = \infty$ the line $x = 2$ is a vertical asymptote to $y = f(x)$. Since $\lim_{x\to\pm\infty} 4/(x-2)^2 = 0$ the line $y = x + 1$ is an oblique asymptote in $\pm\infty$.

E. Now we can sketch the graph:



□

Answer: A. All $x \neq 2$. B. Strictly increasing in $x < 2$, strictly decreasing in $2 < x < 4$ and strictly increasing in $x > 4$. C. A local minimum at $x = 4$. D. Vertical asymptote $x = 2$, oblique asymptote $y = x + 1$ in $\pm\infty$. E. See above

2. Compute the integrals:

A. $\displaystyle\int_0^2 \frac{x}{(x^2+4)^{1/3}}\,dx$ (you may use the substitution $u = x^2 + 4$)

B. $\displaystyle\int_1^4 \sqrt{x}\ln x\,dx$ (you may want to integrate by parts)

Solution. A. We use the substitution $u = x^2 + 4$, with $du = 2x\,dx$ and the new interval of integration from 4 to 8, and obtain:

$$\int_0^2 \frac{x}{(x^2+4)^{1/3}}\,dx = \frac{1}{2}\int_4^8 \frac{du}{u^{1/3}} = \left[\frac{3u^{2/3}}{4}\right] = 3 - \frac{3}{2^{2/3}}.$$

B. We integrate by parts and obtain:

$$\int_1^4 \sqrt{x}\ln x\,dx = \left[\frac{x^{3/2}\ln x}{3/2}\right]_1^4 - \int_1^4 \frac{\sqrt{x}}{3/2}\,dx = \frac{16\ln 4}{3} - \frac{28}{9}.$$

□

Answer: A. $3 - \frac{3}{2^{2/3}}$. B. $\frac{16\ln 4}{3} - \frac{28}{9}$.

3. What is the largest possible area of a right triangle, if the hypotenuse and one of the other sides together has a total length of 1 meter?

   Solution. Let the length of the hypotenuse be $1-x$ meter. Then one side has length x meter and the other side by the Pythagorean theorem has length $\sqrt{1-2x}$ meter. The area is

$$A(x) = \frac{1}{2}x\sqrt{1-2x}, \qquad \text{where } 0 < x < 1/2.$$

   We differentiate and obtain

$$A'(x) = \frac{1}{2}\left(\sqrt{1-2x} - x \cdot \frac{2}{2\sqrt{1-2x}}\right) = \frac{1-3x}{2\sqrt{1-2x}}.$$

   We see that $A'(x) = 0 \iff x = 1/3$.
   If $0 < x < 1/3$ then $A'(x) > 0$ and hence $A(x)$ is increasing.
   If $1/3 < x < 1/2$ then $A'(x) < 0$ and hence $A(x)$ is decreasing
   It follows that the largest possible are is $A(1/3) = \sqrt{3}/18$ square meters.

   □

   Answer: $\sqrt{3}/18$ square meters

DEL B

4. Let $f(t) = e^t - \sin t - \cos t$..
   A. Find the Taylor polynomial of degree 2 around the point $t = 0$ to the function f.
   B. State the error term (in some form)
   C. Compute the limit $\lim_{t\to 0} \frac{f(t)}{t^2}$..

Solution. A. Using the known expansions for e^t, $\sin t$ and $\cos t$ (or by differentiating etcetera) we get the second degree Taylor polynomial around the origin to f to be $p(t) = t^2$..
B. The error is $B(t)t^3$ for some function $B(t)$ bounded near the origin.
C. Using the above we obtain, for some B bounded near the origin:

$$\lim_{t\to 0} \frac{f(t)}{t^2} = \lim_{t\to 0} \frac{t^2 + B(t)t^3}{t^2} = \lim_{t\to 0}(1 + B(t)t) = 1.$$

□

Answer: A. $p(t) = t^2$. B. $B(t)t^3$ for some B bounded near the origin. C. 1

5. Compute the integral $\int_0^1 \arcsin x \, dx$.

   (For a maximum score, an exact computation is required, but an approximate computation might give some points. Simplify your answer.)

   Solution. We compute the integral using integration by parts in the first step:

$$\int_0^1 \arcsin x \, dx = [x \arcsin x]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} \, dx = \frac{\pi}{2} + [\sqrt{1-x^2}]_0^1 = \frac{\pi}{2} - 1.$$

   (If you want to approximate the integral instead, you can use for instance a Riemann sum, the trapezoidal rule or Taylor expansion of the integrand.)

   □

   Answer: $\pi/2 - 1$

6. The charge $q(t)$ in the capacitor in a certain alternating current circuit satisfies the differential equation

$$\frac{d^2q}{dt^2} + 2\frac{dq}{dt} + 2q = 5\cos t$$

with initial values $q(0) = 1$ and $q'(0) = 3$.

A. Determine the charge in the capacitor at time t.

B. Determine the long term behavior of the charge in the capacitor.

Solution. We observe that $q(t) = q_h(t) + q_p(t)$ where q_h is the full solution to the corresponding homogeneous differential equation and q_p is some particular solution to the given equation.

We seek q_h. The characteristic equation $r^2 + 2r + 2 = 0$ has solution $r = -1 \pm i$ and so

$$q_h(t) = e^{-t}(A\cos t + B\sin t), \quad A, B \text{ arbitrary constants.}$$

We seek q_p and guess $q_p(t) = c\cos t + d\sin t$. After differentiating, and substituting into the differential equation and identifying constants we see that a particular solution is

$$q_p(t) = \cos t + 2\sin t.$$

Therefore the solution to the given differential equation is

$$q(t) = e^{-t}(A\cos t + B\sin t) + \cos t + 2\sin t, \quad A, B \text{ arbitrary constants.}$$

The initial condition $q(0) = 1$ gives $A = 0$ and the condition $q'(0) = 3$ gives $B = 1$ and so the charge q at the time t is given by

$$q(t) = e^{-t}\sin t + \cos t + 2\sin t.$$

B. Since $\lim_{t\to\infty} e^{-t}\sin t = 0$ the long term behavior is $\cos t + 2\sin t$.

□

Answer: A. $q(t) = e^{-t}\sin t + \cos t + 2\sin t$.

B. $\cos t + 2\sin t$.

DEL C

7. This is about the theory of differentiation and integration.
   A. Formulate the product rule of differentiation.
   B. Prove the product rule of differentiation.
   C. Formulate the rule for integration by parts.
   D. Prove the rule for integration by parts.

   Solution. See the text book.

   □

   Answer:

8. Let $F(x) = \int_0^x e^{-t^2} \cos t\, dt$ with domain of definition $D = [0, \pi]$.
   A. On what intervals of D is F increasing and decreasing, respectively.
   B. Find points a och b in D such that
      $F(a) \leq F(x)$ for all $x \in D$,
      $F(b) \geq F(x)$ for all $x \in D$.

Solution. We see that F is continuous on the closed and bounded interval $[0, \pi]$ and so the existence of points a and b with the stated properties is granted. They can be critical points, endpoints or singular points. We differentiate and obtain $F'(x) = e^{-x^2} \cos x$ that exists for all x such that $0 < x < \pi$. No singular points exist. We study $F'(x)$:
+ on $0 < x < \pi/2$ we have that $F'(x) > 0$
+ at $x = \pi/2$ we have that $F'(x) = 0$
+ on $\pi/2 < x < \pi$ we have that $F'(x) < 0$

It follows that F is strictly increasing on $[0, \pi/2]$ and strictly decreasing on $[\pi/2, \pi]$.

The above shows that F has a local and global maximum at $x = \pi/2$, so if we choose $b = \pi/2$ we have that $F(b) \geq F(x)$ för alla $x \in D$.

The minimum value of F must be attained at one of the endpoints of the interval, and so we need to compare $F(0)$ and $F(\pi)$. We observe that $F(0) = 0$. Further:

$$F(\pi) = \int_0^\pi e^{-t^2} \cos t\, dt = \int_0^{\pi/2} e^{-t^2} \cos t\, dt + \int_{\pi/2}^\pi e^{-t^2} \cos t\, dt.$$

Since $e^{-t^2} \cos t > 0$ on $[0, \pi/2)$ we get $\int_0^{\pi/2} e^{-t^2} \cos t\, dt > 0$ and since $e^{-t^2} \cos t < 0$ on $(\pi/2, \pi]$ we get $\int_{\pi/2}^\pi e^{-t^2} \cos t\, dt < 0$. Furthermore

$$\left| \int_0^{\pi/2} e^{-t^2} \cos t\, dt \right| > \left| \int_{\pi/2}^\pi e^{-t^2} \cos t\, dt \right|$$

because $\cos t$ is symmetric around $\pi/2$ and e^{-t^2} is decreasing. It follows that $F(\pi) > 0$. If we choose $a = 0$ we therefore have $F(a) \leq F(x)$ for all $x \in D$.

□

Answer: A. F is strictly increasing in $[0, \pi/2]$ and strictly decreasing in $[\pi/2, \pi]$.
B. Choose $a = 0$ and $b = \pi/2$

9. Compute the limit $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2} \arctan \frac{k}{n}$.

Solution. The sum $\sum_{k=1}^{n} \frac{k}{n^2} \arctan \frac{k}{n}$ is a Riemann sum with n equal subintervals to the integral

$$\int_0^1 x \arctan x \, dx.$$

Since the integrand is continuous on the closed interval of integration, the sequence of Riemann sums converges to the integral as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2} \arctan \frac{k}{n} = \int_0^1 x \arctan x \, dx.$$

We compute the integral using integration by parts in the first step:

$$\begin{aligned}
\int_0^1 x \arctan x \, dx &= \left[\frac{x^2}{2} \arctan x\right]_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} \, dx \\
&= \frac{\pi}{8} - \frac{1}{2} \int_0^1 \left(1 - \frac{1}{1+x^2}\right) dx \\
&= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} \\
&= \frac{\pi - 2}{4}.
\end{aligned}$$

□

Answer: $\frac{\pi-2}{4}$