Solutions to Exam in EL2745 Principles of Wireless Sensor Networks, October 24, 2014

1. **Bernoulli vs Gilbert-Elliot Model of the Wireless Channel**

(a) For Bernoulli model, the message loss rate is p by definition. Let π_0 and π_1 be the steady state probabilities of being in the reception and loss states, respectively. In the steady state, we have

$$\begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix} = \begin{bmatrix} 1-p_{01} & p_{01} \\ p_{10} & 1-p_{10} \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix} .$$

it follows that the stationary probability of being in the reception and loss states are

$$\pi_0 = \frac{p_{10}}{p_{01}+p_{10}} ,$$

and

$$\pi_1 = \frac{p_{01}}{p_{01}+p_{10}} .$$

Then, loss rate is π_1.

Gilbert-Elliot model reduces to Bernoulli model if its loss probability does not depend on the previous state. This condition implies that

$$\Pr\{\text{loss} \mid \text{previous loss}\} = \Pr\{\text{loss} \mid \text{previous reception}\} ,$$

which reduces to $p_{11} = 1-p_{10} = p_{01}$, or equivalently $p_{01}+p_{10} = 1$.

(b) For Bernoulli model, due to independency of error events, the probability of a burst of size l is

$$\Pr\{\text{loss burst of length } l\} = p^l .$$

For Gilbert-Elliot model,

$$\Pr\{\text{loss burst of length } l\} = p_{11}^{l-1} p_{10} = (1-p_{10})^{l-1} p_{10} ,$$

because we have l losses if we have a loss, and after $(l-1)$ other losses, we transit to reception state.

(c) We denote by ABL the average burst length. For Bernoulli model, recalling the previous item, we have

$$\text{ABL} = \sum_{l=1}^{\infty} l \times \Pr\{\text{loss burst of length } l\} = \sum_{l=1}^{\infty} l p^l = \frac{p}{(1-p)^2} .$$

For Gilbert model,

$$\text{ABL} = \sum_{l=1}^{\infty} l \times \Pr\{\text{loss burst of length } l\} = \sum_{l=1}^{\infty} l\,(1-p_{10})^{l-1} p_{10} = \frac{1}{p_{10}} .$$

(d) The probability of loss of the Bernoulli model is p. Thus we have $p = P(\gamma)^f$, where $P(\gamma)$ is the probability of error for the BPSK digital modulation as a function of E_b/N_0, γ, in the AWGN channel. Let the channel amplitude be denoted by the random variable α, and let the average SNR normalized per bit be denoted by $\gamma^\star = \mathbf{E}[\alpha^2]\mathbf{E_b}/\mathbf{N_0}$. Then to obtain $P(e)$ for a Rayleigh fading channel $P(\gamma)$ must be integrated over the distribution of γ:

$$P(e) = \int_0^\infty P(\gamma)p(\gamma)d\gamma,$$

For Rayleigh fading,

$$p(\gamma) = \frac{1}{\gamma^\star}e^{-\gamma/\gamma^\star}.$$

In the case of coherent BPSK, the integration can actually be computed yielding

$$P(e) = \frac{1}{2}\left[1 - \sqrt{\frac{\gamma^\star}{1+\gamma^\star}}\right].$$

At high SNR such as OQPSK systems, the approximation $(1+x)^{1/2} \sim 1 + x/2$ can be used, giving

$$P(e) \sim \frac{1}{4\gamma^\star}.$$

2. **Slotted ALOHA with Forward Error Correction**

(a) The probability of loss is equal to the probability of having, at least, another arrival in the same time slot. Hence,

$$\Pr(\text{loss}) = 1 - \Pr(\text{no arrival at this time slot}) = 1 - e^{-T\lambda}. \tag{1}$$

For $T = 1$, we have that

$$1 - e^{-\lambda} \leq 0.1\,, \tag{2}$$

thus $\lambda \leq -\ln(0.9) = 0.105$.

(b) If $T = 2$, then,

$$1 - e^{-2\lambda} \leq 0.1\,,$$

thus

$$\lambda \leq -\ln(0.9)/2 = 0.052\,,$$

which is reasonable, since data rate *per slot* should remain unchanged to achieve the same packet loss probability.

(c) FEC doubles the load in the system, meaning that the new arrival rate is 2λ, however now messages have two opportunities to get successfully received. We say a message is lost when transmission is unsuccessful in both cases. It implies that

$$\Pr(\text{loss}) = \Pr(\text{collision at one transmission})^2 = \left(1 - e^{-2\lambda}\right)^2. \tag{3}$$

(d) Comparing (1) to (3), FEC is efficient if

$$\left(1 - e^{-2\lambda}\right)^2 < \left(1 - e^{-\lambda}\right)\,, \tag{4}$$

which gives $\lambda < 0.48$. the final result indicates that FEC is efficient for low load. The main reason is that FEC increases the load, which may lead to the well-known throughput collapse of ALOHA.

3. **Detection** All estimators we want to compute are dependent on the posterior pdf $p(\theta|x)$. In order to find it, we first compute the joint distribution:

$$\begin{aligned}
p(x,\theta) &= p(x|\theta)p(\theta) \\
&= (\frac{1}{\theta}U(\theta - x)U(x))(\theta \exp(-\theta)U(\theta)) \\
&= \exp(-\theta)U(\theta)U(x)U(\theta - x),
\end{aligned}$$

where function $U(x) = x$ if $x \geq 0$, otherwise it equals to 0. From the joint pdf, we can obtain $p(x)$ as

$$\begin{aligned}
p(x) &= \int_{-\infty}^{\infty} p(x,\theta)d\theta \\
&= \int_{-\infty}^{\infty} \exp(-\theta)U(\theta)U(x)U(\theta - x)d\theta \\
&= \int_{x}^{\infty} \exp(-\theta)d\theta, x \geq 0 \\
&= \exp(-x)U(x)
\end{aligned}$$

   (a) We can find the posterior pdf $p(\theta|x)$ (Bayes rule)

$$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p(x)} = \begin{cases} \frac{\exp(-\theta)}{\exp(-x)}, & 0 < x \leq \theta \\ 0, & \text{otherwise}. \end{cases}$$

   From here we can find that the MAP: $\hat{\theta}_{\text{MAP}} = x$.

   (b) The MMSE can be found by $\mathbf{E}[\theta|x]$:

$$\begin{aligned}
\mathbf{E}[\theta|x] &= \exp(x)\int_{x}^{\infty} \theta \exp(-\theta)d\theta \\
&= \exp(x)(x+1)\exp(-x) \\
&= x+1.
\end{aligned}$$

   Thus, $\hat{\theta}_{\text{MMSE}} = x+1$ for $x > 0$.

   (c) The minimum mean absolute error estimator is given by the median of $p(\theta|x)$:

$$\begin{aligned}
& \hat{\theta}_{\text{MED}} = \text{avg}\int_{x}^{\theta} \exp(x-\theta')d\theta' = \frac{1}{2} \\
\Rightarrow \quad & \text{avg} e^{x-\theta} = \frac{1}{2} \\
\Rightarrow \quad & \hat{\theta}_{\text{MED}} = \ln 2 + x
\end{aligned}$$

   (d) The MMSE and MAP are the same for Gaussian distributions. See the chapter ”Distributed Estimation” of the draft book ”An Introduction to WSNs”.

4. **Localization and Synchronization**

(a) The estimator of the position is the following:

$$\begin{aligned}\hat{x} &= \arg\min (y_1 - \cos(x))^2 + (y_2 - \sin(x))^2 \\ &= \arg\min y_1^2 + y_2^2 + 1 - 2y_1 \cos(x) - 2y_2 \sin(x)\,.\end{aligned}$$

Differentiating and setting equal to zero gives

$$\hat{x} = \arctan(y_2/y_1)\,.$$

(b) Define $y = (y_1, y_2, y_3)^T = (r_1 - r_2, r_1 - r_3, r_2 - r_3)^T$. Then, we have

$$\begin{aligned}r_i = h_i(x) + e_i &= \sqrt{(p_{i,1} - x_1)^2 + (p_{i,2} - x_2)^2} + r_0 + e_i\,, \\ y_k &= \bar{h}_k(x) + \bar{e}_k\,, \\ \bar{h}_1(x) &= h_1(x) - h_x(x)\,, \\ \bar{h}_2(x) &= h_1(x) - h_3(x)\,, \\ \bar{h}_3(x) &= h_2(x) - h_3(x)\,, \\ \bar{h}(x) &= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} h(x)\,.\end{aligned}$$

Numerically, we have

$$\begin{aligned}h(x_0) &= (5, 4, 5)^T\,, \\ \bar{h}(x_0) &= (1, 0, -1)^T\,, \\ \bar{R} = T\mathrm{Cov}(e)T^T &= \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}\end{aligned}$$

(c) Since the measurements are perfect, we have the expression as follows:

$$\tan(\varphi_1) = \frac{y - y_1}{x - x_1}\,, \qquad \tan(\varphi_2) = \frac{y - y_2}{x - x_2}\,.$$

Then by inverting the measurement model we get the relation:

$$\begin{aligned}x &= \frac{x_2 \tan(\varphi_2) - x_1 \tan(\varphi_1) + y_1 - y_2}{\tan(\varphi_2) - \tan(\varphi_1)}\,, \\ y &= \frac{(x_2 - x_1)\tan(\varphi_1)\tan(\varphi_2) + y_1 \tan(\varphi_2) - y_2 \tan(\varphi_1)}{\tan(\varphi_2) - \tan(\varphi_1)}\,.\end{aligned}$$

Since $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 0)$, we have

$$\begin{aligned}x &= \frac{\tan(\varphi_2)\tan(\varphi_1)}{\tan(\varphi_2) - \tan(\varphi_1)}\,, \\ y &= \frac{\tan(\varphi_1)\tan(\varphi_2)}{\tan(\varphi_2) - \tan(\varphi_1)}\,.\end{aligned}$$

(d) To find the best fit to the data according to the MMSE criterion, we form

$$\begin{bmatrix} C_i(t_1) \\ C_i(t_2) \\ C_i(t_3) \end{bmatrix} = \begin{bmatrix} 1 & C_s(t_1) \\ 1 & C_s(t_2) \\ 1 & C_s(t_3) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$$\begin{bmatrix} 2700 \\ 2810 \\ 2920 \end{bmatrix} = \begin{bmatrix} 1 & 2000 \\ 1 & 2100 \\ 1 & 2200 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Thus by using the MMSE $X = A^{\dagger}Y$, we have $(a_0, a_1) = (500, 1.1)$.

5. **Networked Control System**

(a) The sampled system is given by

$$
\begin{aligned}
x(kh+h) &= \Phi x(xh) + \Gamma u(kh) \\
y(kh) &= Cx(kh)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi &= e^{-ah} \\
\Gamma &= \int_0^h e^{-as} ds b = \frac{b}{a}\left(1-e^{-ah}\right).
\end{aligned}
$$

Thus the sampled system is

$$
\begin{aligned}
x(kh+h) &= e^{-ah}x(kh) + \frac{b}{a}\left(1-e^{-ah}\right)u(kh) \\
y(kh) &= cx(kh)\,.
\end{aligned}
$$

The poles of the sampled system are the eigenvalues of Φ. Thus there is a real pole at e^{-ah}. If h is small $e^{-ah} \approx 1$. If $a > 0$ the the pole moves towards the origin as h increases, if $a < 0$ it moves along the positive real axis.

(b) The sampled system is

$$
\begin{aligned}
x(k+1) &= \Phi x(k) + \Gamma u(k) \\
y(k) &= x(k)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi &= \begin{pmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{pmatrix} \\
\Gamma &= \begin{pmatrix} 1-e^{-1} \\ \frac{1-e^{-2}}{2} \end{pmatrix}.
\end{aligned}
$$

(c) The characteristic polynomial of $\Phi - \Gamma L$ is

$$
z^2 - 0.3z + 0.02 = (z-0.1)(z-0.2)\,.
$$

Thus the poles are at 0.1 and 0.2. The system is stable.

(d) We use the following result to study the stability of the system:

Theorem 1 *Consider the system given in Fig. 2. Suppose that the closed-loop system without packet losses is stable. Then*

- *if the open-loop system is marginally stable, then the system is exponentially stable for all* $0 < r \leq 1$.

- *if the open-loop system is unstable, then the system is exponentially stable for all*

$$\frac{1}{1-\gamma_1/\gamma_2} < r \leq 1\,,$$

where $\gamma_1 = \log[\lambda^2_{\max}(\Phi - \Gamma K)]$, $\gamma_2 = \log[\lambda^2_{\max}(\Phi)]$

Thus, the stability of this system depends on the values of K, h, A. When the conditions are not satisfied, from a control theory point of view we may choose different K for controller, or different sampling time h for the system to make the system stable. Instead, from a networking point of view, we may change the protocol parameters to so have a packet loss probability that meets the stability conditions.