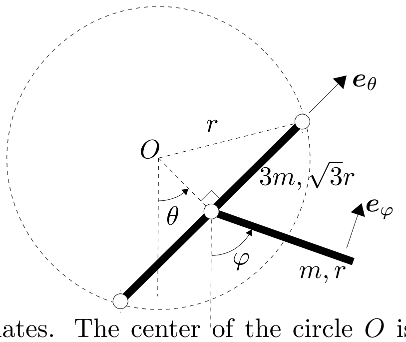
Rigid Body Dynamics (SG2150)
Solutions to Exam, 2015-10-29, 8.00-12.00

Problem 1.



Use θ and φ as generalised coordinates. The center of the circle O is a center of rotation for the longer rod. The center of mass G_1 of the longer rod is found to be a distance $\sqrt{r^2 - (\sqrt{3}r/2)^2} = r/2$ from O. The moment of inertia of the longer rod about O is then

$$J_1 = 3m\frac{(\sqrt{3}r)^2}{12} + 3m\left(\frac{r}{2}\right)^2 = \frac{3}{2}mr^2$$

and the kinetic energy of the longer rod is $T_1 = J_1\dot{\theta}^2/2$. The velocity of the center of mass G_2 of the shorter rod is

$$\boldsymbol{v}_{G_2} = \frac{r}{2}\left(\dot{\theta}\boldsymbol{e}_\theta + \dot{\varphi}\boldsymbol{e}_\varphi\right)$$

and the kinetic energy of the shorter rod is

$$T_2 = \frac{m}{2}|\boldsymbol{v}_{G_2}|^2 + \frac{mr^2}{12}\frac{\dot{\varphi}^2}{2}.$$

Expanding and noting that $\boldsymbol{e}_\theta \bullet \boldsymbol{e}_\theta = \cos(\varphi - \theta)$ we find

$$T = T_1 + T_2 = mr^2\left[\frac{7}{8}\dot{\theta}^2 + \frac{1}{4}\cos(\varphi - \theta)\dot{\theta}\dot{\varphi} + \frac{1}{6}\dot{\varphi}^2\right].$$

The potential energies of the rods in the gravity field are

$$V_1 = -3mg\frac{r}{2}\cos(\theta), \quad V_2 = -mg\frac{r}{2}\left(\cos(\theta) + \cos(\varphi)\right)$$

giving

$$V = V_1 + V_2 = -mgr\left[2\cos(\theta) + \frac{1}{2}\cos(\varphi)\right].$$

Lagrange's equations become

$$mr^2\left[\frac{7}{4}\ddot{\theta} + \frac{1}{4}\cos(\varphi - \theta)\ddot{\varphi} - \frac{1}{4}\sin(\varphi - \theta)\dot{\varphi}^2\right] + mgr2\sin(\theta) = 0$$

$$mr^2\left[\frac{1}{4}\cos(\varphi - \theta)\ddot{\theta} + \frac{1}{3}\ddot{\varphi} + \frac{1}{4}\sin(\varphi - \theta)\dot{\theta}^2\right] + mgr\frac{1}{2}\sin(\varphi) = 0$$

Equilibrium solutions are constant in time: $\theta(t) = \theta_0$, $\varphi(t) = \varphi_0$ and when this inserted into Lagrange's equations we find $\sin(\theta_0) = 0$, $\sin(\varphi_0) = 0$ giving four equilibrium points $\theta_0 = \{0, \pi\}$, $\varphi_0 = \{0, \pi\}$. The stable one is (obviously?) $\theta_0 = \varphi_0 = 0$.

Studying motion near the equilibrium point we write

$$\theta(t) = \theta_0 + u_\theta(t) = u_\theta(t), \quad \varphi(t) = \varphi_0 + u_\varphi(t) = u_\varphi(t)$$

and when inserted into Lagrange's equation and expanded to linear order in u, we find the linearized equations

$$\boldsymbol{M} \begin{bmatrix} \ddot{u}_\theta \\ \ddot{u}_\varphi \end{bmatrix} + \boldsymbol{K} \begin{bmatrix} u_\theta \\ u_\varphi \end{bmatrix} = \mathcal{O}(u)^3$$

where the mass and stiffness matrices are

$$\boldsymbol{M} = mr^2 \begin{bmatrix} \frac{7}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} \end{bmatrix}, \quad \boldsymbol{K} = mgr \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

The eigenvalue problem leads to

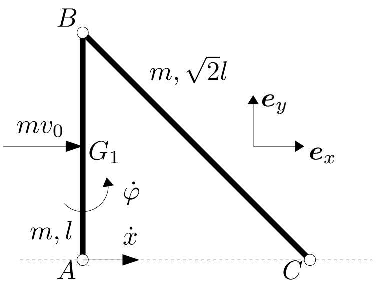
$$\det(\boldsymbol{K} - \lambda \boldsymbol{M}) = 0 \Rightarrow \frac{25}{48}\mu^2 - \frac{37}{24}\mu + 1 = 0$$

where $\mu = r\lambda/g$. Solving the equation we find

$$\lambda_1 = \frac{24}{25}\frac{g}{r}, \quad \lambda_2 = 2\frac{g}{r}$$

since both $\lambda_i > 0$, the solutions are oscillations with angular frequencies $\omega_i = \sqrt{\lambda_i}$.

Problem 2.



Since the impacting particle is brought to a stop efter the impact, the impluse on it must have been $\mathbf{0} - mv_0\boldsymbol{e}_x = -mv_0\boldsymbol{e}_x$. Thus the impulse on the rod AB must have been $mv_0\boldsymbol{e}_x$.

Use the velocity $\dot{x}$ of the point A in the $\boldsymbol{e}_x$ direction and the counter clockwise angular velocity $\dot{\varphi}$ as generalized velocities. The connection formula for velocities in a rigid body gives $\boldsymbol{v}_{G_1} = (\dot{x} - l\dot{\varphi}/2)\boldsymbol{e}_x$ and $\boldsymbol{v}_B = (\dot{x} - l\dot{\varphi})\boldsymbol{e}_x$. Since both points of the rod BC have velocities in the $\boldsymbol{e}_x$ direction and the rod is not perpendicular to the $\boldsymbol{e}_x$ direction, the connection formula shows that the angular velocity of the rod BC must be zero, and thus all points of the rod BC have velocity $(\dot{x} - l\dot{\varphi})\boldsymbol{e}_x$.

The kinetic energy is

$$T = m|\boldsymbol{v}_{G_1}|^2/2 + J_{G_1}\omega_{AB}^2/2 + m|\boldsymbol{v}_{BC}|^2/2 = m\left[(\dot{x} - l\dot{\varphi}/2)^2/2 + l^2\dot{\varphi}^2/24 + (\dot{x} - l\dot{\varphi})^2/2\right].$$

From the velocity of the point G_1: $\boldsymbol{v}_{G_1} = (\dot{x} - l\dot{\varphi}/2)\boldsymbol{e}_x$ we can identify the contributions from $\dot{x}$ and $\dot{\varphi}$ respectively, and thus compute the generalized impulses

$$I_x = \boldsymbol{e}_x \bullet mv_0\boldsymbol{e}_x = mv_0, \quad I_\varphi = -(l/2)\boldsymbol{e}_x \bullet mv_0\boldsymbol{e}_x = -(l/2)mv_0.$$

Lagrange's equations for impact

$$\frac{\partial T}{\partial \dot{q}_a}|_{\text{final}} - \frac{\partial T}{\partial \dot{q}_a}|_{\text{initial}} = I_a$$

here becomes

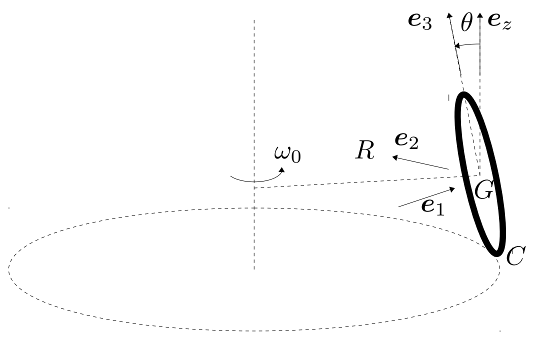
$$m(\dot{x}_f - (l/2)\dot{\varphi}_f) + m(\dot{x}_f - l\dot{\varphi}_f) = m(2\dot{x}_f - 3l\dot{\varphi}_f/2) = mv_0$$
$$-m(l/2)(\dot{x}_f - (l/2)\dot{\varphi}_f) + (ml^2/12)\dot{\varphi}_f - ml(\dot{x}_f - l\dot{\varphi}_f) = m(-3l\dot{x}_f/2 + 4l^2\dot{\varphi}_f/3) = -(l/2)mv_0.$$

This gives

$$\dot{x}_f = 7v_0/5, \quad l\dot{\varphi} = 6v_0/5 \Rightarrow \boldsymbol{v}_{G_1,f} = (\dot{x}_f - l\dot{\varphi}_f/2)\boldsymbol{e}_x = (4/5)v_0\boldsymbol{e}_x.$$

Comment: Thus the normal Newton coefficient of restitution is 4/5 in this case.

Problem 3.



Introduce a triad of basis vectors: $\boldsymbol{e}_1$ outward along the symmetry axis, $\boldsymbol{e}_2$ horizontally in the positive direction around ev_z, and $\boldsymbol{e}_3$ in the direction of the vector $\boldsymbol{r}_{CG}$. The angular velocity of the ring is

$$\boldsymbol{\omega} = \omega_0\boldsymbol{e}_z + \omega_1\boldsymbol{e}_1 = \{\boldsymbol{e}_z = \cos(\theta)\boldsymbol{e}_3 + \sin(\theta)\boldsymbol{e}_1\} = \omega_0\cos(\theta)\boldsymbol{e}_3 + (\omega_0\sin(\theta) + \omega_1)\boldsymbol{e}_1$$

where ω_1 must be such that the rolling condition is satisfied. The center of mass velocity $\boldsymbol{v}_G = R\omega_0\boldsymbol{e}_2$. Thus rolling requires

$$\boldsymbol{0} = \boldsymbol{v}_G + \boldsymbol{\omega} \times \boldsymbol{r}_{GC} = R\omega_0\boldsymbol{e}_2 + (\omega_0\cos(\theta)\boldsymbol{e}_3 + (\omega_0\sin(\theta) + \omega_1)\boldsymbol{e}_1) \times (-r\boldsymbol{e}_3) = (R\omega_0 + r(\omega_0\sin(\theta) + \omega_1))\boldsymbol{e}_2.$$

Thus $(\omega_0\sin(\theta) + \omega_1) = -(R/r)\omega_0$ and we get the angular velocity

$$\boldsymbol{\omega} = \left(\cos(\theta)\boldsymbol{e}_3 - \frac{R}{r}\boldsymbol{e}_1\right)\omega_0.$$

To avoid having to introduce the contact forces on the ring at C, we formulate the angular momentum balance equation about the moving point C:

$$\dot{\boldsymbol{L}}_C + \boldsymbol{v}_c \times m\boldsymbol{v}_G = \dot{\boldsymbol{L}}_C = \left(\frac{d\boldsymbol{L}_C}{dt}\right)_{1,2,3} + \boldsymbol{\omega}_{1,2,3} \times \boldsymbol{L}_C = \boldsymbol{M}_C$$

since both C and G have velocities parallel to $\boldsymbol{e}_2$ and where the angular velocity of the triad is $\boldsymbol{\omega}_{1,2,3} = \omega_0\boldsymbol{e}_z = \omega_0(\cos(\theta)\boldsymbol{e}_3 + \sin(\theta)\boldsymbol{e}_1)$.

Since the point C is an instantaneous center of rotation for the ring, we have $\boldsymbol{L}_C = \boldsymbol{J}_C(\boldsymbol{\omega})$. The required moments of inertia for the ring are $J_{C1} = J_{G1} + mr^2 = 2mr^2$, and $J_{C3} = J_{G3} = mr^2/2$. We find

$$\boldsymbol{L}_C = mr^2\omega_0 \left(\frac{1}{2}\cos(\theta)\boldsymbol{e}_3 - 2\frac{R}{r}\boldsymbol{e}_1\right)$$

and

$$\dot{\boldsymbol{L}}_C = \left(\frac{d\boldsymbol{L}_C}{dt}\right)_{1,2,3} + \boldsymbol{\omega}_{1,2,3} \times \boldsymbol{L}_C = mr^2\dot{\omega}_0 \left(\frac{1}{2}\cos(\theta)\boldsymbol{e}_3 - 2\frac{R}{r}\boldsymbol{e}_1\right) - mr^2\omega_0^2\cos(\theta)\left(\frac{1}{2}\sin(\theta) + 2\frac{R}{r}\right)\boldsymbol{e}_2.$$

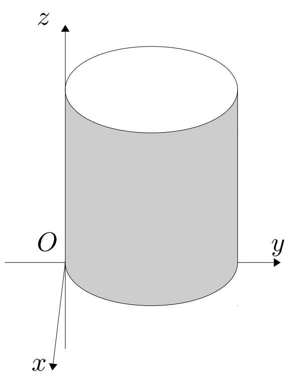
The torque about C is supplied by the gravity force $-mg\boldsymbol{e}_z$ acting at the center of mass G, so

$$\boldsymbol{M}_C = \boldsymbol{r}_{CG} \times (-mg\boldsymbol{e}_z) = -mgr\sin(\theta)\boldsymbol{e}_2.$$

Angular momentum balance now gives $\dot{\omega}_0 = 0$ from the $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$ components, and from $\boldsymbol{e}_2$ we get

$$-mr^2\omega_0^2\cos(\theta)\left(\frac{1}{2}\sin(\theta) + 2\frac{R}{r}\right) = -mgr\sin(\theta) \Rightarrow \omega_0^2\cos(\theta)(4R + r\sin(\theta)) = 2g\sin\theta.$$

Problem 4.



It is clear that the x, y, and z axes are principal axes for the cylindrical shell about its center of mass G. Further $J_{Gz} = mr^2$. To compute $J_{Gx} = J_{Gy}$, we slice the shell into thin rings of constant z. Each ring has moment of inertia $r^2 dm/2$ about a horizontal axis through its center of mass, and by the parallel axis theorem

$$J_{Gx} = J_{Gy} = \int \left(r^2/2 + (z-r)^2\right) dm = mr^2/2 + m(2r)^2/12 = 5mr^2/6$$

since first term in the integral gives a contribution equal to a ring of mass m and radius r, while the second term gives a contribution like a rod of mass m and length $2r$.

The point G has coordinates $x_G = 0$, $y_G = z_G = r$, so again by the parallel axis theorem we get for the moment of inertia about the point O

$$\boldsymbol{J}_O = mr^2 \begin{bmatrix} \frac{5}{6} & 0 & 0 \\ 0 & \frac{5}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix} + mr^2 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = mr^2 \begin{bmatrix} \frac{17}{6} & 0 & 0 \\ 0 & \frac{11}{6} & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

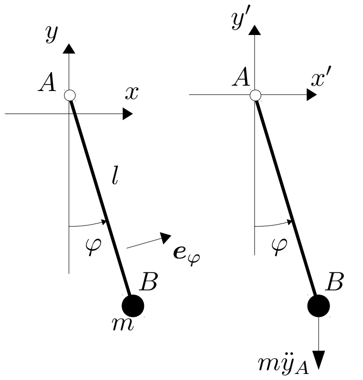
It is clear that the x axis is still a principal axis (also since the y–z plane is a reflection plane), with moment of inertia $J_1 = 17mr^2/6$.

Comment: The other two principal moments of inertia are found by solving the eigenvalue problem, and are found to be

$$J_{2,3} = \frac{23 \pm \sqrt{145}}{12} mr^2$$

with one smaller and one slightly greater than J_1.

Problem 5.



In the inertial frame, the velocity of the mass is $\boldsymbol{v}_B = \dot{y}_A \boldsymbol{e}_y + l\dot{\varphi}\boldsymbol{e}_\varphi$ and the kinetic energy is

$$T = \frac{m}{2}|\boldsymbol{v}_B|^2 = \frac{m}{2}\left(\dot{y}_A^2 + l^2\dot{\varphi}^2 + 2l\boldsymbol{e}_y \bullet \boldsymbol{e}_\varphi \dot{y}_A\dot{\varphi}\right) = \frac{m}{2}\left(\dot{y}_A^2 + l^2\dot{\varphi}^2 + 2l\sin(\varphi)\dot{y}_A\dot{\varphi}\right)$$

and the potential energy is

$$V = mgy_B = mg(y_A - l\cos(\varphi)).$$

The Lagrange function is

$$L = T - V = \frac{m}{2}\left(\dot{y}_A^2 + l^2\dot{\varphi}^2 + 2l\sin(\varphi)\dot{y}_A\dot{\varphi}\right) - mg(y_A - l\cos(\varphi)).$$

We find

$$\frac{\partial L}{\partial \dot{\varphi}} = m\left(l^2\dot{\varphi} + l\sin(\varphi)\dot{y}_A\right), \quad \frac{\partial L}{\partial \varphi} = ml\cos(\varphi)\dot{y}_A\dot{\varphi} - mgl\sin(\varphi)$$

and Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = ml^2\ddot{\varphi} + m(g + \ddot{y}_A)l\sin(\varphi) = 0.$$

Now consider the accelerated frame instead. The velocity of the mass is now just $\boldsymbol{v}'_B = l\dot{\varphi}\boldsymbol{e}_\varphi$ and the kinetic energy is

$$T' = \frac{m}{2}l^2\dot{\varphi}^2$$

while the potential energy is the sum of gravitational and inertial force energies

$$V' = mgy'_B + m\ddot{y}_A y'_B = -m(g + \ddot{y}_A)l\cos(\varphi).$$

The Lagrange function is

$$L' = \frac{m}{2}l^2\dot{\varphi}^2 + m(g + \ddot{y}_A)l\cos(\varphi)$$

giving the same Lagrange's equation.

Comment: We note that

$$L - L' = m\left(\dot{y}_A^2/2 + l\sin(\varphi)\dot{y}_A\dot{\varphi} - gy_A - l\ddot{y}_A\cos(\varphi)\right) = \frac{d}{dt}\left[m\left(-l\dot{y}_A\cos(\varphi) + \int(\dot{y}_A^2/2 - gy_A)dt\right)\right].$$

Problem 6. Euler's dynamic equations for a free body are

$$\begin{aligned} J_1\dot{\omega}_1 + (J_3 - J_2)\omega_2\omega_3 &= 0, \\ J_2\dot{\omega}_2 + (J_1 - J_3)\omega_3\omega_1 &= 0, \\ J_3\dot{\omega}_3 + (J_2 - J_1)\omega_1\omega_2 &= 0. \end{aligned}$$

For the derivation see the text "Dynamics of Bodies" chapter 4.

If all ω_i are constant in time this reduces to

$$\begin{aligned} (J_3 - J_2)\omega_2\omega_3 &= 0, \\ (J_1 - J_3)\omega_3\omega_1 &= 0, \\ (J_2 - J_1)\omega_1\omega_2 &= 0. \end{aligned}$$

- If all J_i are different, then at most one of ω_i can be non-zero, and the size of that components is arbitrary.
- If $J_1 = J_2 \neq J_3$ then either $\omega_3 = 0$ with ω_1 and ω_2 arbitrary, or $\omega_1 = \omega_2 = 0$ with ω_3 arbitrary.
- If $J_1 = J_2 = J_3$ then any values of ω_i are allowed.

This can be summarized: $\boldsymbol{\omega}$ is constant in the body frame if and only if is is an eigenvector of the inertia operator $\boldsymbol{J}_G$.

AN 2015-10-29