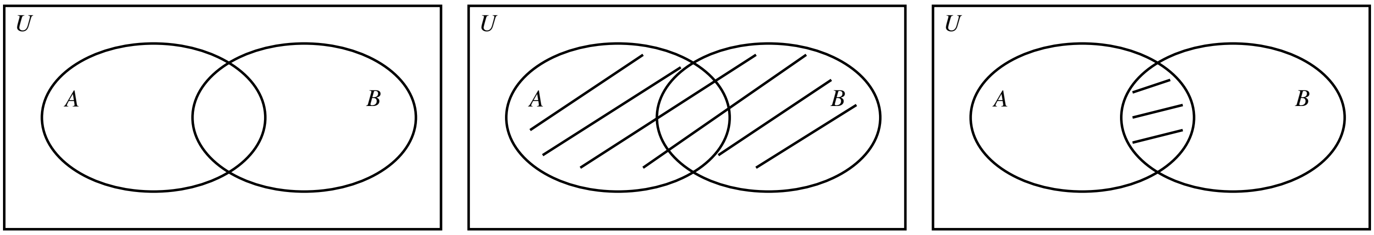
MEETING 18 - COUNTING AND COMBINATORICS

We will start to count and combine all manner of things, the basic thing is to count the number of elements in a set, but since this is such a general notion, what we do here can be done in a very broad range of contexts.

The principle of inclusion and exclusion

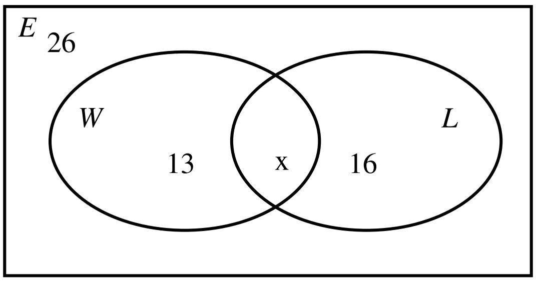
The basic problem is to find number of elements in a union of sets, we will study this by example. When we deal with this it is very helpful to have Venn diagrams at hand, the Venn diagram for two sets A, B in a universal set U can be depicted as follows:



We could shade various part of this diagram to illustrate different sets, for example the set $A \cup B$, the union of A, B, would be the set described by shading the interiors of both ellipses, this is done in the middle figure. To illustrate $A \cap B$ we would shade the area that is contained in both of the ellipses depicting the sets A, B, this is done in the figure to the right. Now we wish to count the number of elements in $A \cup B$, when both sets are finite part of a finite universal set U.

Example: Consider a situation where a company has 26 employees. Assume that 13 of these are women and that 16 of the employees have been employed with the company for 10 years or more. How many of the men have been employed more than 10 years?

Solution: We will not be able to answer the question fully since the data is insufficient, but we can make som estimates based on the information given. The universal set E, is the set of all employees in the company and this set has 26 elements, and of this set 13 are women so the total number of male employees is 13 and the number of men having been employed more than 10 years is less than 13. We now start by drawing a Venn diagram over all the employees in the company, we set $W =$*the set of all women*, $L =$*the set of all long-term emplyees* (who have been employed more than 10 years.) Based on the information given, we can introduce numbers in the diagram:



Here we have introduced the numbers 26, 13, and 16 at the appropriate places. We have also introduced the number x to denote the number of elements in the set $|W \cap L|$, that is x is the number of female long-term employees.

How does $|W \cup L|$ relate to $|W|$ and $|L|$? If we think about it, we must have the relationship

$$|W \cup L| = |W| + |L| - |W \cap L|.$$

Why? Well, $|W \cup L|$ is the number of elements in the union of the sets W, L, if we wish to count the number of elements in the union of the sets W, L and therefore form the number $|W| + |L|$ then we must remember that every element that is *only* in W (and not in L) is counted once, every element that *only* is in L (and not in W), but every element that is an element of both W, L, that is, every element in $W \cap L$ (we have denoted the number of such elements by x) will be counted twice, once because it is an element of W and once because it is an element of L, hence to obtain the number of elements of $W \cup L$ we must subtract the quantity $|W \cap L|$ from $|W| + |L|$. This is the simplest form of the principle of inclusion and exclusion so we formulate that as a proposition:

Proposition: *The Principle of Inclusion and Exclusion, simplest form* Let A, B be finite sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

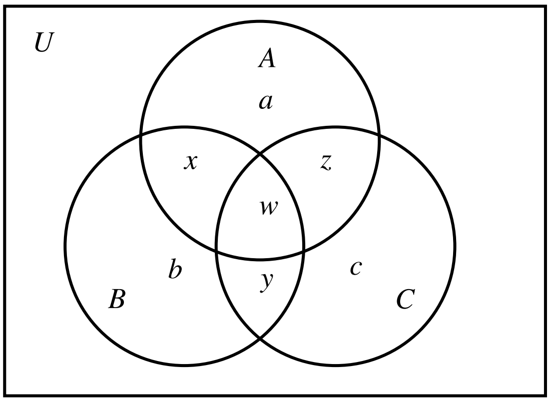
Applied to our situation above we would get the equation $|W \cup L| = |W| + |L| - |W \cap L| = 13 + 16 - x = 29 - x \Leftrightarrow$

$$x = 29 - |W \cup L|$$

How can we use this equality? The number we are seeking is $16 - x$ and this can be written $16 - (29 - |W \cup L|) = |W \cup L| - 13$. Since $L \subseteq W \cup L$, and $|W \cup L| \geq |L| = 16$ we get the estimate $16 - x = |W \cup L| - 13 \geq 16 - 13 = 3$. If the number of male long-term employees is denoted by y we have, in saummary, found the following estimates:

$$3 \leq y \leq 13.$$

We can do this with three sets too: We want to find out what $|A \cup B \cup C|$ is, so we first draw a Venn diagram with three sets:



In this diagram, we have depicted the three sets A, B, C and introduced the numbers a, b, c, x, y, z, w to denote the number of elements in each region of the diagram. Hence, the number a denotes the number of elements that are only in A *not* in B or C, similarly the number b denotes the number of elements that are only in B and c those that are only in C. The number x denotes the number of elements that are in $A \cap B$ but not in $A \cap B \cap C$. The numbers y, z have similar meanings related to B, C and finally the number w denotes the number of elements that are in all three sets, that is the number of elements in the set $A \cap B \cap C$.

Now to count the number of elements in $A \cup B \cup C$ we need to form the number $a + b + c + x + y + z + w$. If we begin experimenting in analogy to the simplest form of the principle of inclusion and exclusion we could form the number

$$q = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$$

and if we utilize $|A| = a + x + w + z$, $|B| = b + x + w + y$, $|C| = c + y + w + z$, $|A \cap B| = x + w$, $|B \cap C| = y + w$, $|A \cap C| = z + w$ we get that this number is

$$(a + x + w + z) + (b + x + w + y) + (c + y + w + z) - (x + w) - (y + w) - (z + w)$$

and, after cancellation we obtain that $q = a + b + c + x + y + z$. Now we wanted to obtain the number $|A \cup B \cup C| = a + b + c + x + y + z + w$ and obviously q is not this number, but it is only $w = |A \cap B \cap C|$ that is lacking. Hence $|A \cup B \cup C| = a + b + c + x + y + z + w = q + w$ or

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

which is the principle of inclusion and exclusion for *three* sets, A, B, C. The general form of the principle for inclusion and exclusion concerning n sets, $A_1, A_2, \ldots, A_n$, is formed in an analogous way:

$$|A_1 \cup \ldots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap \ldots \cap A_n|.$$

We will now study a long example which will not really be a direct application of the principle of inclusion and exclusion, it will be an example that illutrates *the setting* in which we will be able to better study the principle of inclusion and exclusion. So bear this in mind when studying this example: we will study another example afterwards which involves a more direct application of the principle of inclusion and exclusion.

We now state a proposition concerning relations among cardinalities (that is how many elements a set has) of compositions involving two finite sets in a finite universe. All of the statements in the proposition are easy to prove and we have walked through the thinking of the proof of one of the statements.

Proposition: Let A, B be subsets of a finite universal set U. Then

(a) $|A \cup B| = |A| + |B| - |A \cap B|$ (*the principle of inclusion and exclusion in the simplest form.*)
(b) $|A \cap B| \leq \min\{|A|, |B|\}$
(c) $|A - B| = |A| - |A \cap B| \geq |A| - |B|$
(d) $|A^c| = |U| - |A|$
(e) $|A \cup B| - |A \cap B| = |A| + |B| - 2|A \cap B| = |A - B| + |B - A|$
(f) $|A \times B| = |A||B|$

Now we come to the example which is a more direct application of the principle of inclusion and exclusion.

Example: How many integers are there in the set $\Omega = \{1, 2, 3, \ldots, 10000\}$ *not* divisible by 2, 3, or 5?

Solution: Set $A = \{x \in \Omega; 2|x\}$, $B = \{x \in \Omega; 3|x\}$ and $C = \{x \in \Omega; 5|x\}$. We wish to find $|\Omega - (A \cup B \cup C)|$. This number will be $10000 - |A \cup B \cup C|$ which is why we first seek the number $k = |A \cup B \cup C|$. The principle of inclusion and exclusion with three sets states that this number is formed as

$$k = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |B \cap C| + |A \cap B \cap C|$$

and we will study each term separately. Now each term in this sum is on the form $|\{x \in \Omega; d|x\}|$, where d is a divisor, it is one of the numbers $2, 3, 5, 6, 15, 10, 30$. ($|A \cap B|$ is the set of all integers in Ω divisble by 6 (divisible by *both* $2, 3$) and so on.) Let us find $|\{x \in \Omega; d|x\}|$ for a general d. The division algorithm states that there is a q such that $10000 = q \cdot d + r$, where $0 \leq r < d$. The numbers in $\Omega = \{1, 2, 3, \ldots, 10000\}$ that are divisible by by d are precisely the numbers $1 \cdot d, 2 \cdot d, 3 \cdot d, \ldots, q \cdot d$ and there is exactly q of these numbers. Now, according to the division algorithm, the number q is just $10000/d$, where the division is integer division (the result is the integer part of the quotient), that is q is the integer part of $10000/d$. This applied to the terms above yields

$$|A| = 10000/2 = 5000, \quad |B| = 10000/3 = 3333, \quad |C| = 10000/5 = 2000,$$

$$|A \cap B| = 10000/6 = 1666, \quad |B \cap C| = 10000/15 = 666, \quad |B \cap C| = 10000/10 = 1000$$

$$|A \cap B \cap C| = 10000/30 = 333$$

and these numbers plugged into the formula gives $k = 5000 + 3333 + 2000 - 1666 - 666 - 1000 + 333 = 7334$ and hence $|\Omega| = 10000 - k = 10000 - 7334 = 2666$.

The addition and multiplication rules

If we have n pairwise disjoint sets, that is n sets that have no elements in common, then we have

$$|A_1 \cup \ldots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n| = \sum_i |A_i|.$$

This is because all terms with intersecting sets vanish (becomes 0) since the sets had no elements in common. This is also completely natural since $\{A_1, A_2, \ldots, A_n\}$ forms a partition of $A_1 \cup \ldots \cup A_n$.

In mathematical statistics we speak of an *event* which is just something that can happen which we will later attach something called a *probability* to. Events can be described as sets and they are called *mutually exclusive* if their intersection is empty, which is another way of saying that they cannot occur at the same time. For example if we roll two dice, an even can be described as *getting the sum 4*, this event could then be described as the set of die-pair-outcomes $\{(1, 3), (2, 2), (3, 1)\}$. Another, mutually exclusive event could be *getting the sum 5*, which could be described as the set of die-pair-outcomes $\{(1, 4), (2, 3), (3, 2), (4, 1)\}$. We can call the individual die-pair-outcomes the *ways* in which an event can occur, and the number of ways in which a set of mutually exclusive events can occur is then the sum of all the ways in which each event can occur. This called *the addition rule* and, given our example above, if we study the combined event *getting the sum 4 or gettting the sum 5*, the number of ways this event can occur would, according to the addition rule, be $3 + 4 = 7$, since the events are mutually exclusive and the 3 i s the number of ways one of them can occur and 4 is the number of ways that the other one can occur.

The addition rule is one way of counting in how many ways an individual event can occur. We will now turn to studying how many ways *many* event can happen. Therefore we make the following proposition:

Proposition: Let $A_1, A_2, \ldots, A_n$ be n events and let $|A_i|$ denote the number of ways that $|A_i|$ can happen. The total number of ways in which the sequence of events $A_1, A_2, \ldots, A_n$ can occur will then be the number $|A_1 \times A_2 \times \ldots \times A_n| = \Pi_{i=1}^{n}|A_i|$.

Proof: We will not prove this, but study an example.

Example: In how many ways can we get the sum 5 twice in a row when we roll two dice?

Solution: The number of ways that we can get the sum 5 once is 4, as we saw above. Hence the event A_1=*Getting the sum 5 the first time* can occur in 4 ways. For each of these outcomes, when we roll the dice a second time, we can still get the sum 5 in 4 times, so the event A_2=*Getting the sum 5 the second time* can also occur in 4 ways. The total number ways in which we can get 5 two times in a rows will therefore be $4 \cdot 4 = 16$.

We will apply this in the proof of a very important proposition:

Proposition: Let A be a finite set with n elements. The the number of subsets of A is 2^n.

Proof: Assume that the set A is given by $\{a_1, a_2, \ldots, a_n\}$. Then consider in how many ways we can choose a subset from this set. The number of ways in which we can choose a subset from A must of course be the number of subsets of A. Consider the event of choosing whether the first element a_1, should be in the subset or not. In how many ways can this occur? In precisely 2 ways, we can either choose to include a_1 in the subset we are forming or we can choose to not include a_1. Then we turn to a_2 and consider the event of choosing whether it is going to be in the subset or not. This can also be done in 2 ways. In summary, choosing whether element a_i is going to be in the subset or not can be done in precisely 2 ways, since forming a subset involves n events of this kind occuring, according to the multiplication rule, this can happen in precisely 2^n ways. The proof is complete.

Example: The number of subsets of $\{1, 2, 3\}$ is $2^3 = 8$ and the subsets are

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$$

This means that the powerset of $\{1, 2, 3\}$ is the set $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Observe that this is a set where the elements themselves are sets.

Example: The number of subsets of $\{1, 2, 3\}$ which contains 1 is 4 since there are 4 ways of forming such a subset. We could view this as a process in three steps: Choose 1, can be done in $|A_1| = 1$ way. Choose whether 2 is going to be in the subset, this can be done in $|A_2| = 2$ ways. And similarly, we can choose to include or not include the element 3 in the subset we are forming in $A_3 = 2$ ways. Total number of ways to choose a set containing 1 is therefore $A_1 \cdot A_2 \cdot A_3 = 1 \cdot 2 \cdot 2 = 4$ ways, and each way correspond to exactly one subset containing 1.

The Pigeonhole Principle

The Pigeon hole principle concerns questions of how much room we have. If we fill a certain space, what is the minimum distance between objects? If we express ourselves in terms of functions, we can state it like this:

The Pigeonhole principle: If A, B are finite sets with $|A| > |B|$, and $f : A \to B$ is a function, then f cannot be one-to-one (injective). In earlier terms, since there are more elements in A than in B ($|A| > |B|$), then there is not room enough in B so that each element in A can have a unique image in B through f.

Example: If we choose 11 arbitrary numbers, the there must exist a two among these whose difference is divisible by 10.

Solution: To see this we can first observe that if two of the numbers coincide, then their difference is 0 and this is certainly divisible by 10. So we can assume that the 11 numbers are different, this means that 11 numbers form the set $A = \{a_1, a_2, \ldots, a_{11}\}$. Now form the function $f : A \to B$ by defining $f(a) = r$, where r is the remainder when a is divided by 10. Then this function is a function from a set A containing 11 elements to a set $B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, a set containing 10 elements. As $|B| = 11 > 10 = |A|$, by the Pigeonhole principle f cannot be one-to-one so there must be two numbers a_i, a_j such that $f(a_i) = f(a_j)$, but this means that $a_i \equiv a_j(mod 10)$, which shows that the difference between two numbers in A must be divisible by 10, which is what we wanted to prove.

Of course this will apply to any other integer than 10, we could say that among $n+1$ arbitrarily chosen integers, there must alway be two whose difference will be divisible by n.

Example: The Pigeonhole principle can also be applied to geometric situations: Prove that if we put 9 points inside a cube with a side whose length is 2, two of these points must lie closer to each other than $\sqrt{3}$.

Solution: To see this, we imagine the cube being described as $C = [0,2] \times [0,2] \times [0,2]$, that is the crossproduct between the three intervals $[0,2],[0,2],[0,2]$. Now this cube can be viewed as the union between eight cubes,

$$C_1 = [0,1] \times [0,1] \times [0,1], \quad C_2 = [0,1] \times [0,1] \times [1,2], \quad C_3 = [0,1] \times [1,2] \times [0,1], \quad C_4 = [0,1] \times [1,2] \times [1,2],$$

$$C_5 = [1,2] \times [0,1] \times [0,1], \quad C_6 = [1,2] \times [0,1] \times [1,2], \quad C_7 = [1,2] \times [1,2] \times [0,1], \quad C_8 = [1,2] \times [1,2] \times [1,2],$$

and, for any set of nine distinct points in C, say $\{a_1, a_2, \ldots, a_9\}$ we can form the function $f(a_i) = j$, where j is the number of the cube that a_i is contained in, that is it is a number in the set $B = \{1,2,3,4,5,6,7,8\}$, which has 8 elements. Again, we have a function $f : A \to B$ where $|A| = 9 > 8 = |B|$ and this function cannot be one-to-one, two of the points must then have $f(a_i) = f(a_j)$, that is they lie in the same cude, and as the diameter of a cube with a side of 1 is $\sqrt{3}$, a_i, a_j must lie at a distance from each other that is less than $\sqrt{3}$.

Ok, when we have one more element in the set A, than in the set B, we can be sure that each function maps two of these element to the same image. But if we have a lot more element in A, then we have a situation in which $f(A)$ is very crowded in B indeed. We say like this:

The Pigeonhole Principle, strong form: If n objects are put into m boxes, and $n > m$, then some box must contain at least $n/m + 1$ objects (again this is integer division, $n/m + 1$ is then the integer part of n/m plus 1, this is sometimes called the *ceiling* of n/m, the smallest integer greater than n/m).
Proof: Denote by $c(x)$, the ceiling of x, then we always have $x \leq c(x) \leq x + 1$ so that we always have

$$c\left(\frac{n}{m}\right) < \frac{n}{m} + 1 \Rightarrow c\left(\frac{n}{m}\right) - 1 < \frac{n}{m}$$

and now, if *all* boxes contain fewer than $c\left(\frac{n}{m}\right)$ objects, that is less than, or equal to $c\left(\frac{n}{m}\right) - 1 < \frac{n}{m}$ objects, in total we would have strictly less than $m \cdot \frac{n}{m} = n$ objects, but we *have* n objects and this is a contradiction. This means that there must be a box with $c\left(\frac{n}{m}\right)$ or more objects, which is what we wanted to prove.
We will not be so concerned with the strong form of the Pigeonhole principle but it is interesting to study the proof as an example of a proof where the contrapositive is established (proof by contradiction).

Permutations

A permutation can *loosely* be defined as "the way to do or arrange someting". Many problems in so-called combinatorics deals with finding out in how many ways a certain thing can be done or arranged, often we ask the question "how many permutations are there", and then we state a number of conditions that the permuations must be subject to. A prime instrument in finding the number of permutations is the multiplication rule, described earlier. We will take a number of examples.

Example: A man courts a woman and she has said to him that she really likes flowers, particularly roses, tulips, lilies, lupins and dandelions. She has also said that she admires people who can make good flower arrangements. The night before a meeting between them, he lies sleepless through the night and ponders her words. Good flower arrangements, good flower arrangements ... What does that mean? In the morning, before their metting, he goes to a florists to buy a bouquet for her. The florist indeed has roses, tulips, lilies, lupins and dandelions, and the man is thrilled. He thinks that he will be able to construct a "flower arrangement" for his courtee. As he is studying a course in linear algebra, he thinks that a good flower arrangement would be to lay out the flowers in a straight line. He can afford to buy three flowers, and he then asks himself, in how many ways can a construct a flower arrangement (a line of flowers) with three flowers? Earlier he has also taken a course in dicrete mathematics and recalls the product rule and he applies this: constructing a line of three flowers is a process in three steps, in the first step I can choose one of the five flowers and place it first in the line, this can be done in 5 ways. In the second step I can choose one of the four remaining flowers, and this can be done in 4 ways. And choosing the third and last flower can be done in 3 ways, the total number of ways will therefore be $5 \cdot 4 \cdot 4 \cdot 3$. If we use the notation $n!$ to denote the number $n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$ we can write this number as $5 \cdot 4 \cdot 3 = \frac{5!}{2!}$ since

$$5 \cdot 4 \cdot 3 = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{5!}{2!}.$$

This number is 60 so the man buys a lupin, a tulip and a rose and arranges them in a row $lupin - tulip - rose$ and hopes that the woman also appreciates his linear flower arrangement.

Example: A couple of years later, the man and the woman is going to get married. She definitely has remembered his ability to arrange stuff in linear order, so he gets the assignment to arrange the table placement of the guests at the reception. The table has 10 places but he has only received instructions to place 7 guests. The three remaining seats will be filled later. In how many ways can he form a placement of the guests? Again he turns to discrete mathematics and the multiplication rule. He realises that choosing the place for the first guest can be done in 10 ways, the place for the second guest can be done in 9 ways (since one place is taken by the first guest), the third choice can be done in 7 ways, and so on until the 7th guest's place is to be chosen which can be done in 4 ways. In summary the number of different placements of the 7 reception guest would be

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = \frac{10!}{3!} = 604800.$$

(This is a very large number so he decides to be extra careful when choosing the placements.)

When dealing with questions of the above kind, the number $\frac{n!}{(n-r)!}$ reappears. It appears when we wish to select r objects from a set of n available objects and place these r objects in a certain order. In the first example the man selected 3 flowers from a set of 5 available ones and placed them in a linear order, this involved forming the number $\frac{5!}{2!} = \frac{5!}{(5-3)!}$. In the second example he selected 7 chairs out of 10 and decided which guest would be assigned to which chair, this involves deciding on an internal order of the set of selected chairs. The number $\frac{10!}{3!} = \frac{10!}{(10-7)!}$ was then involved. This number is so important that we give it a name:

Definition: Let n be a postive integer and let $0 \leq r \leq n$. The symbol $P(n, r)$ is then defined to be the number

$$P(n, r) = n \cdot (n-1) \cdots (n-(r-1)) = \frac{n!}{(n-r)!}.$$

We will introduce some terms here:

Definition: Consider a set A of n distinct objects, (for example the first n natural numbers, or 5 flowers to choose from, or 10 chairs at wedding reception). A *permutation* of A is then an ordering of all of the objects of A. An r-permutation is an arrangement of r objects of the n available in a line. (For example the arrangement of 3 flowers out of 5 in a line, or the assignment of 7 guests to 7 out of 10 chairs at a wedding reception.)

By our examples above, we find the following to be true:

Proposition: The number of permutations of n symbols (or objects, or whatever we want to call the elements) is $n!$. The number of r-permuations of n symbols (or objects ... flowes, wedding guests etc.) is $P(n, r)$.

Combinations

The last section dealt with the problem of finding the number of r-permutations from a set of n given objects. That could equivalently be formulated, how many different ***lines*** of length r of n objects can be formed. Then, *the individual order* between the r individual objects were important. Now we will consider the problem of choosing ***sets*** of size r from a set of n available objects. We will apporach this by taking an example.

Example: Consider 7 different creatures, a man, a woman, a boy, a girl, a dog, a cat and a fish. These are members in a family and the family has won the lottery and four creatures are allowed to go on a cruise and since the family has 7 members, they must decide by internal lottery which four creatures that get to go on the cruise. In how many ways can this be done? It is equivalent to asking, how many ways are there to select 4 elements out of a set of 7? This is denoted $\binom{7}{4}$ and our task is to find this number.

Solution: We can turn to the multiplication rule. As we said above, we denote by $\binom{7}{4}$ the number of sets with 4 elements out of 7 possible. Now consider this alternative process of forming $P(7, 4)$, the number of *lines* of 4 elements out of a set of 7 possible.

Step 1. Choose a set of 4 out of 7 (that is choose which four creatures get to go on the cruise). This can be done in $\binom{7}{4}$ ways. Perhaps this could be the *set* $\{girl, fish, cat, dog\}$. This choice would not be a different choice from the *set* $\{fish, girl, dog, cat\}$ since sets do not have an internal order of elements.

Step 2. Choose which element in this set is the first one, this can be done in 4 ways.

Step 3. Choose which element in this set is the second one, this can be done in 3 ways
Step 4. Choose which element in this set is the third one, this can be done in 2 ways.
Step 5. Choose which element in this set is the last one, this can be done in 1 ways.

In summary, the number of ways to form 4 *lines* of elements out of a set of 7 possible are

$$P(7,4) = \binom{7}{4} \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \binom{7}{4} \cdot 4!$$

but then we can divide by 4! to obtain the quantity we are seeking,

$$\binom{7}{4} = \frac{1}{4!} P(7,4).$$

In this example, the numbers 4 and 7 were chosen so that they would illustrate the general proposition: there is of course nothing that stops us from reasoning in the exact same way if we have a set of n elements (instead of 7) and wish to choose a set of r elements from this set. We have shown the following:

Proposition: Let n and r be given integers with $0 \leq r \leq n$. The of ways to choose r elements from a set of n elements is

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}.$$

Remark: If we fix n, r as in the proposition, then $\binom{n}{r}$ is the number of sets of elements of size r. And this is also the number of ways to choose a subset of r integers from the set $\{1, 2, 3, \ldots, n\}$ which must be the number of *ways* to make this selection. But just as well as we can select which elements are *in* the set, the set could also be established by choosing which elements are *not* in the set. This means that the number of sets of size r, chosen from a set of n elements must coincide with the number of sets of $n-r$ elements chosen from a set of n elements. That is, we must have the formula

$$\binom{n}{r} = \binom{n}{n-r}.$$

We have now motivated this identity with a combinatorial argument, but we can also establish the validity of this formula by pure algebra:

$$\binom{n}{r} = \frac{1}{r!} \frac{n!}{(n-r)!} = \frac{1}{(n-r)!} \frac{n!}{r!} = \frac{1}{(n-r)!} \frac{n!}{(n-(n-r))!} = \binom{n}{n-r}.$$

Some authors uses the term *combination* for a subset. And also the term r-*combination* for a subset of r elements chosen our of a larger set of elements. Another way to express the proposition above could then be "the number of r-combinations of a set of n objects is $\binom{n}{r}$". The number $\binom{n}{r}$ is also called a *binomial coefficient* and the reasons for this name will become apparent later on. We will now study some examples of uses of binomial coefficients.

Example: How many committees of 6 people can be chosen from a group of people consisting of 20 women and 10 men if the committee should consist of three members of each sex?

Solution: We are to select three women and three men, it is not relevant in which order we do this we can describe a general way of selecting the committee by first selecting three women and then three men. The first step, selecting the women, can be done in $\binom{20}{3}$ ways. And the second step, selecting the men, can be done in $\binom{10}{3}$ ways. According to the multiplication principle, the total number of ways to make both these steps will be

$$\binom{20}{3} \cdot \binom{10}{3} = \frac{20!}{3!17!} \cdot \frac{10!}{3!7!} = \frac{20 \cdot 19 \cdot 18 \cdot 10 \cdot 9 \cdot 7}{3 \cdot 2 \cdot 3 \cdot 2} = 119700.$$

Example: A man remarks that the entire group consists of 2/3 men but only 1/3 of women, he demands that the committee should therefore also reflect these proportions so that 4 are men and 2 are women in the committee. How many committees can now be formed?

Solution: We simply redo the calculations with 4 and 2 in place of the two 3's, the number of committees now becomes

$$\binom{20}{4} \cdot \binom{10}{2} = \frac{20!}{4!16!} \cdot \frac{10!}{2!8!} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 218025.$$