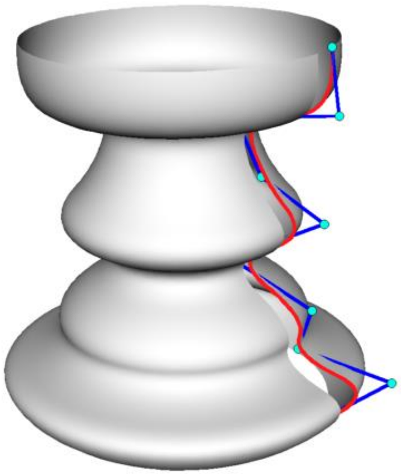
Introduction to Visualization and Computer Graphics
DH2320, Fall 2015

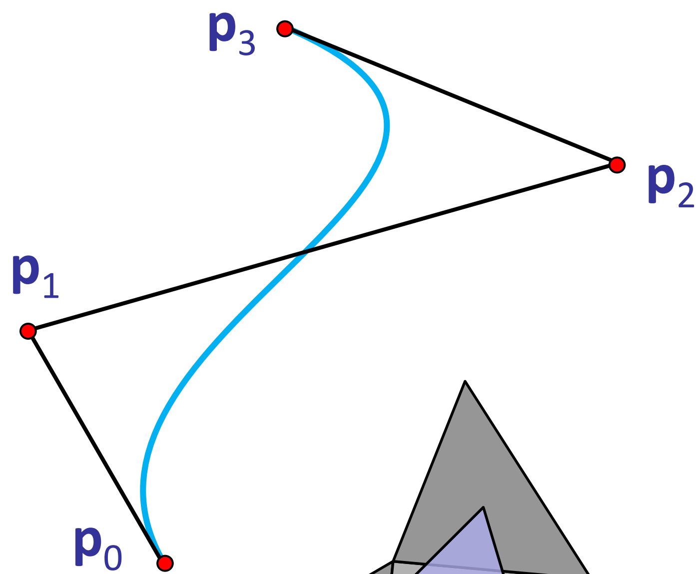
Prof. Dr. Tino Weinkauf

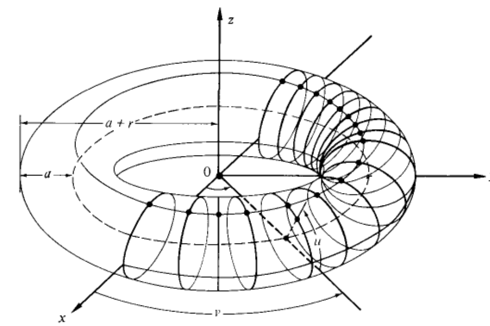
Geometric Modeling

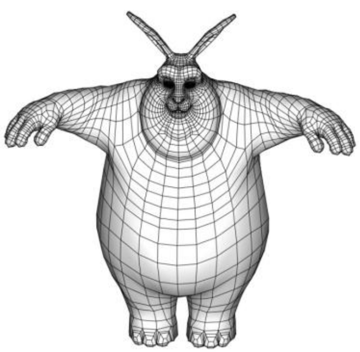
Introduction

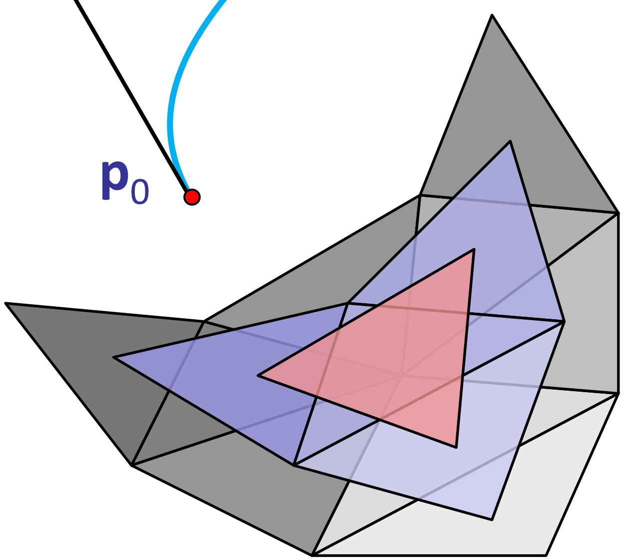
- There are many ways for creating graphical data.
- Classic way: **Geometric Modeling**







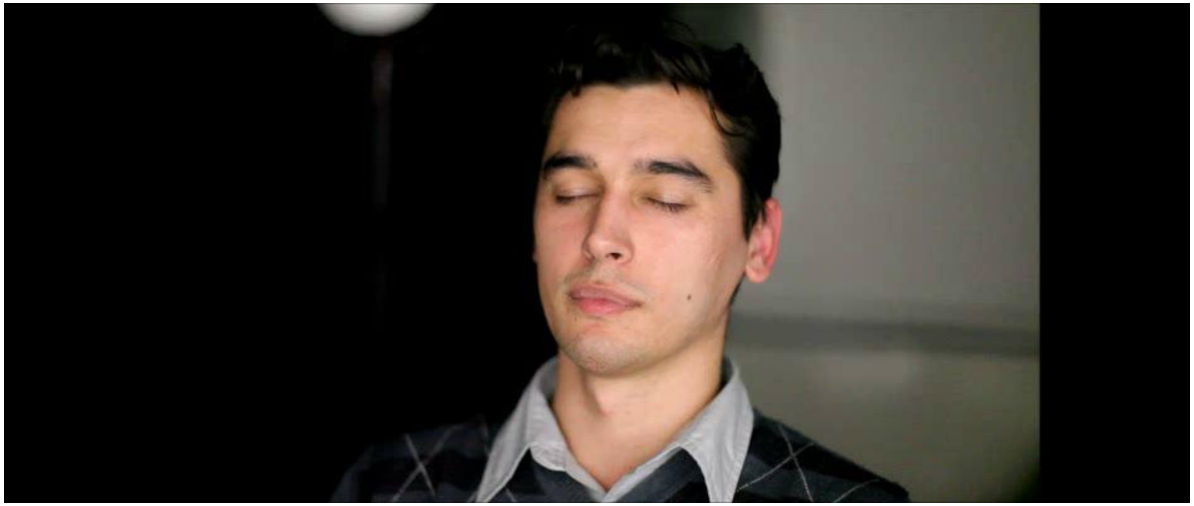




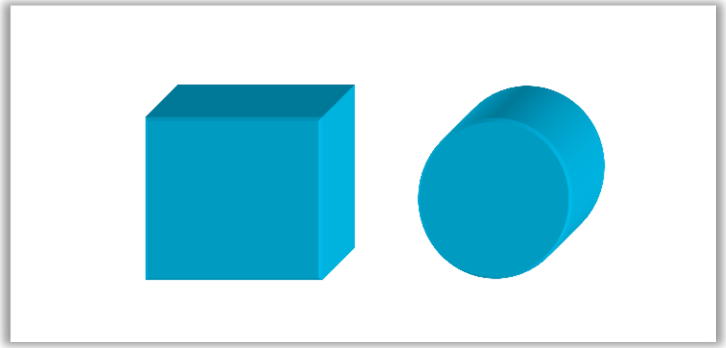
- There are many ways for creating graphical data.

- Other approaches:
  - **3D scanners**
  - Photography for measuring optical properties
  - Simulations, e.g., for flow data

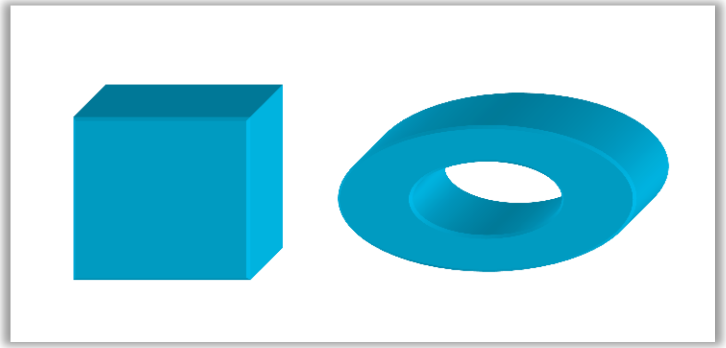
3D Scanning



- Geometric objects convey a part of the real or theoretical world; often, something tangible
- They are described by their **geometric** and **topological** properties:
  - Geometry describes the form and the position/orientation in a coordinate system.
  - Topology defines the fundamental structure that is invariant against continuous transformations.



Different geometry
Same topology



Different geometry
Different topology

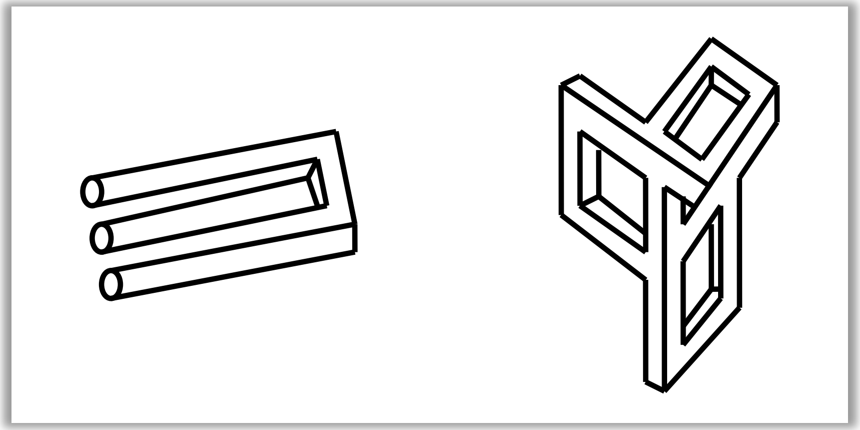
- Geometric Modeling is the computer-aided design and manipulation of geometric objects. (CAD)

- It is the basis for:
  - computation of geometric properties
  - rendering of geometric objects
  - physics computations (if some physical attributes are given)

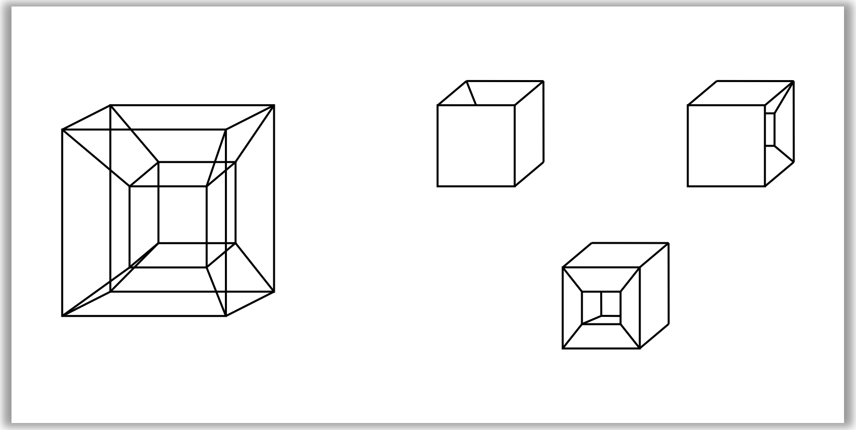
- 3D models are geometric representations of 3D objects with a certain level of abstraction.
- We distinguish between three types of models:
- Wire Frame Models
  - describe an object using boundary lines
- Surface Models
  - describe an object using boundary surfaces
- Solid Models
  - describe an object as a solid

Wire Frame Models

- Describe an object using boundary curves
- No relationship between these curves
    - Surfaces between them are not defined
- Properties:
    - simple, traditional
    - non-sense objects possible
    - visibility of curves cannot be decided
    - solid object intersection cannot be computed
    - surfaces between the curves cannot be computed automatically
    - not useable for CAD/CAM



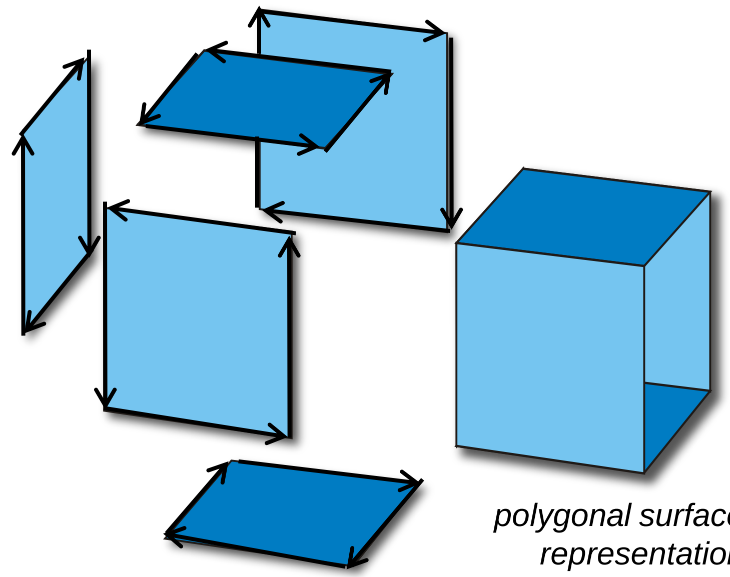
non-sense objects (Ernst, 1987)



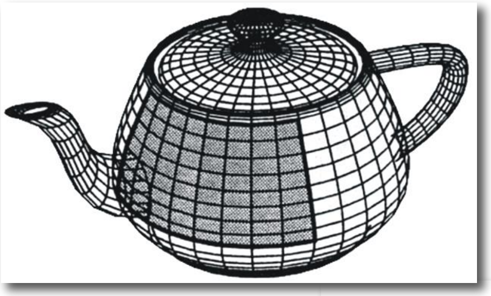
ambiguity of wire frame models

Surface Models

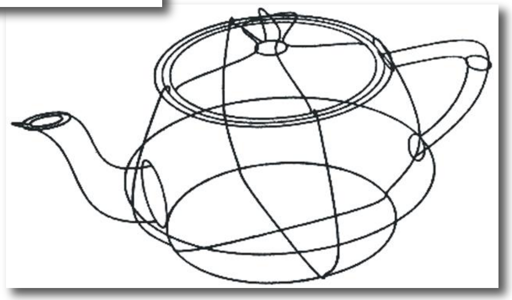
- Defines surfaces between boundary curves
- Describes the hull, but not the interior of an object
- Often implemented using polygons, hull of a sphere or ellipsoid, free-form surfaces, …
- No relationship between the surfaces
  - The interior between them is not defined
- Visibility computations: yes
  Solid intersection comp.: no
- Most often used type of model



polygonal surface representation

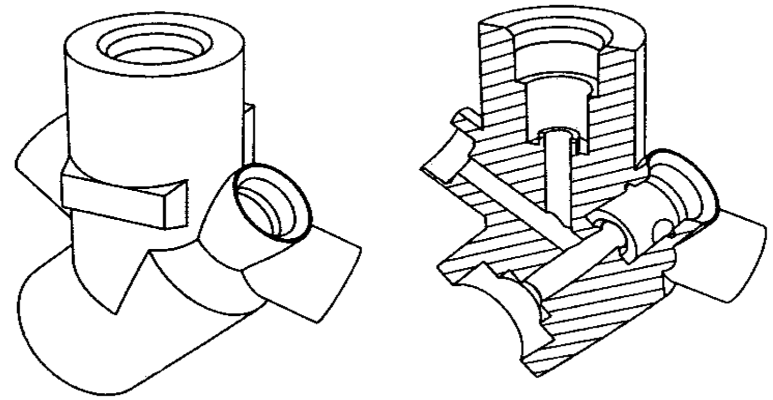


parametric surface representation using 32 Bezier patches

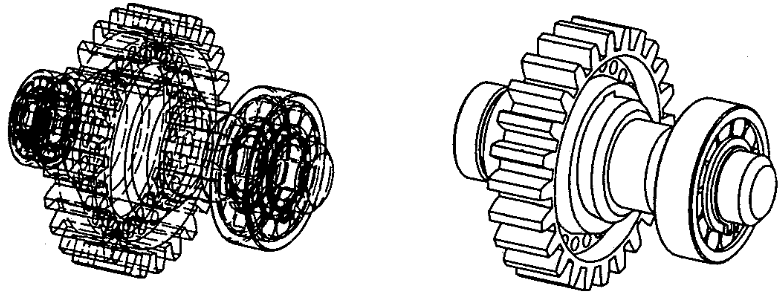


Solid Models

- Describe the 3D object completely by covering the solid
- For every point in 3D, we can decide whether it is inside or outside of the solid.
- Visibility and intersection computations are fully supported
- Basis for creating solid objects using computer-aided manufacturing



solid model and a cut through it (Werkbild Strässle, from Ockert, 1993)



visibility computation for lines using a solid model

Chrome-cobalt disc with crowns for dental implants, manufactured using WorkNC CAM
Sescoi CAD/CAM

Introduction to Visualization and Computer Graphics
DH2320, Fall 2015
Prof. Dr. Tino Weinkauf

Geometric Modeling

Bezier Curves, Splines and Surfaces

de Casteljau Algorithm
Bernstein Form
Bezier Splines
Tensor Product Surfaces
Total Degree Surfaces

Bezier Curves
de Casteljau algorithm

- Paul de Casteljau (1959) @ Citroën
- Pierre Bezier (1963) @ Renault

Meine Zeit bei Citroën / My time at Citroën
see the PDF deCasteljau_de.pdf and deCasteljau_en.pdf in the download area of the webpage

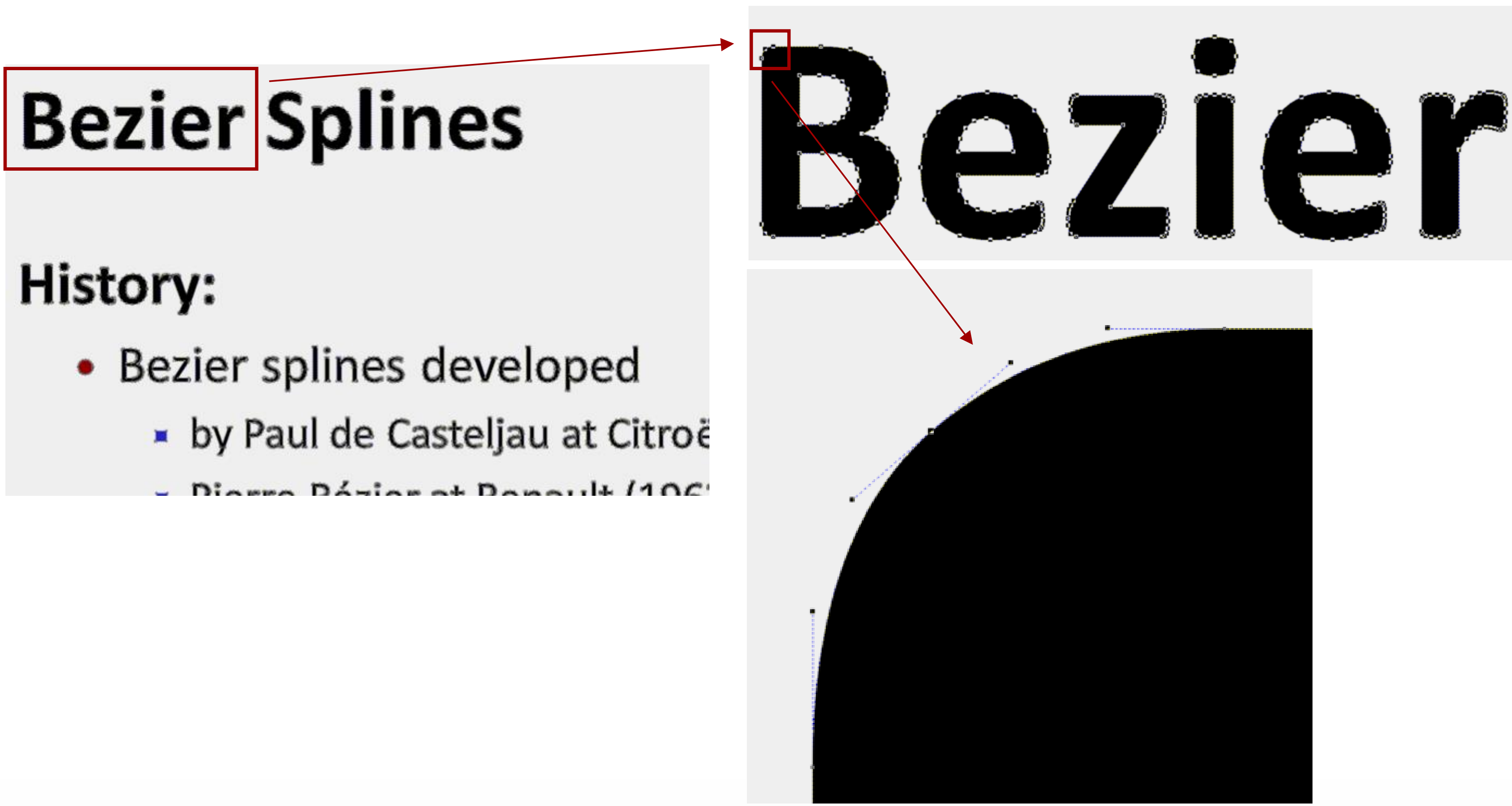
Bezier curves

History:

- Bezier curves/splines developed by
  - Paul de Casteljau at Citroën (1959)
  - Pierre Bézier at Renault (1963)

  for free-form parts in automotive design
- Today: Standard tool for 2D curve editing
- Cubic 2D Bezier curves are everywhere:
  - Postscript, PDF, Truetype (quadratic curves), Windows GDI...
  - Inkscape, Corel Draw, Adobe Illustrator, Powerpoint, ...
- Widely used in 3D curve & surface modeling as well

All You See is Bezier Curves...

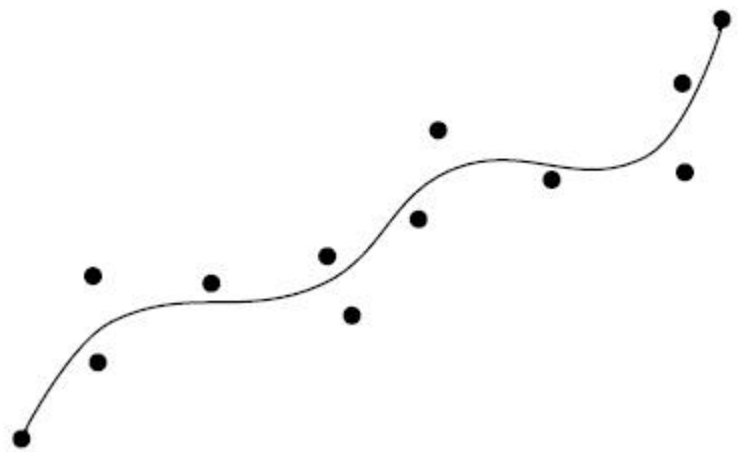


De Casteljau algorithm

Approximation setting:

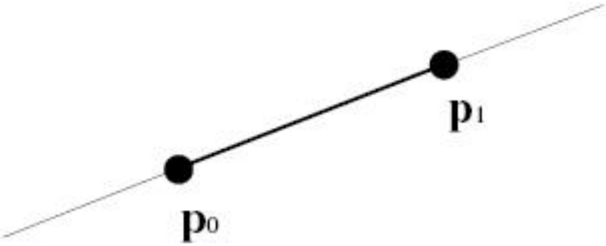
Given: p_0, …, p_n

Wanted: smooth, approximating curve



De Casteljau algorithm

Linear interpolation



$$\mathbf{x}(t) = (1-t)\cdot\mathbf{p}_0 + t\cdot\mathbf{p}_1$$

De Casteljau algorithm

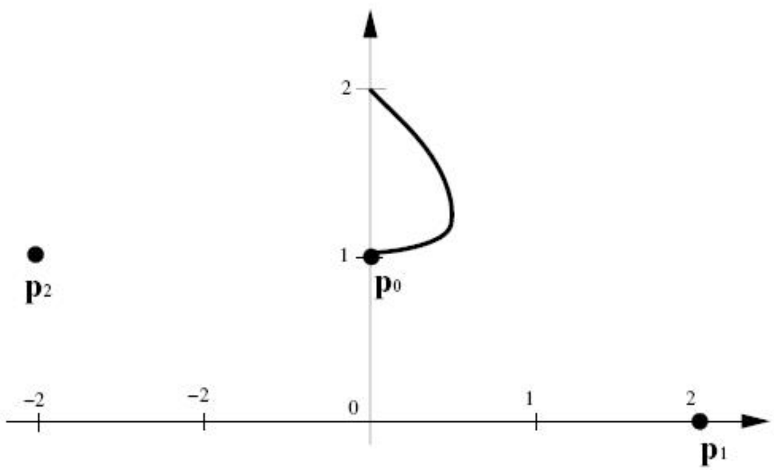
Parabolas

$$\mathbf{x}(t) = \mathbf{p}_0 + t \cdot \mathbf{p}_1 + t^2 \cdot \mathbf{p}_2$$

➔ planar curve, even if defined in R³

Example:

$$\mathbf{p}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{p}_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{p}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$



De Casteljau algorithm

Another parabola construction

given: 3 points b_0, b_1, b_2

$$\mathbf{b}_0^1 = (1-t)\cdot\mathbf{b}_0 + t\cdot\mathbf{b}_1$$
$$\mathbf{b}_1^1 = (1-t)\cdot\mathbf{b}_1 + t\cdot\mathbf{b}_2$$

$$\mathbf{b}_0^2 = (1-t)\cdot\mathbf{b}_0^1 + t\cdot\mathbf{b}_1^1$$

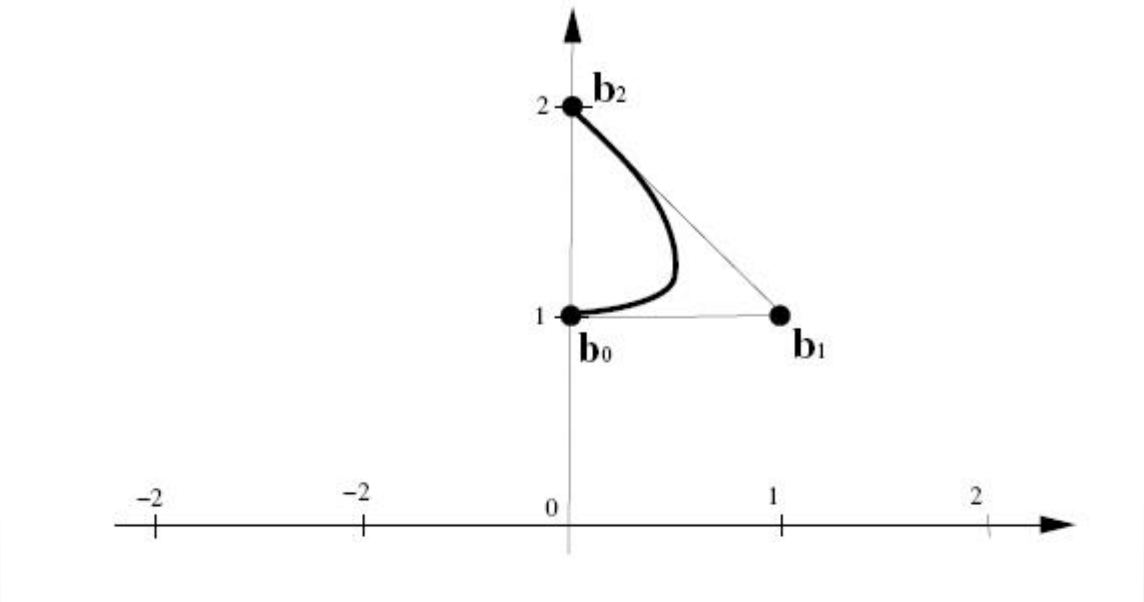
parabola x(t)

$$\mathbf{x}(t) = (1-t)^2\cdot\mathbf{b}_0 + 2\cdot t\cdot(1-t)\cdot\mathbf{b}_1 + t^2\cdot\mathbf{b}_2$$

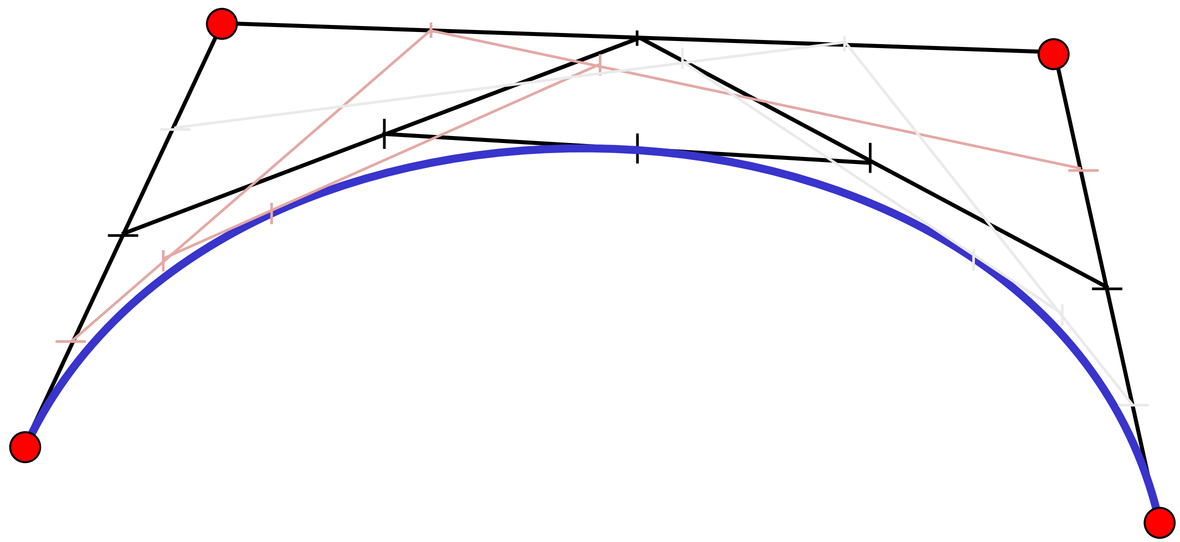
De Casteljau algorithm

Example

$$\mathbf{b}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$



De Casteljau algorithm



De Casteljau Algorithm: Computes $x(t)$ for given t

- Bisect control polygon in ratio $t : (1 - t)$
- Connect the new dots with lines (adjacent segments)
- Interpolate again with the same ratio
- Iterate, until only one point is left

De Casteljau algorithm

Description of the de Casteljau algorithm

- given: points $\mathbf{b}_0, \mathbf{b}_1, ..., \mathbf{b}_n \in \mathbb{R}^3$
- wanted: curve $\mathbf{x}(t), \quad t \in [0,1]$

- geometric construction of the point x(t) for given t:

$$\mathbf{b}_i^0(t) = \mathbf{b}_i \qquad \text{für } i = 0, ..., n$$

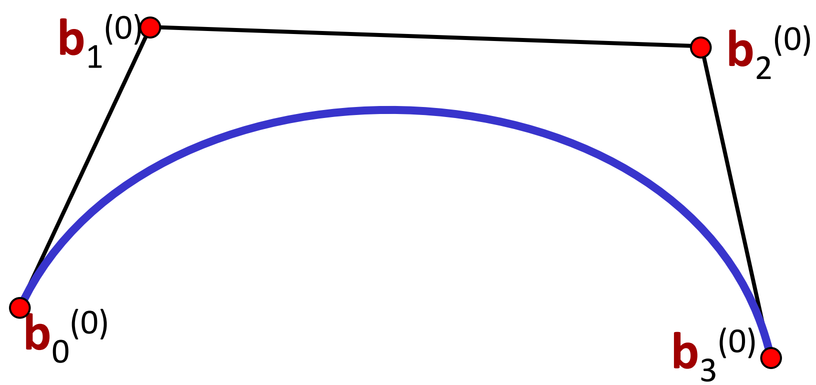
$$\mathbf{b}_i^r(t) = (1-t) \cdot \mathbf{b}_i^{r-1}(t) \; + \; t \cdot \mathbf{b}_{i+1}^{r-1}(t)$$
$$\text{für } r = 1, ..., n \quad ; \quad i = 0, ..., n-r.$$

- Then, $\mathbf{b}_0^n(t)$ is the searched curve point x(t) at the parameter value t

De Casteljau algorithm

repeated convex combination of control points

$$\mathbf{b}_i^{(r)} = (1-t)\cdot\mathbf{b}_i^{(r-1)} + t\cdot\mathbf{b}_{i+1}^{(r-1)}$$



$\mathbf{b}_0^{(0)}$

$\mathbf{b}_1^{(0)}$

$\mathbf{b}_2^{(0)}$

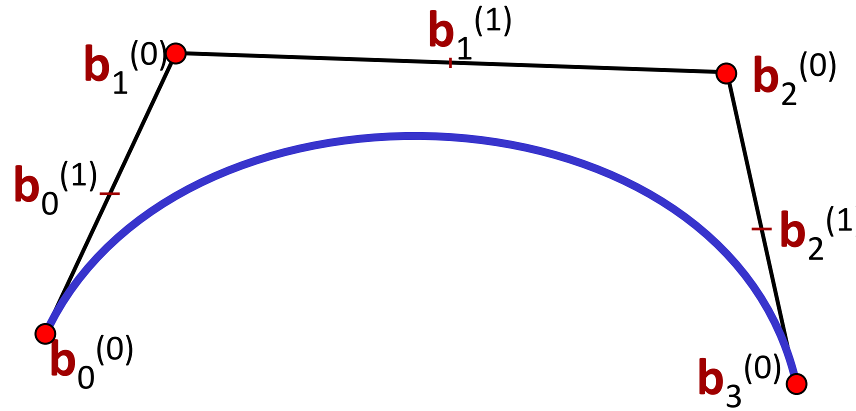
$\mathbf{b}_3^{(0)}$

De Casteljau algorithm

repeated convex combination of control points

$$\mathbf{b}_i^{(r)} = (1-t)\cdot\mathbf{b}_i^{(r-1)} + t\cdot\mathbf{b}_{i+1}^{(r-1)}$$

$$\begin{tikzcd}
\mathbf{b}_0^{(0)} \arrow[dr, "1-t"] & \\
\mathbf{b}_1^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \mathbf{b}_0^{(1)} \\
\mathbf{b}_2^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \mathbf{b}_1^{(1)} \\
\mathbf{b}_3^{(0)} \arrow[r, "t"] & \mathbf{b}_2^{(1)}
\end{tikzcd}$$

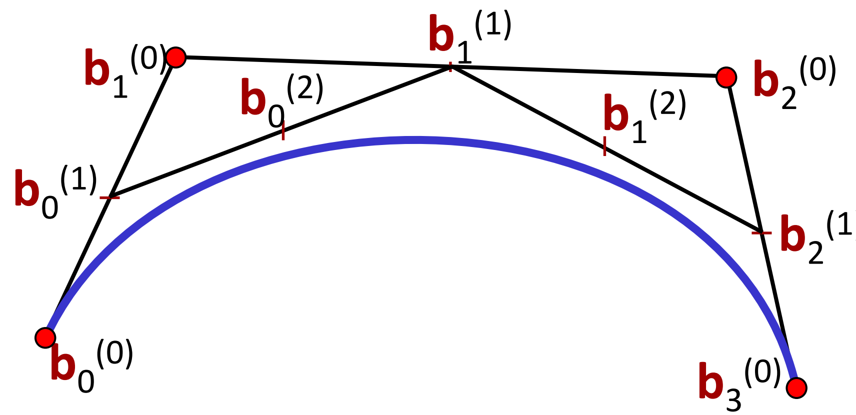


De Casteljau algorithm

repeated convex combination of control points

$$\mathbf{b}_i^{(r)} = (1-t)\cdot\mathbf{b}_i^{(r-1)} + t\cdot\mathbf{b}_{i+1}^{(r-1)}$$

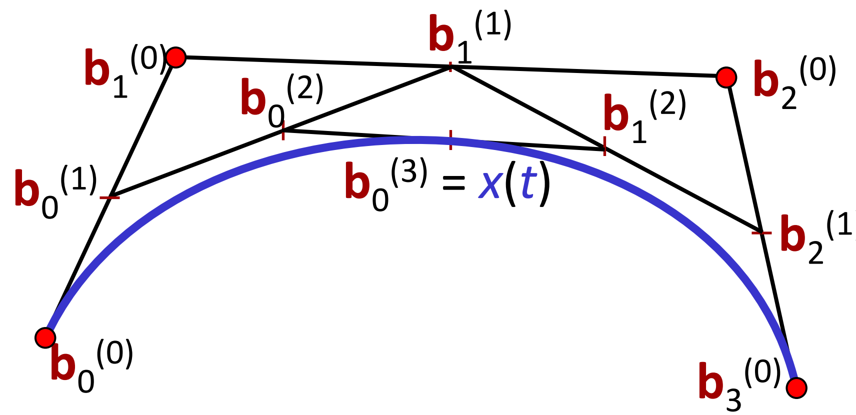
$$\begin{tikzcd}
\mathbf{b}_0^{(0)} \arrow[dr, "1-t"] & & \\
\mathbf{b}_1^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \mathbf{b}_0^{(1)} \arrow[dr, "1-t"] & \\
\mathbf{b}_2^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \mathbf{b}_1^{(1)} \arrow[r, "t"] \arrow[dr, "1-t"] & \mathbf{b}_0^{(2)} \\
\mathbf{b}_3^{(0)} \arrow[r, "t"] & \mathbf{b}_2^{(1)} \arrow[r, "t"] & \mathbf{b}_1^{(2)}
\end{tikzcd}$$

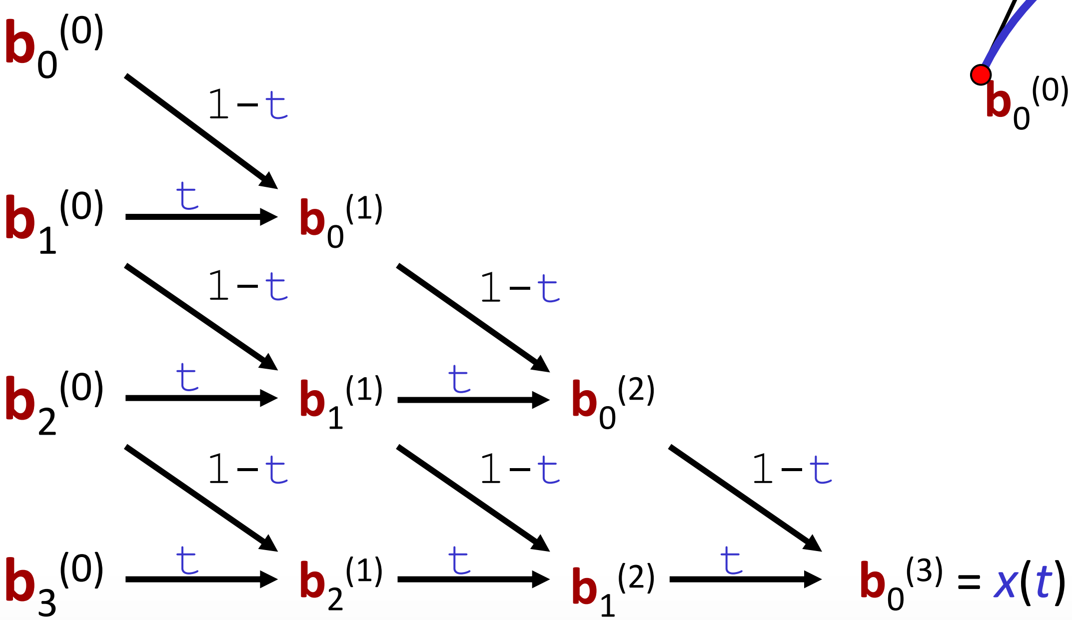


De Casteljau algorithm

repeated convex combination of control points

$$\mathbf{b}_i^{(r)} = (1-t)\cdot\mathbf{b}_i^{(r-1)} + t\cdot\mathbf{b}_{i+1}^{(r-1)}$$





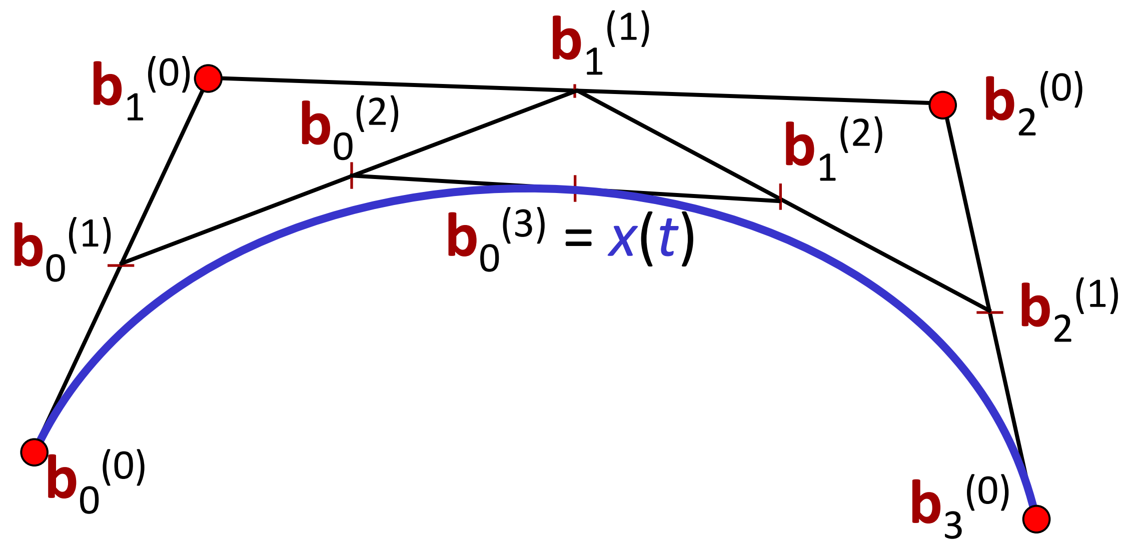
de Casteljau scheme

De Casteljau algorithm

The intermediate coefficients $b_i^r(t)$ can be written in a triangular matrix: the de Casteljau scheme:

$$
\begin{array}{lllll}
\mathbf{b}_0 = \mathbf{b}_0^0 & & & & \\
\mathbf{b}_1 = \mathbf{b}_1^0 & \mathbf{b}_0^1 & & & \\
\mathbf{b}_2 = \mathbf{b}_2^0 & \mathbf{b}_1^1 & \mathbf{b}_0^2 & & \\
\mathbf{b}_3 = \mathbf{b}_3^0 & \mathbf{b}_2^1 & \mathbf{b}_1^2 & \mathbf{b}_0^3 & \\
\vdots & & & & \\
\mathbf{b}_{n-1} = \mathbf{b}_{n-1}^0 & \mathbf{b}_{n-2}^1 & \dots & \mathbf{b}_0^{n-1} & \\
\mathbf{b}_n \;\; = \mathbf{b}_n^0 & \mathbf{b}_{n-1}^1 & \dots & \mathbf{b}_1^{n-1} & \mathbf{b}_0^n = \mathbf{x}(t)
\end{array}
$$

De Casteljau algorithm



Algorithm:

for $r = 1..n$ **do**

  for $i = 0..n-r$ **do**

    $\mathbf{b}_i^{(r)} = (1-t) \cdot \mathbf{b}_i^{(r-1)} + t \cdot \mathbf{b}_{i+1}^{(r-1)}$

  end for

end for

return $\mathbf{b}_0^{(n)}$

The whole algorithm consists only of repeated linear interpolations.

De Casteljau algorithm

The polygon consisting of the points b_0, …, b_n is called Bezier polygon. The points b_i are called Bezier points.

The curve defined by the Bezier points b_0, …, b_n and the de Casteljau algorithm is called Bezier curve.

The de Casteljau algorithm is numerically stable, since only convex combinations are applied.

Complexity of the de Casteljau algorithm

- $O(n^2)$ time
- $O(n)$ memory
- with n being the number of Bezier points

De Casteljau algorithm

Properties of Bezier curves:

- given: Bezier points $\mathbf{b}_0, \ldots, \mathbf{b}_n$

  Bezier curve $\mathbf{x}(t)$
- Bezier curve is polynomial curve of degree n.
- End point interpolation: $\mathbf{x}(0) = \mathbf{b}_0$, $\mathbf{x}(1) = \mathbf{b}_n$. The remaining Bezier points are only generally approximated.
- Convex hull property:

  Bezier curve is completely inside the convex hull of its Bezier polygon.

De Casteljau algorithm

- **Variation diminishing**
  no line intersects the Bezier curve more often than its Bezier polygon.

- **Influence of Bezier points: global, but pseudo-local**
  - *global:* moving a Bezier point changes the whole curve progression
  - *pseudo-local:* $\mathbf{b}_i$ has its maximal influence on $\mathbf{x}(t)$ at $t = i/n$.

- **Affine invariance:**
  Bezier curve and Bezier polygon are invariant under affine transformations

- **Invariance under affine parameter transformations**

De Casteljau algorithm

- **Symmetry:**
  The following two Bezier curves coincide, they are only traversed in opposite directions:

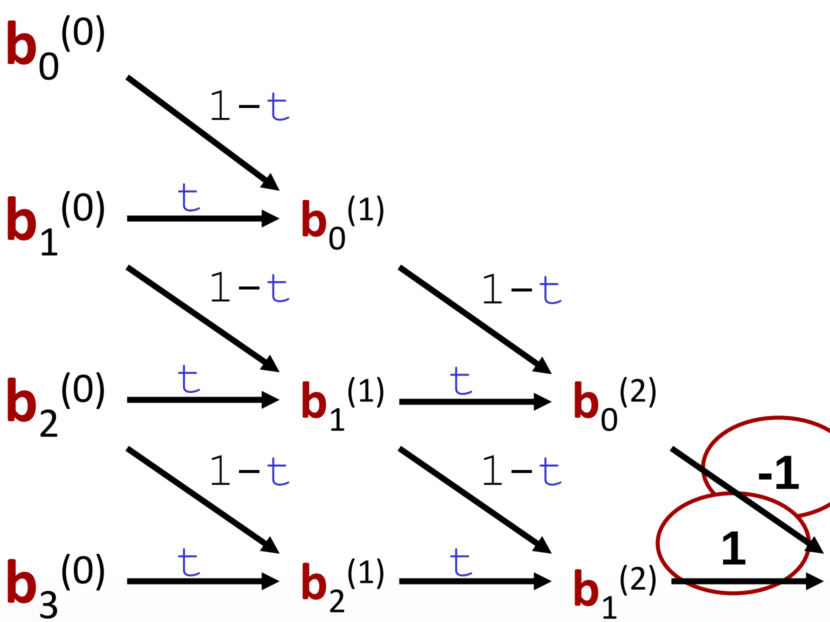
$$\mathbf{x}(t) = [\mathbf{b}_0, \ldots, \mathbf{b}_n] \qquad \mathbf{x}'(t) = [\mathbf{b}_n, \ldots, \mathbf{b}_0]$$

- **Linear precision:**
  Bezier curve is line segment, if $\mathbf{b}_0, \ldots, \mathbf{b}_n$ are collinear

- **Invariant under barycentric combinations**

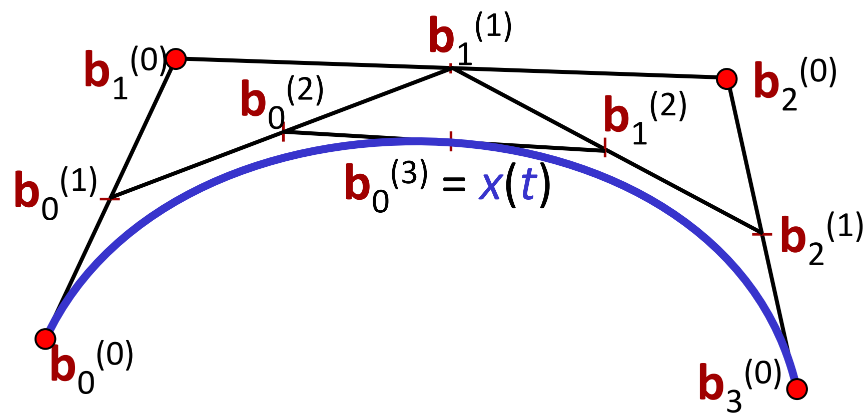
De Casteljau algorithm

- **First derivative of a Bezier curve**
  - **Endpoints:** $\dot{\mathbf{x}}(0) = n \cdot (\mathbf{b}_1 - \mathbf{b}_0)$ $\dot{\mathbf{x}}(1) = n \cdot (\mathbf{b}_n - \mathbf{b}_{n-1})$ $t = 0, t = 1$:



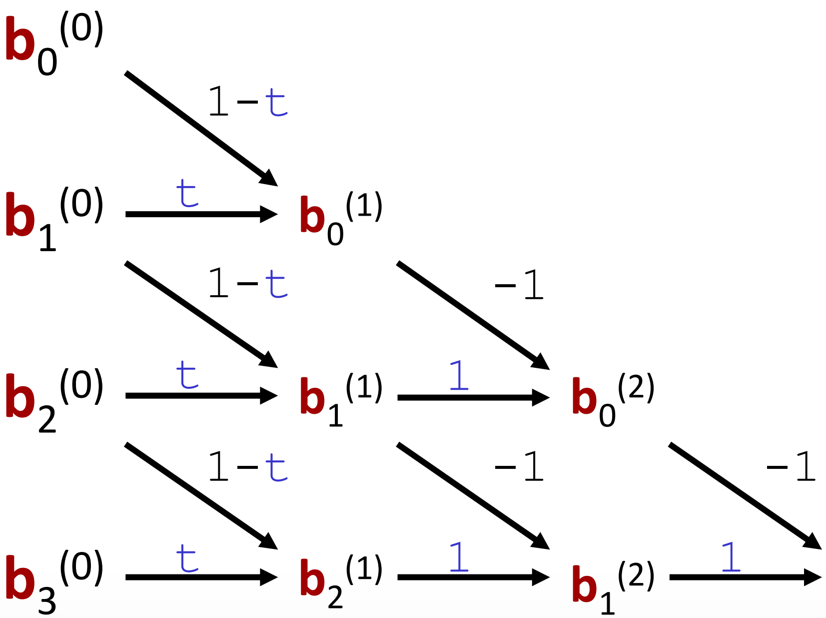
$$\dot{\mathbf{x}}(t) = n\left(\mathbf{b}_1^{(n-1)} - \mathbf{b}_0^{(n-1)}\right)$$

de Casteljau scheme

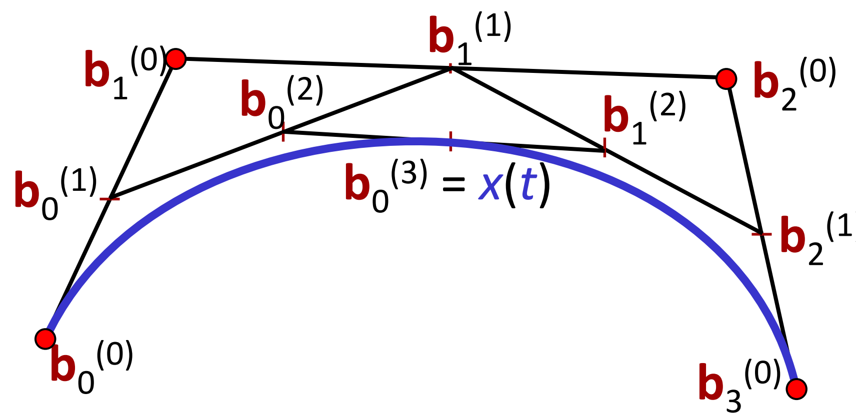


De Casteljau algorithm

- **Second derivative of a Bezier curve**



de Casteljau scheme



$$\ddot{\mathbf{x}}(t) = n(n-1)\left(\mathbf{b}_2^{(n-2)} - 2\mathbf{b}_1^{(n-2)} + \mathbf{b}_0^{(n-2)}\right)$$

Bezier Curves

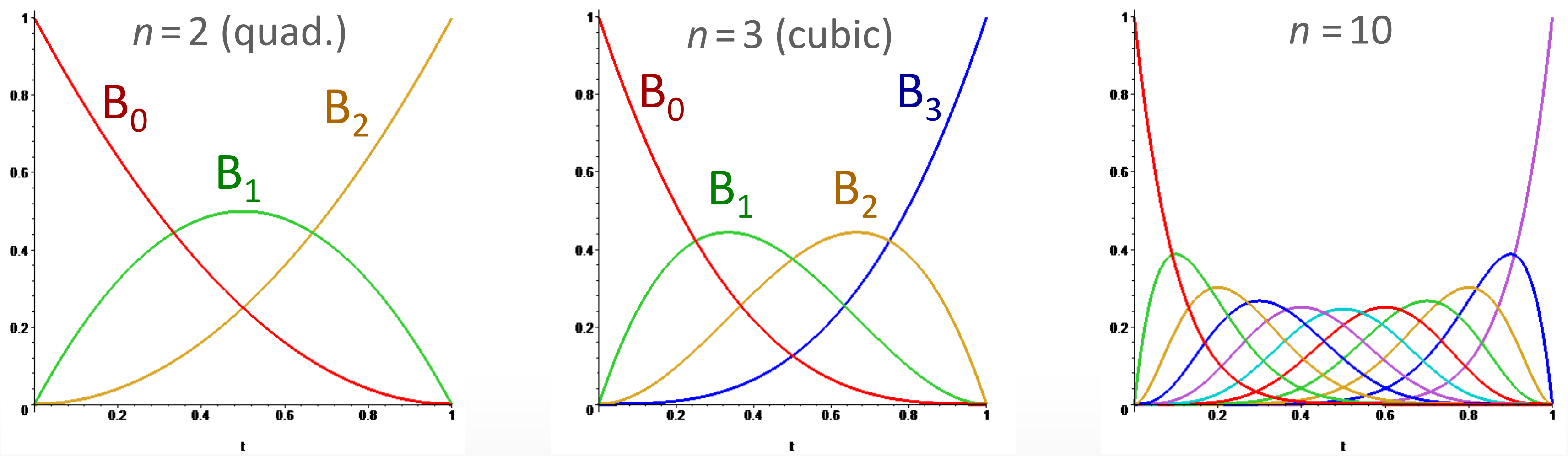
Bernstein form

Bernstein Basis

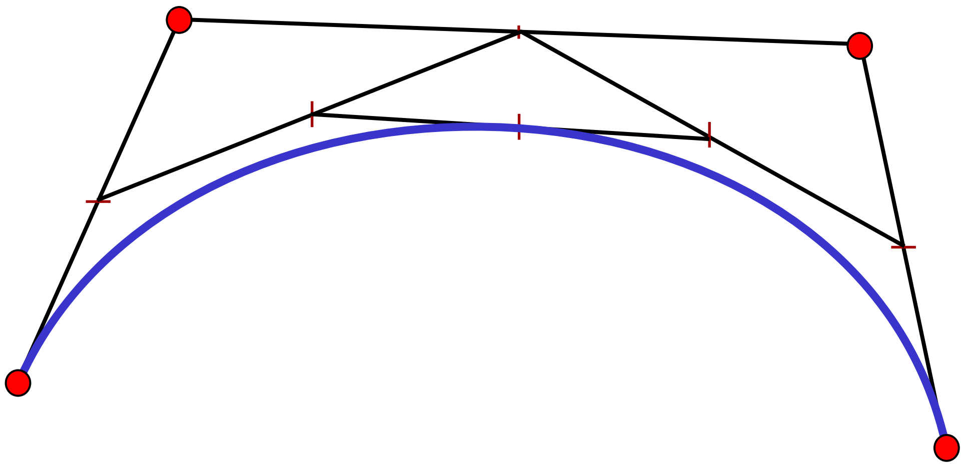
Bezier curves are algebraically defined using the Bernstein basis:

- Bernstein basis of degree *n*: $$B = \left\{ B_0^{(n)}, B_1^{(n)}, \ldots, B_n^{(n)} \right\}$$

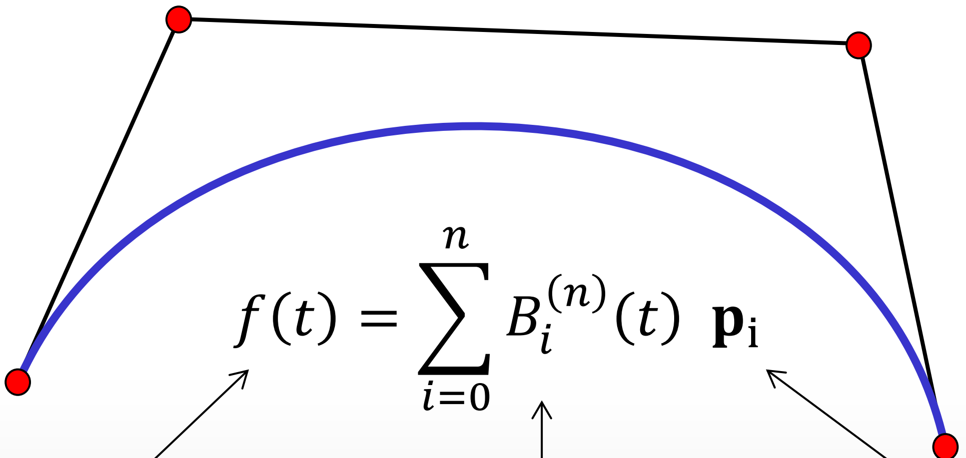
$$B_i^{(n)}(t) := \binom{n}{i} t^i (1-t)^{n-i}$$



Bernstein Basis



de Casteljau algorithm



Bernstein form

$$f(t) = \sum_{i=0}^{n} B_i^{(n)}(t)\ \mathbf{p}_\mathrm{i}$$

curve basis function control point

Examples

The first three Bernstein bases:

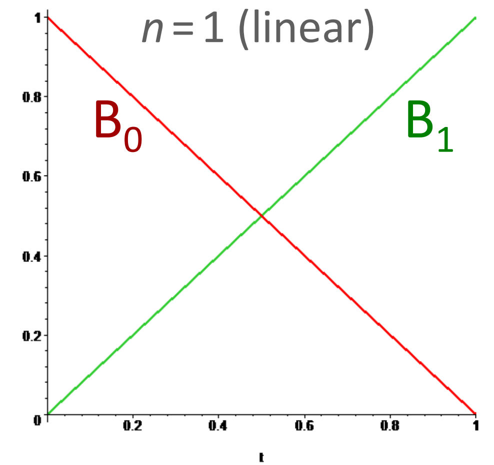
$$B_0^{(0)} := 1$$

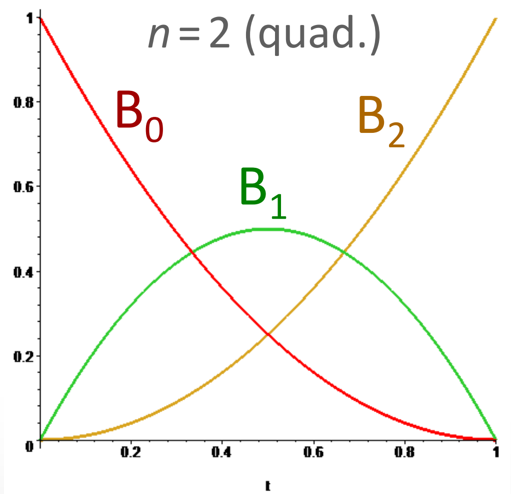
$$B_0^{(1)} := (1-t) \qquad B_1^{(1)} := t$$

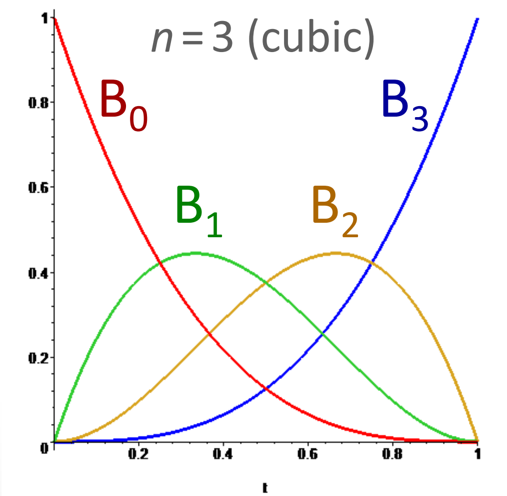
$$B_0^{(2)} := (1-t)^2 \qquad B_1^{(2)} := 2t(1-t) \qquad B_2^{(2)} := t^2$$

$$B_0^{(3)} := (1-t)^3 \qquad B_1^{(3)} := 3t(1-t)^2$$

$$B_2^{(3)} := 3t^2(1-t) \qquad B_3^{(3)} := t^3$$







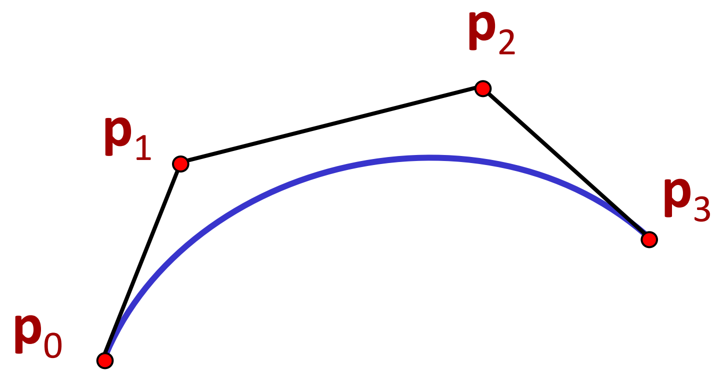
$$B_i^{(n)}(t) := \binom{n}{i} t^i (1-t)^{n-i}$$

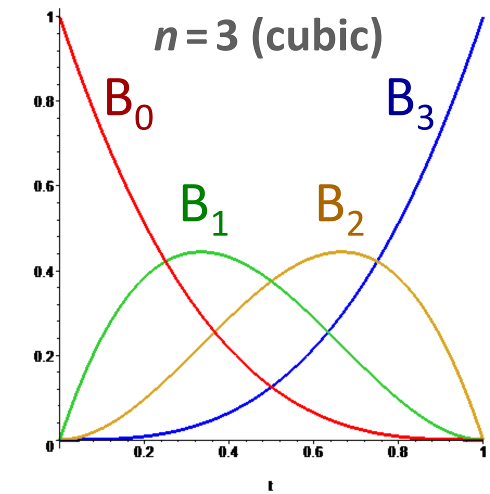
Bezier Curves in Bernstein form

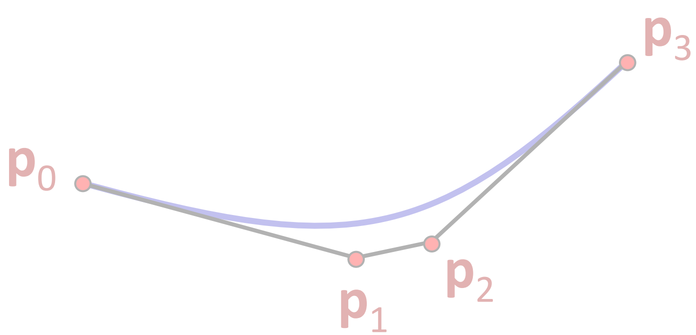
Bezier Curves:

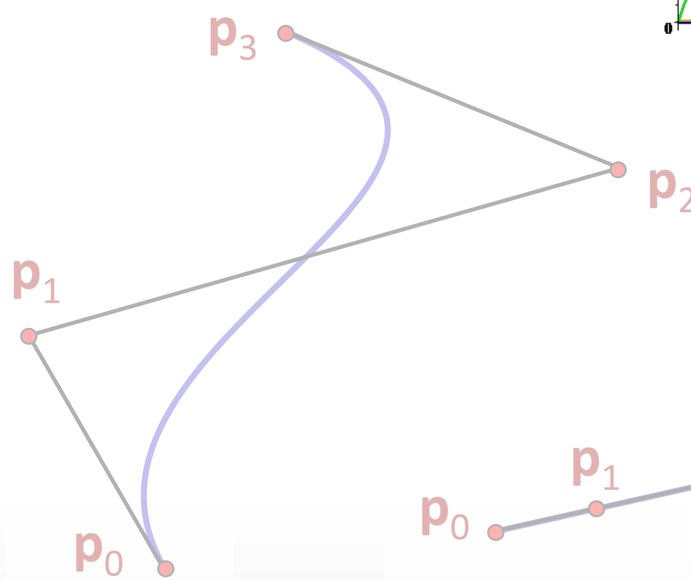
- $\mathbf{f}(t) = \sum_{i=0}^{n} \mathbf{p}_i B_i^{(n)}$

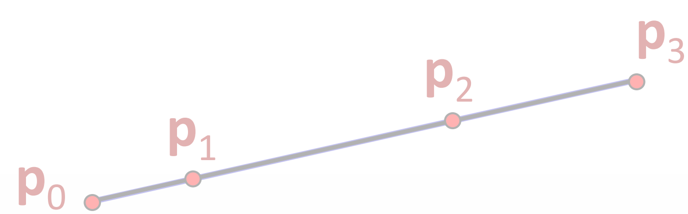
  $t \in [0..1]$







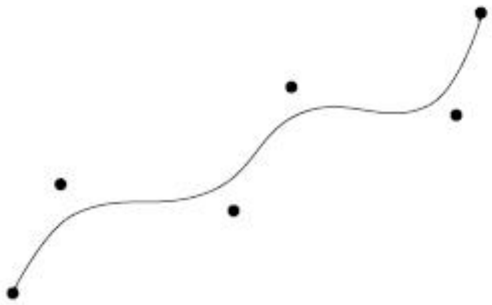




Summary for Bezier Curves

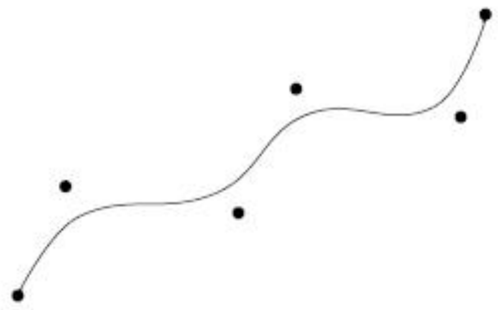
Bezier curves and curve design:

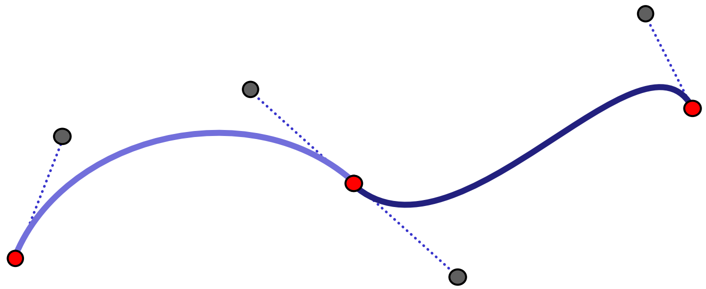
- The rough form is specified by the position of the control points
- Result: smooth curve approximating the control points
- Computation / Representation:
  - de Casteljau algorithm
  - Bernstein form



- Problems:
  - high polynomial degree
  - moving a control point can change the whole curve
  - interpolation of points
  - → **Bezier splines**

Towards Bezier Splines





Approximation → Interpolation

Towards Bezier Splines

Interpolation problem:

- given:

  $\mathbf{k}_0, \ldots, \mathbf{k}_n \in \mathbb{R}^3$ control points

  $t_0, \ldots, t_n \in \mathbb{R}$ knot sequence

  $t_i < t_{i+1}$ für $i = 0, \ldots, n-1$

- wanted:

  interpolating curve $\mathbf{x}(t)$, i.e., $\mathbf{x}(t_i) = \mathbf{k}_i$ for $i = 0, \ldots, n$

- Approach:
  "Joining" of n Bezier curves with certain intersection conditions

Towards Bezier Splines

The following issues arise when stitching together Bezier curves:

- Continuity
- Degree
- (Parameterization)

Bezier Splines

Parametric and Geometric Continuity

Continuity

Joining of curves - continuity

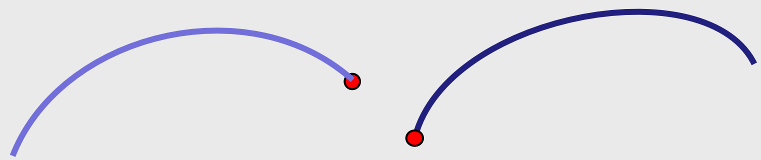
- given: 2 curves

  $\mathbf{x}_1(t)$ over $[t_0, t_1]$

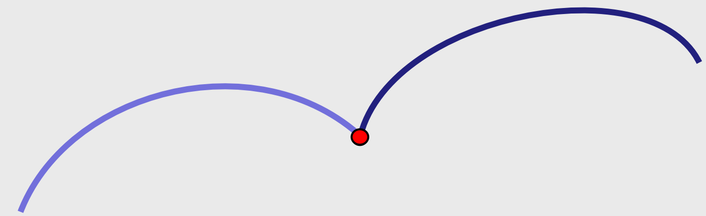
  $\mathbf{x}_2(t)$ over $[t_1, t_2]$

- $\mathbf{x}_1$ and $\mathbf{x}_2$ are C^r continuous in t_1, if they coincide in $0^{th} - r^{th}$ derivative vector in t_1.

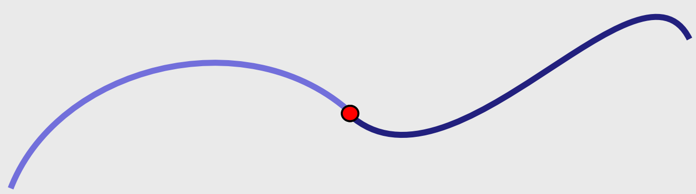
Continuity



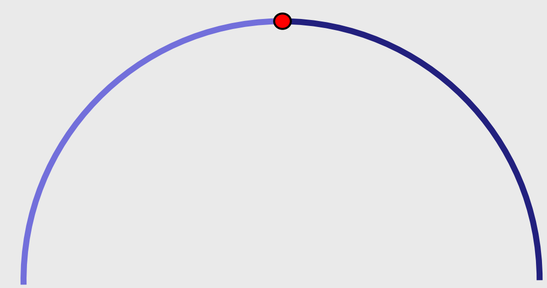
C^{-1} continuity



C^{0} continuity



C^{1} continuity



C^{2} continuity

Continuity

Parametric Continuity C^r**:**

- C^0, C^1, C^2... continuity.
- Does a particle moving on this curve have a smooth trajectory (position, velocity, acceleration,...)?
- Useful for animation (object movement, camera paths)
- **Depends** on parameterization

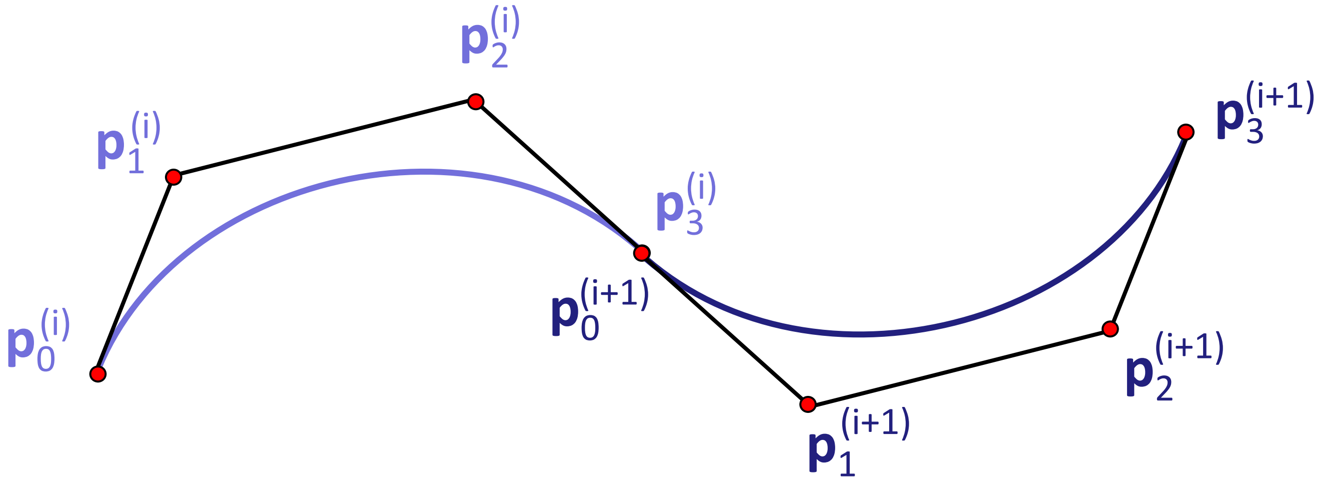
Geometric Continuity G^r**:**

- **Independent** of parameterization
- Is the curve itself smooth?
- More relevant for modeling (curve design)

Bezier Splines

Local control: Bezier splines

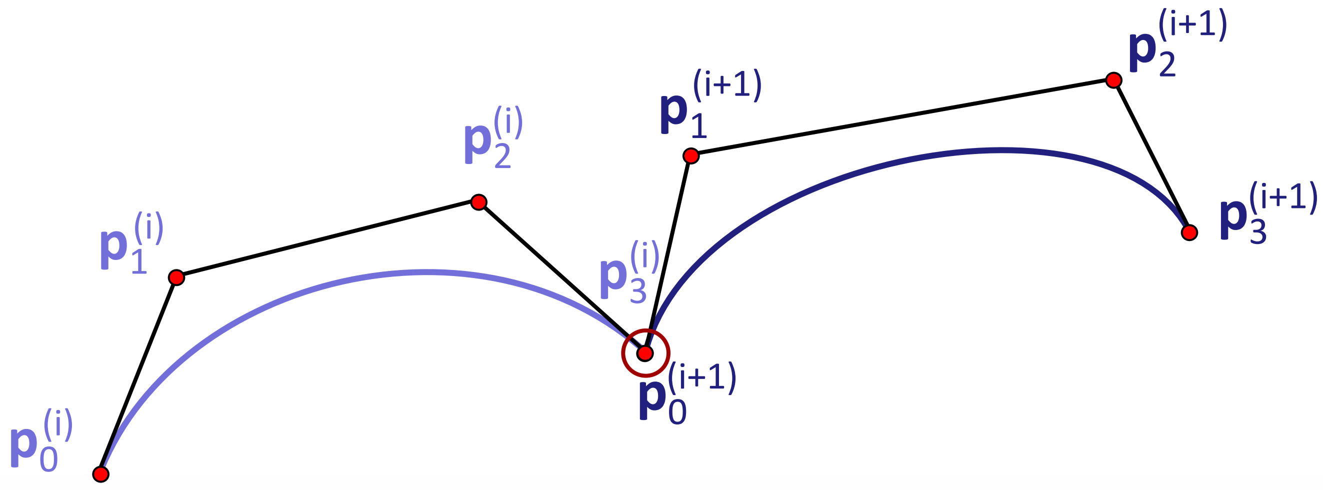
- Concatenate several curve segments
- Question: Which constraints to place upon the control points in order to get C^{-1}, C^0, C^1, C^2 continuity?



Bezier Spline Continuity

Rules for Bezier spline continuity:

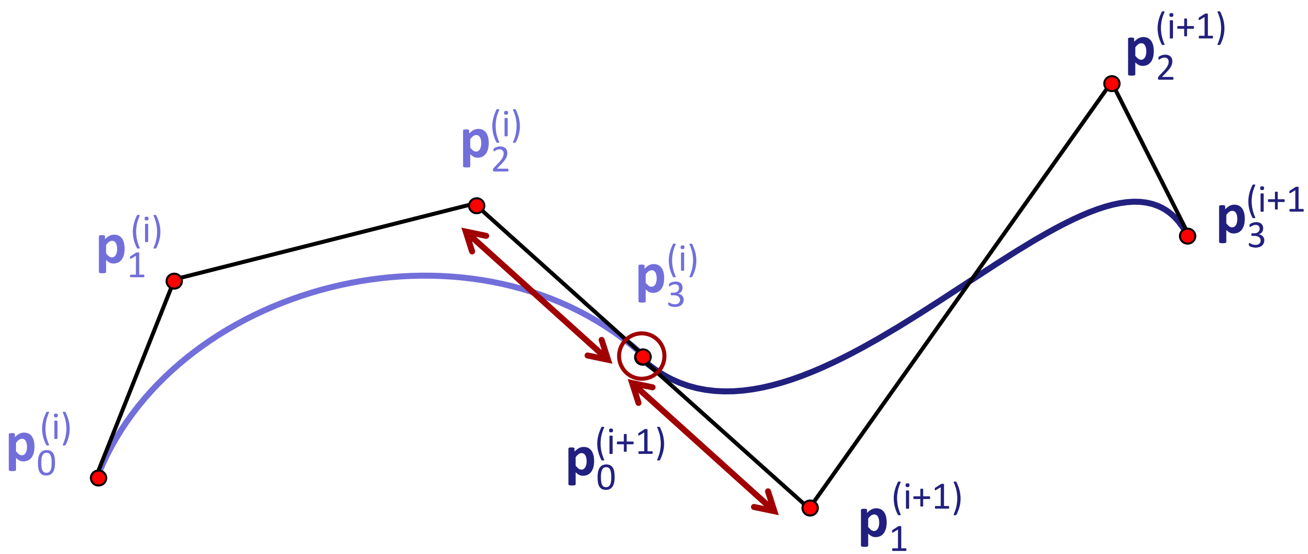
- C^0 continuity:
  - Each spline segment interpolates the first and last control point
  - Therefore: Points of neighboring segments have to coincide for C^0 continuity.



Bezier Spline Continuity

Rules for Bezier spline continuity:

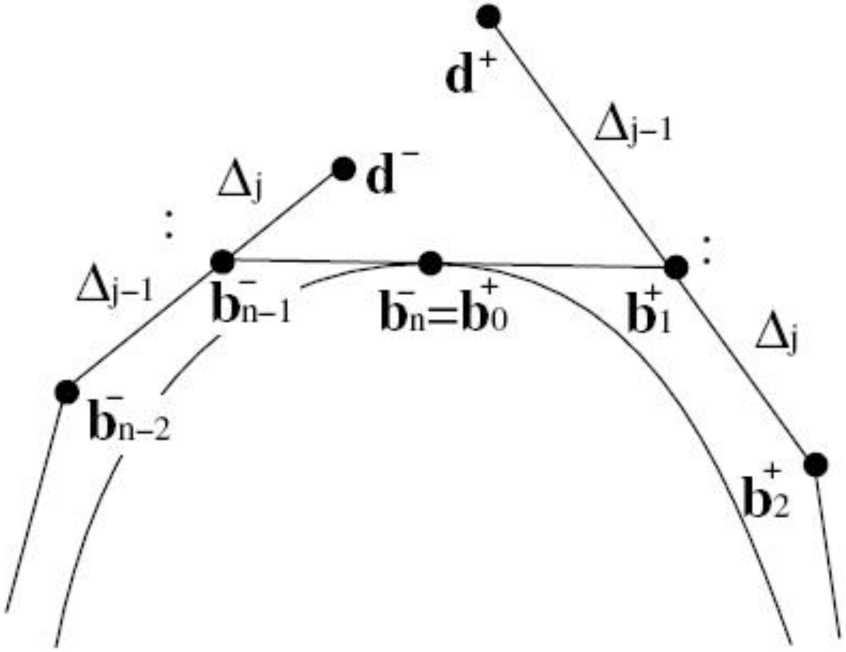
- Additional requirement for C^1 continuity:
  - Tangent vectors are proportional to differences $\mathbf{p}_1 - \mathbf{p}_0$, $\mathbf{p}_n - \mathbf{p}_{n\text{-}1}$
  - Therefore: These vectors must be identical for C^1 continuity



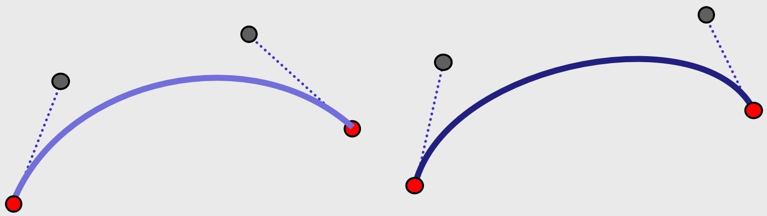
Bezier Spline Continuity

Rules for Bezier spline continuity:

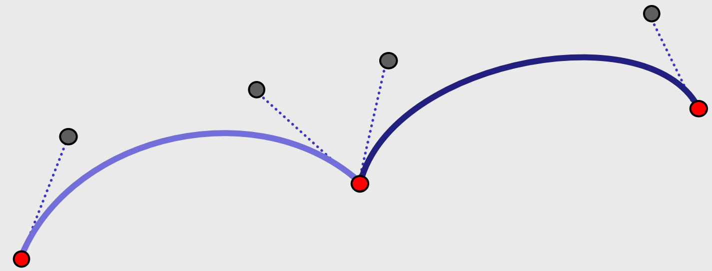
- Additional requirement for C^2 continuity:
  - $\mathbf{d}^- = \mathbf{d}^+$



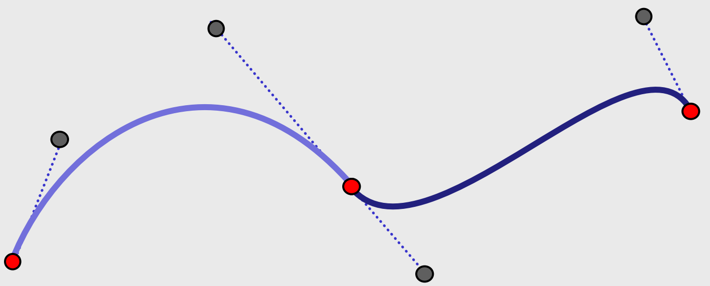
Continuity



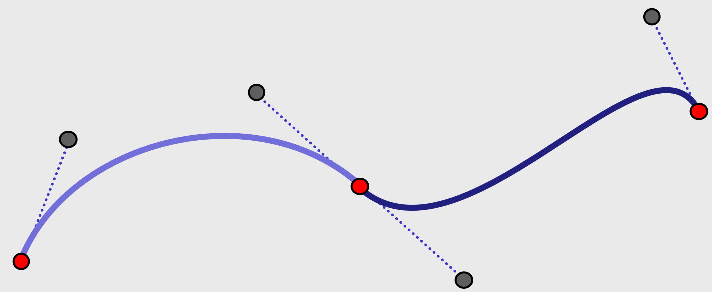
C^{-1} continuity



C^{0} continuity



G^{1} continuity



C^{1} continuity

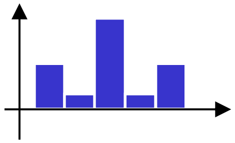
Bezier Splines

Choosing the degree

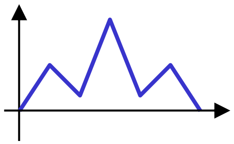
Choosing the Degree...

Candidates:

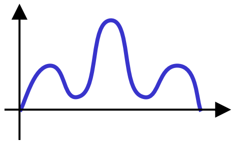
- *d* = 0 (piecewise constant): not smooth



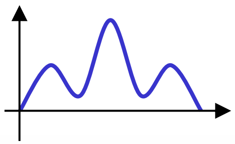
- *d* = 1 (piecewise linear): not smooth enough



- *d* = 2 (piecewise quadratic): constant 2nd derivative, still too inflexible



- *d* = 3 (piecewise cubic): degree of choice for computer graphics applications



Cubic Splines

Cubic piecewise polynomials:

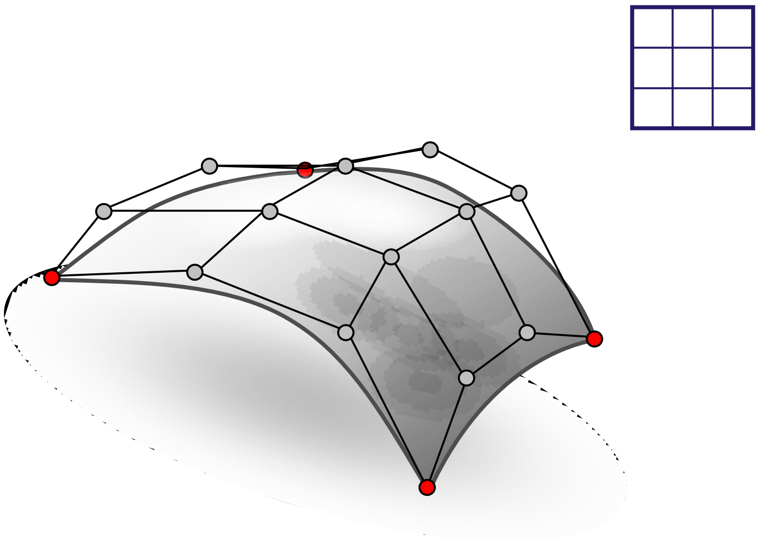
- We can attain C^2 continuity without fixing the second derivative throughout the curve
- C^2 continuity is perceptually important
  - We can see second order shading discontinuities (esp.: reflective objects)
  - Motion: continuous *position*, *velocity* & *acceleration*
    Discontinuous acceleration noticeable (object/camera motion)
- One more argument for cubics:
  - Among all C^2 curves that interpolate a set of points (and obey to the same end conditions), a piecewise cubic curve has the least integral acceleration ("smoothest curve you can get").
    - see `AdditionalMaterial/CubicsMinimizeAcceleration.pdf`

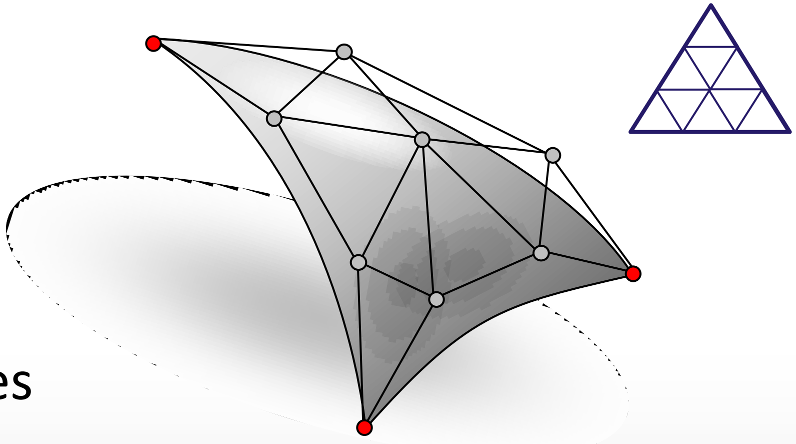
Spline Surfaces

Spline Surfaces

Two different approaches

- Tensor product surfaces
  - Simple construction
  - Everything carries over from curve case
  - Quad patches
  - Degree anisotropy
- Total degree surfaces
  - Not as straightforward
  - Isotropic degree
  - Triangle patches
  - "Natural" generalization of curves



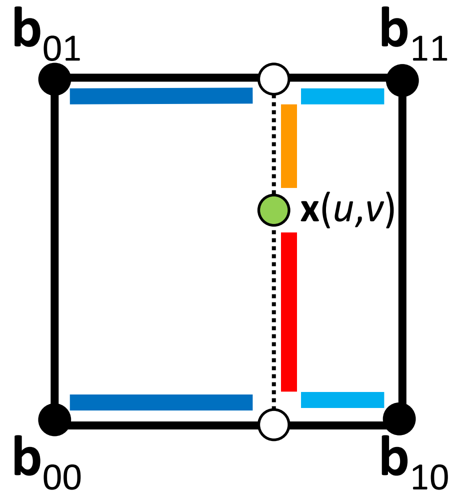


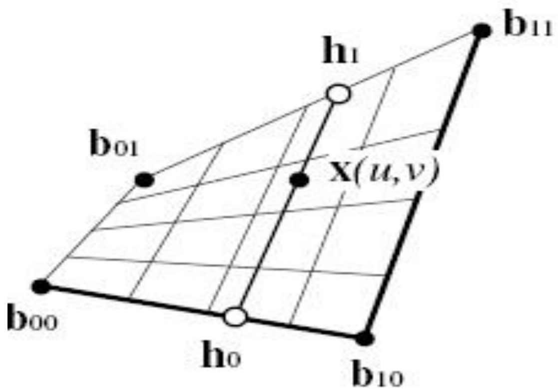
Tensor Product Surfaces

Tensor Product Bezier Surfaces

Bezier curves:
repeated linear interpolation

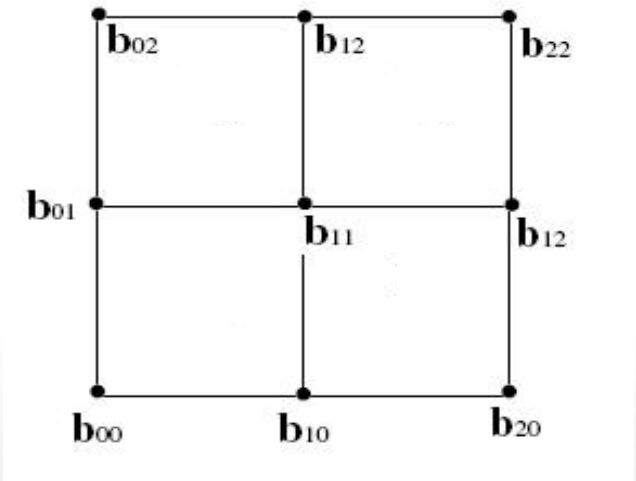
now a different setup:
4 points $\mathbf{b}_{00}$, $\mathbf{b}_{10}$, $\mathbf{b}_{11}$, $\mathbf{b}_{01}$
parameter area $[0,1] \times [0,1]$





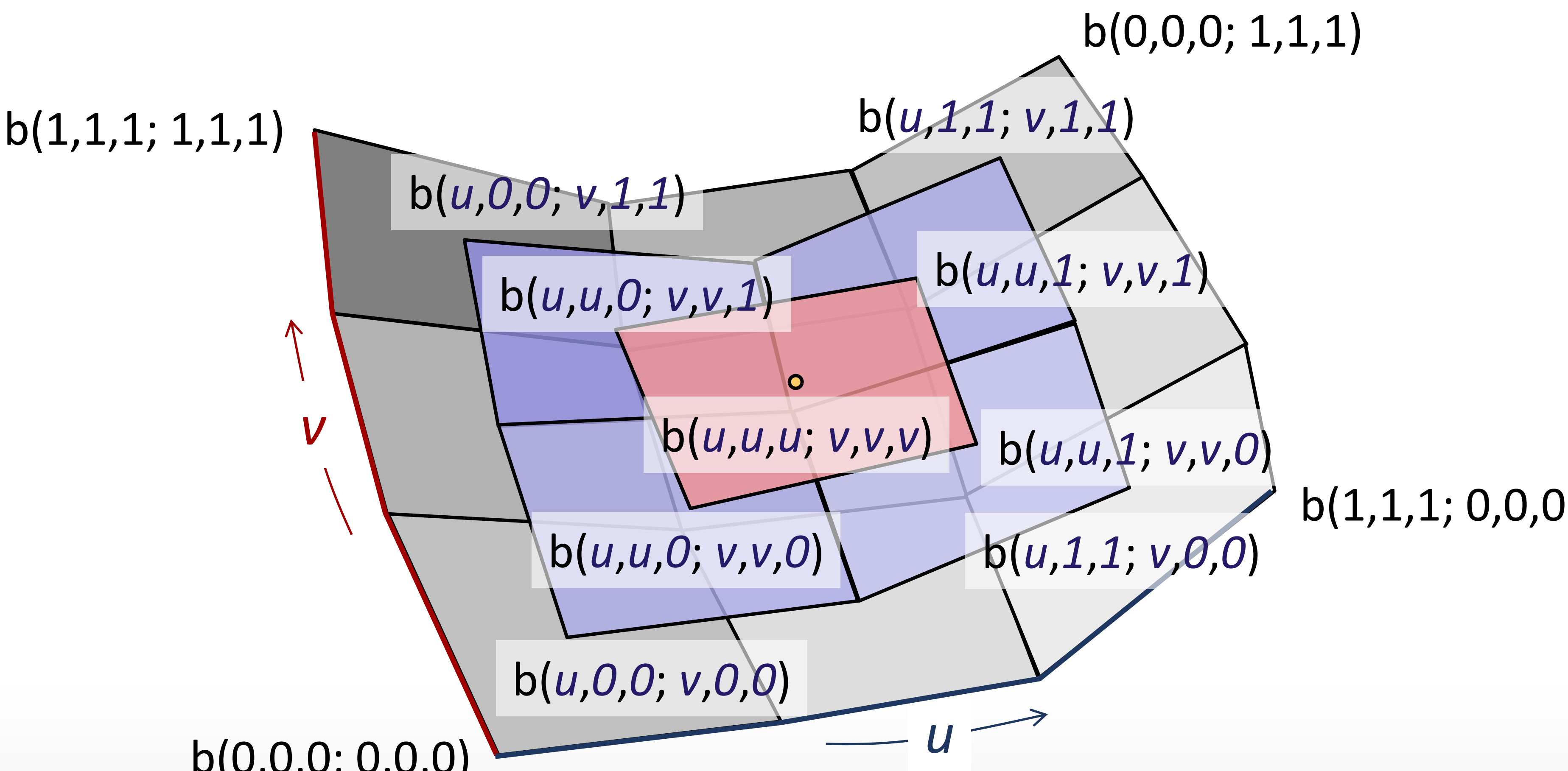
bilinear interpolation:
repeated linear interpolation

repeated bilinear interpolation:
gives us tensor product Bezier surfaces
(example shows quadratic Bezier surface)



De Casteljau Algorithm

De Casteljau algorithm for tensor product surfaces:



Tensor Product Surfaces

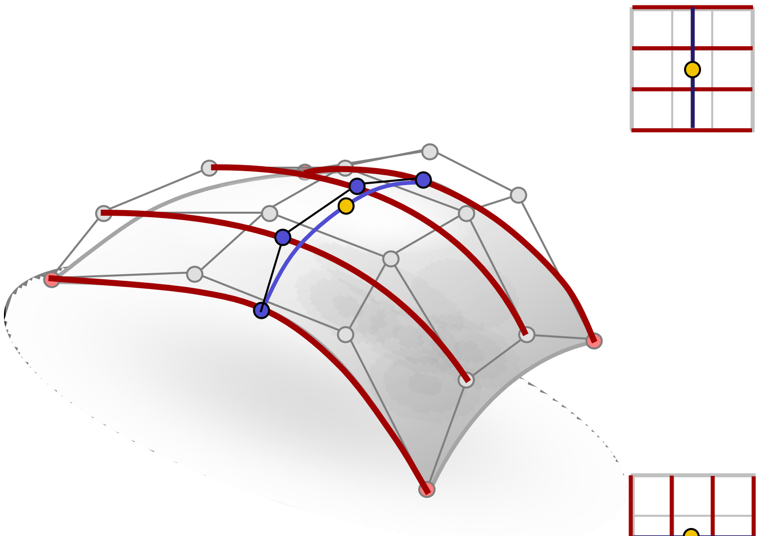
Tensor Product Surfaces:

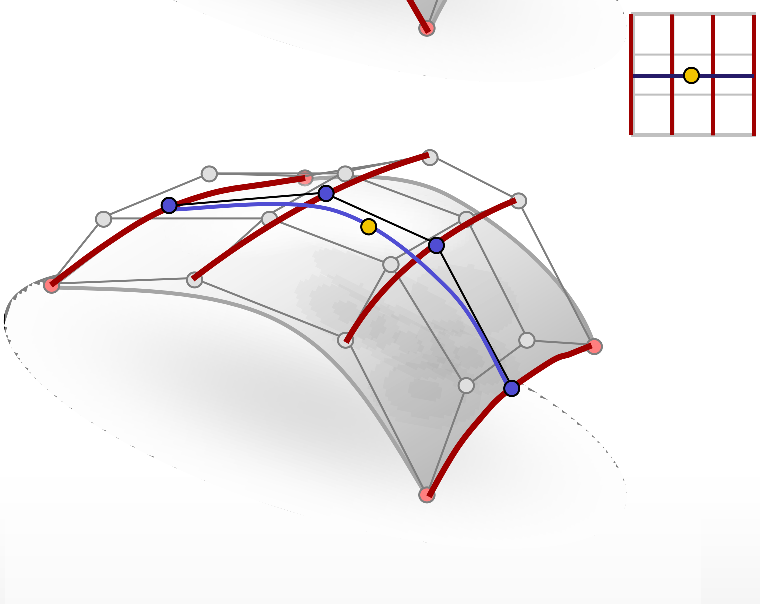
$$\mathbf{f}(u,v) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_i(u) b_j(v) \mathbf{p}_{i,j}$$

$$= \sum_{i=1}^{n} b_i(u) \sum_{j=1}^{n} b_j(v) \mathbf{p}_{i,j}$$

$$= \sum_{j=1}^{n} b_j(u) \sum_{i=1}^{n} b_i(v) \mathbf{p}_{i,j}$$

- “Curves of Curves”
- Order does not matter





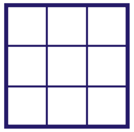
Tensor Product Surfaces

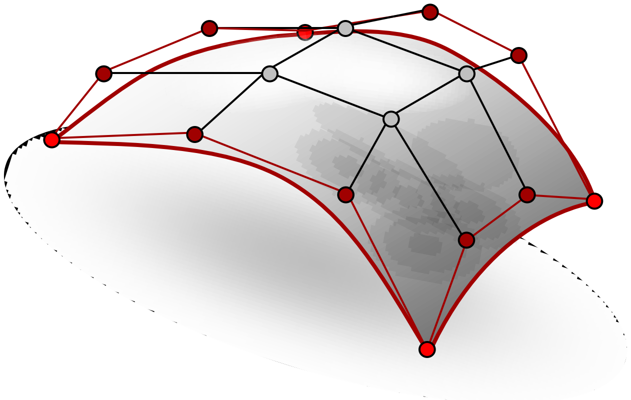
Bezier Patches

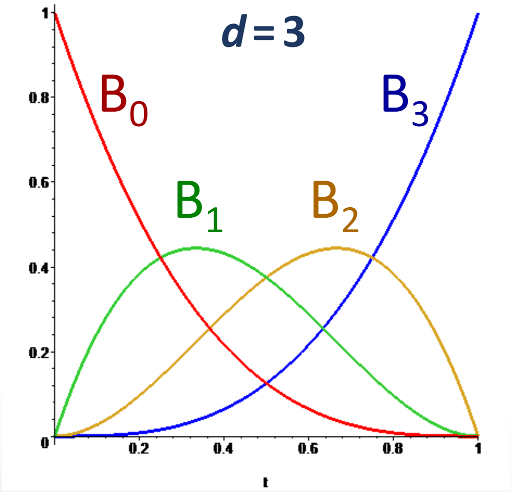
Bezier Patches

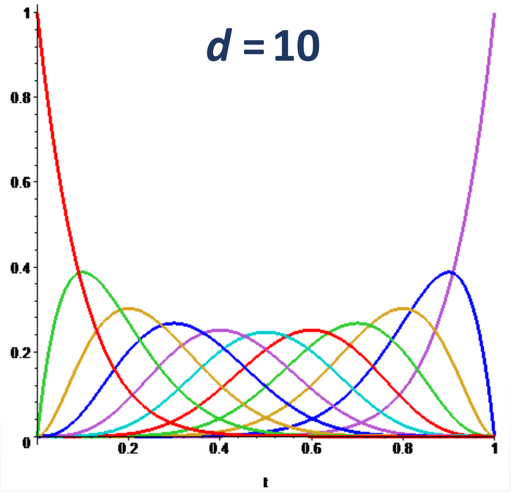
Bezier Patches:

- Remember endpoint interpolation:
  - Boundary curves are Bezier curves of the boundary control points









Continuity Conditions

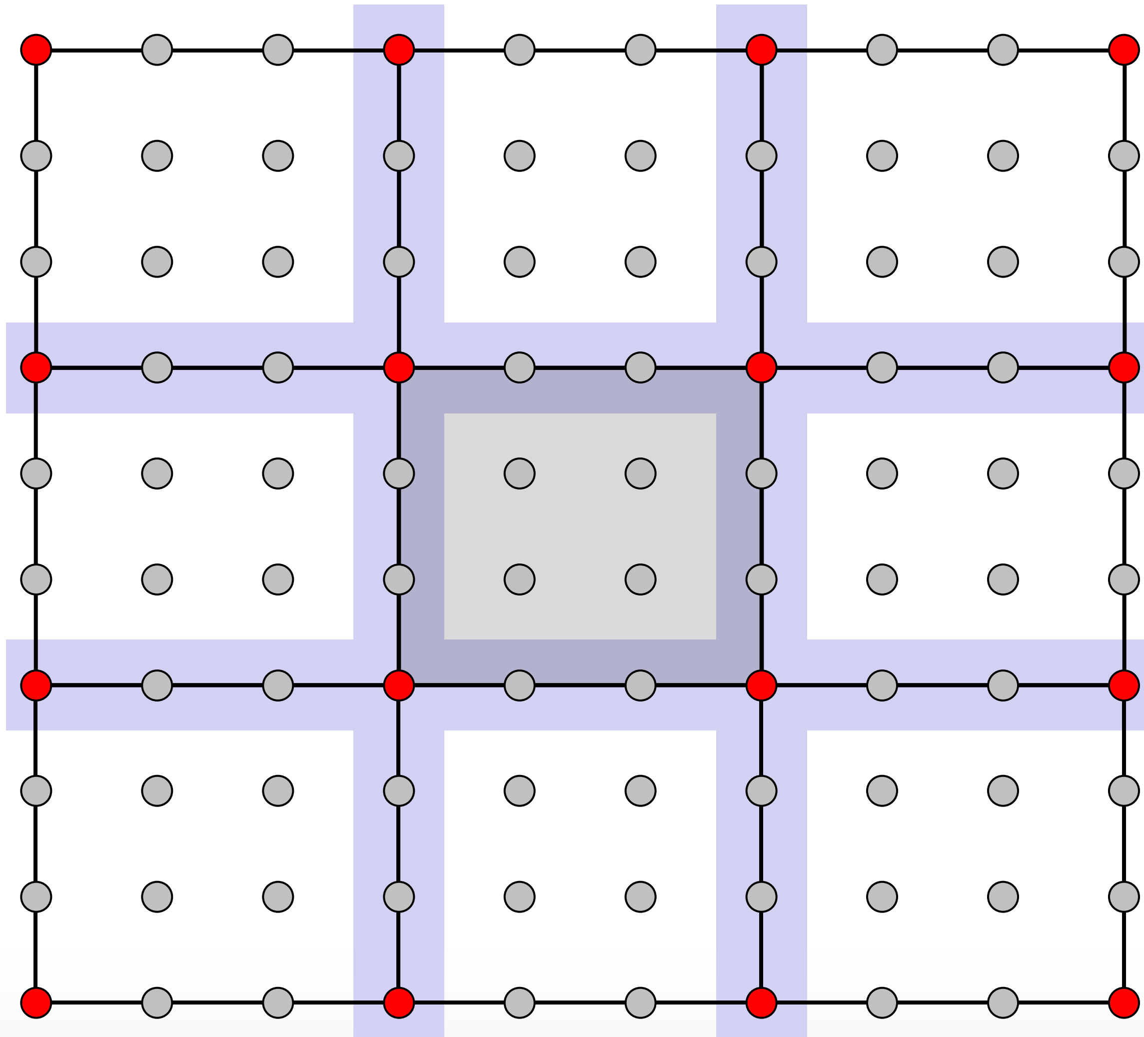
For C^0 continuity:

- Boundary control points must match

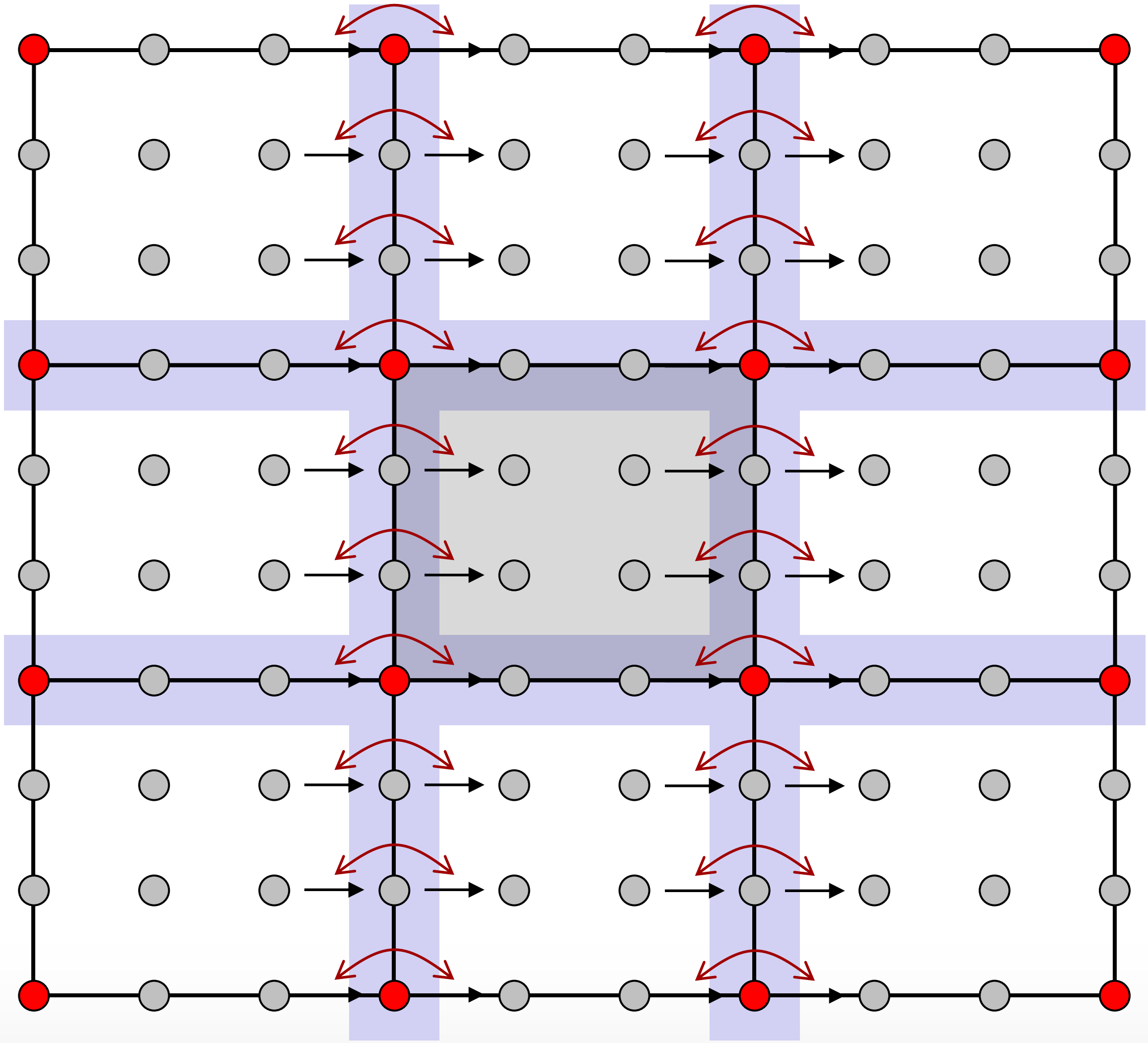
For C^1 continuity:

- Difference vectors must match at the boundary

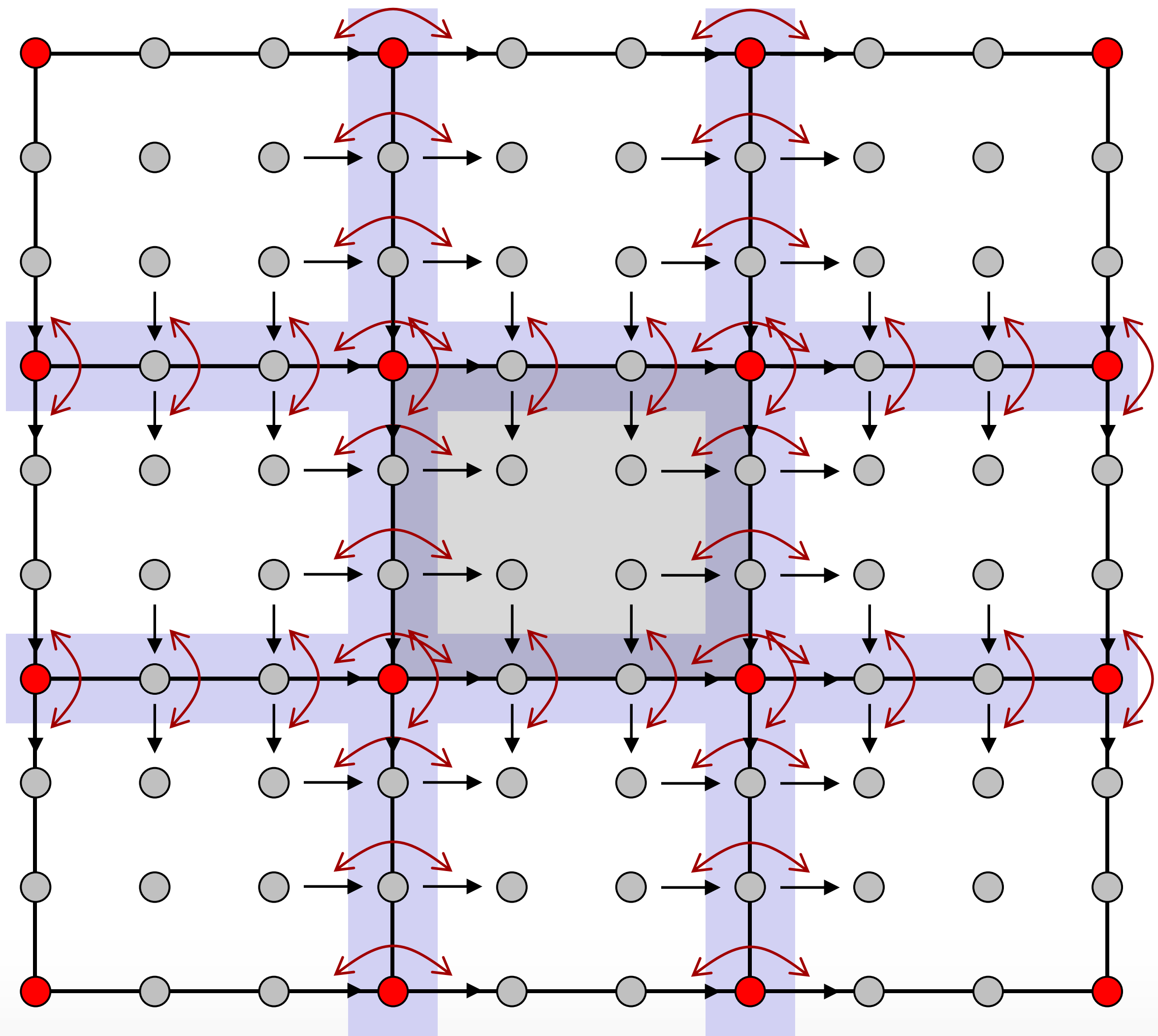
C^0 Continuity



C^1 Continuity



C^1 Continuity

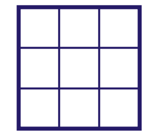


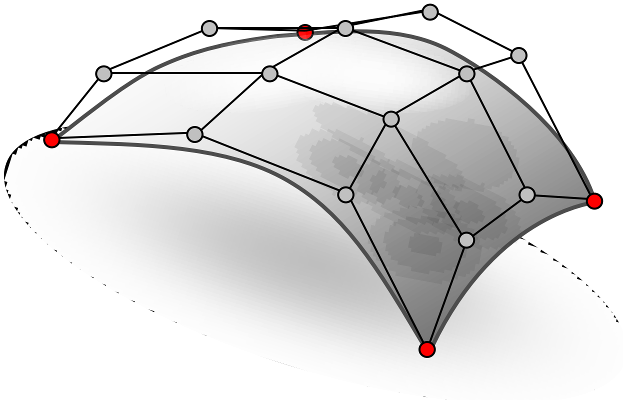
Total Degree Surfaces

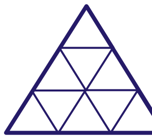
Spline Surfaces

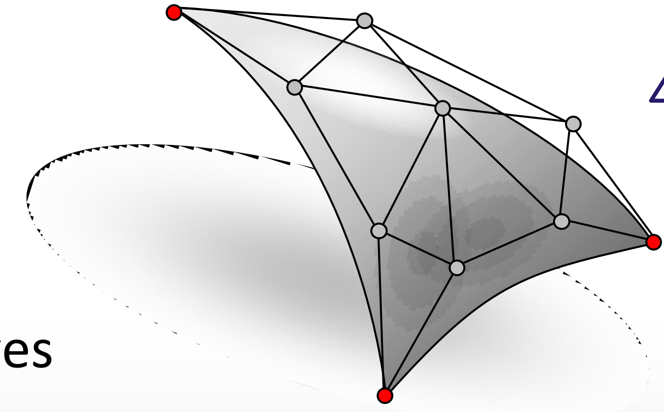
Two different approaches

- Tensor product surfaces
  - Simple construction
  - Everything carries over from curve case
  - Quad patches
  - Degree anisotropy
- Total degree surfaces
  - Not as straightforward
  - Isotropic degree
  - Triangle patches
  - “Natural” generalization of curves





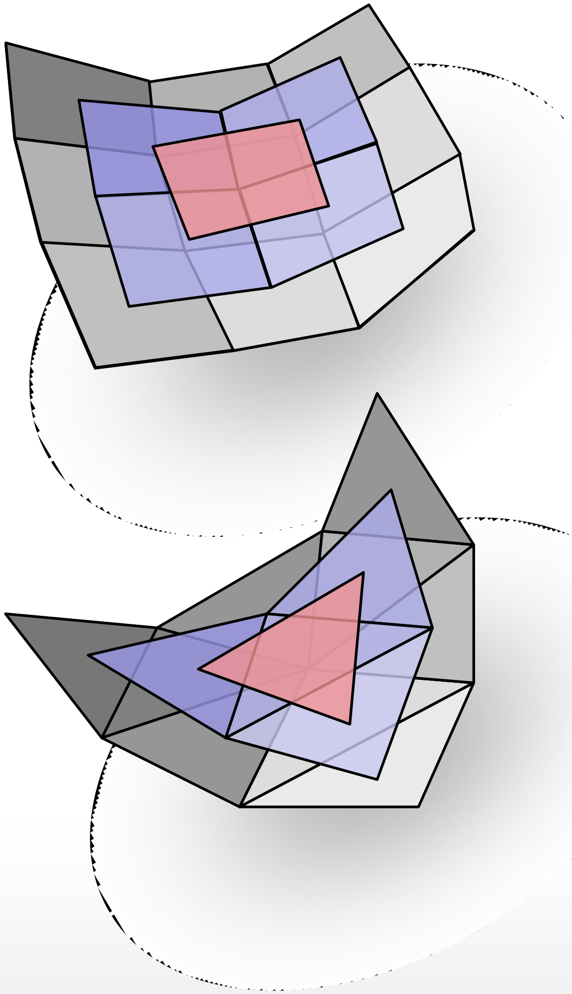




Bezier Triangles

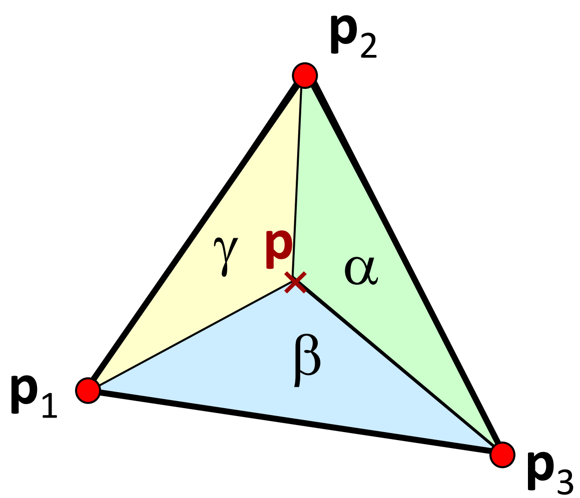
Alternative surface definition: Bezier triangles

- Constructed according to given total degree
  - Completely symmetric: No degree anisotropy
- Can be derived using a triangular de Casteljau algorithm
  - Barycentric interpolation



Barycentric Coordinates

Barycentric Coordinates:



- Planar case:
  Barycentric combinations of 3 points

  $$\mathbf{p} = \alpha\mathbf{p}_1 + \beta\mathbf{p}_2 + \gamma\mathbf{p}_3, \text{with} : \alpha + \beta + \gamma = 1$$

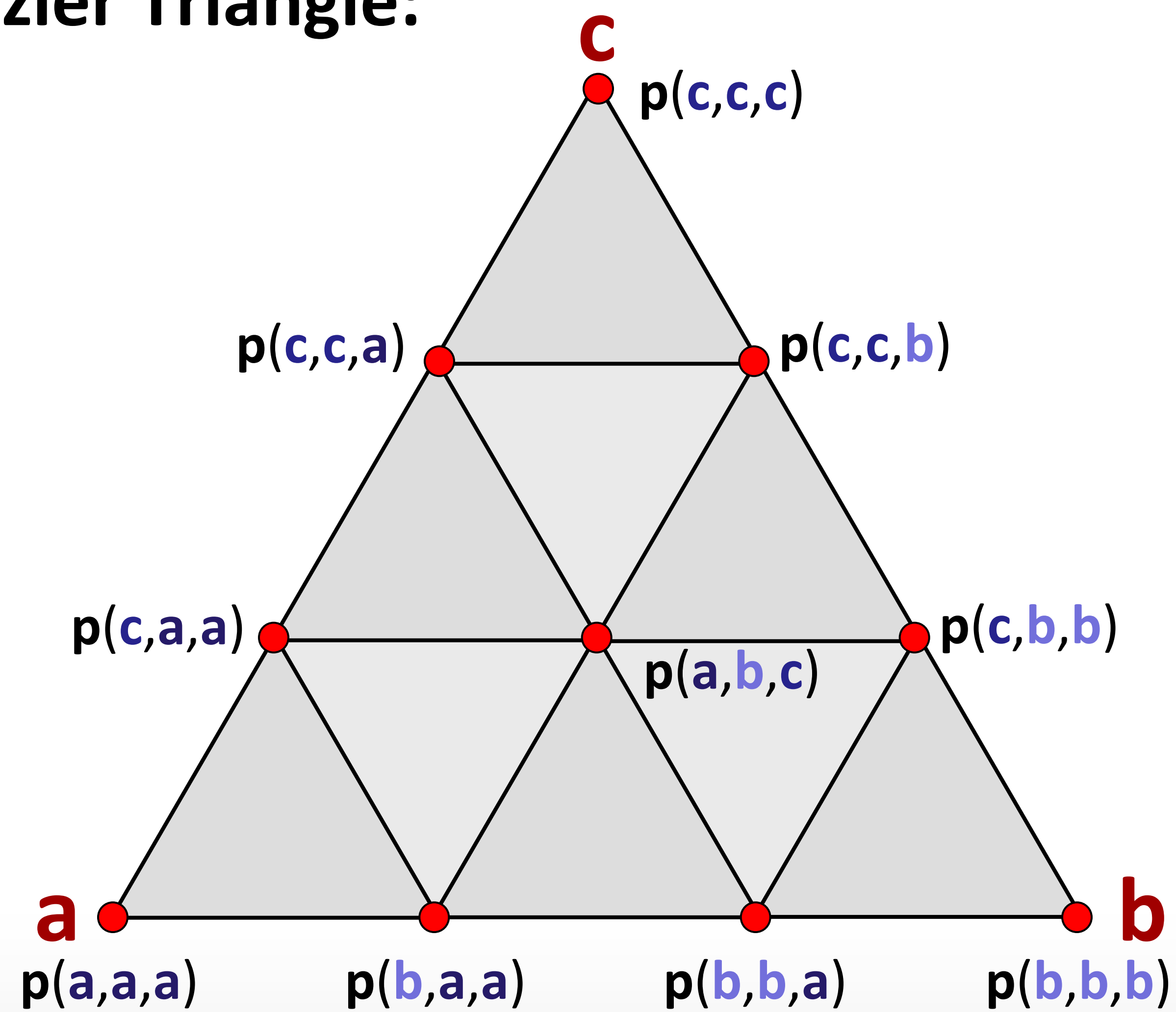
  $$\gamma = 1 - \alpha - \beta$$

- Area formulation:

  $$\alpha = \frac{area(\Delta(\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}))}{area(\Delta(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3))}, \beta = \frac{area(\Delta(\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}))}{area(\Delta(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3))}, \gamma = \frac{area(\Delta(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}))}{area(\Delta(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3))}$$

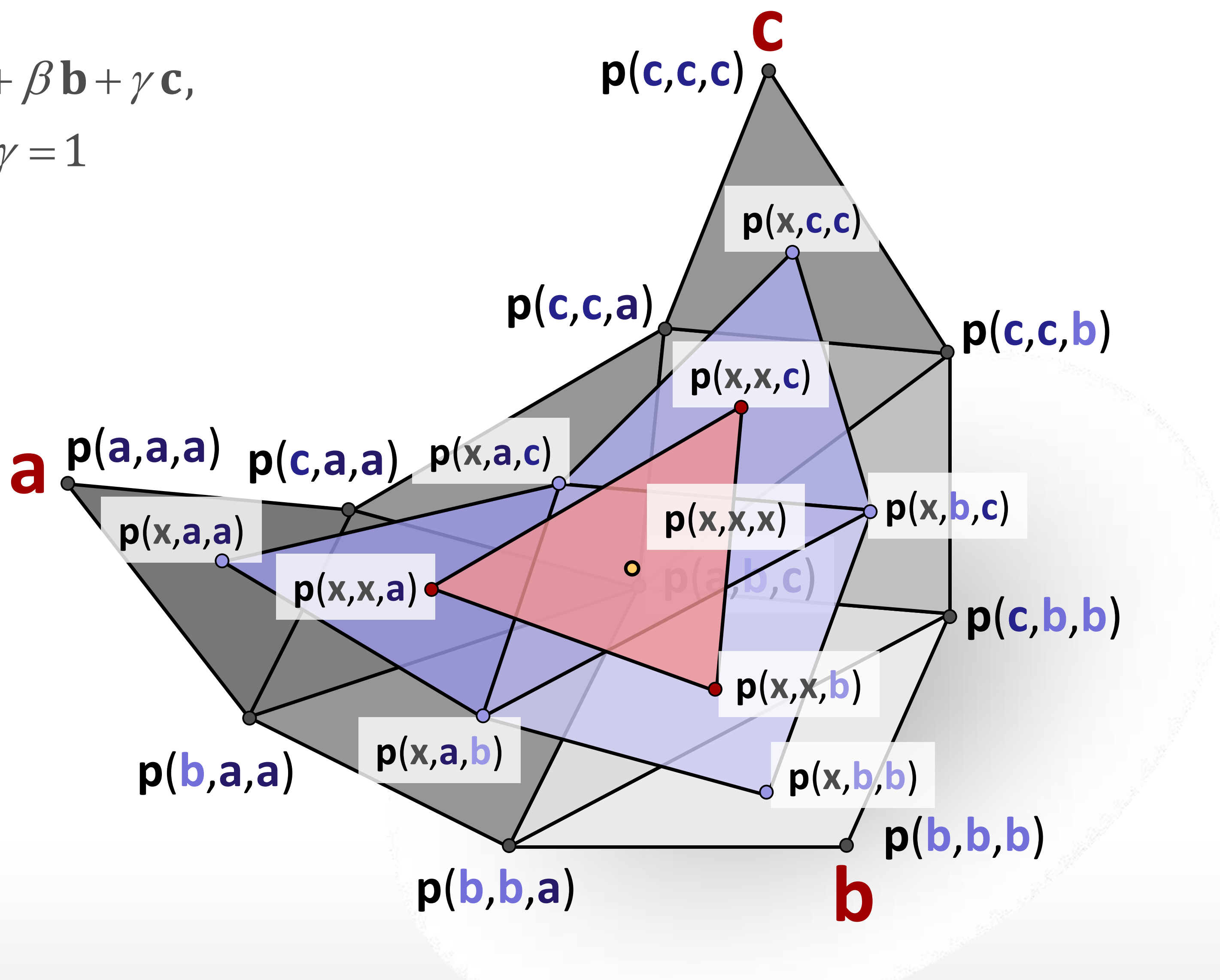
Example

Cubic Bezier Triangle:



De Casteljau Algorithm

$$\mathbf{x} = \alpha\,\mathbf{a} + \beta\,\mathbf{b} + \gamma\,\mathbf{c},$$
$$\alpha + \beta + \gamma = 1$$



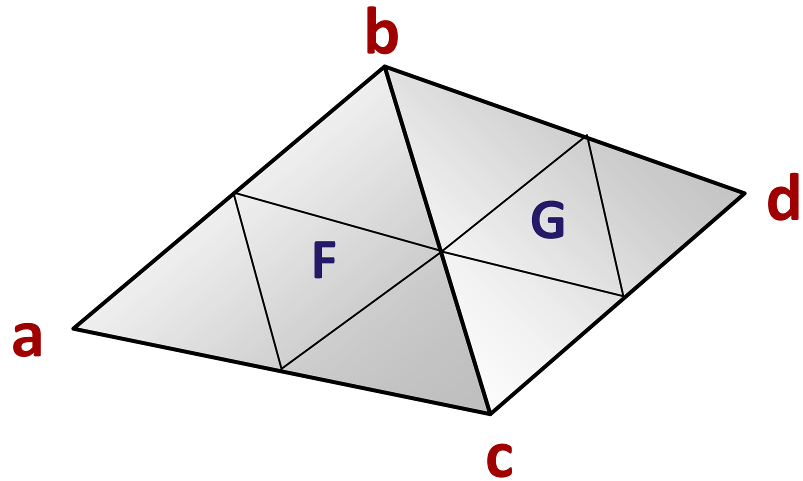
Continuity

We need to assemble Bezier triangles continuously:

- What are the conditions for C^0, C^1 continuity?
- As an example, we will look at the quadratic case...

Continuity

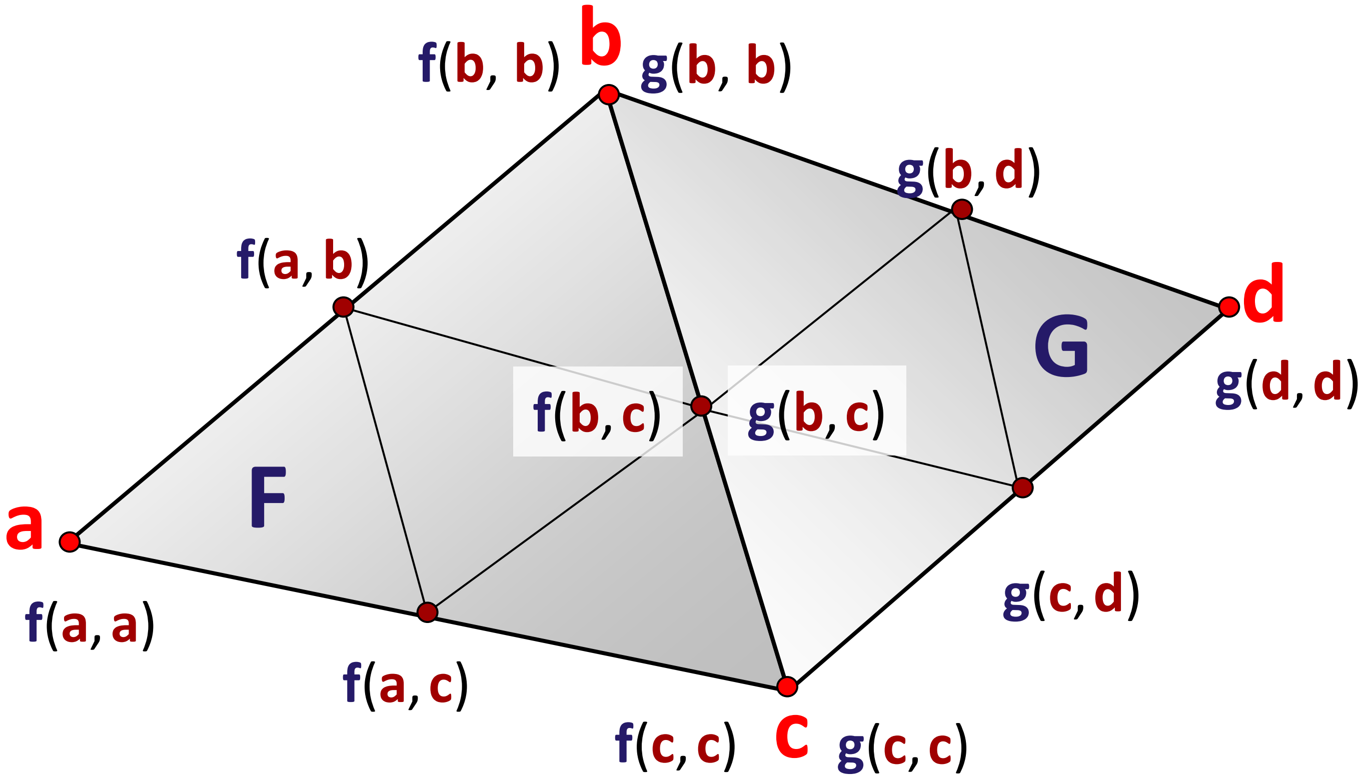
Situation:



- Two Bezier triangles meet along a common edge.
  - Parametrization: $T_1 = \{a, b, c\}$, $T_2 = \{c, b, d\}$
  - Polynomial surfaces $\mathbf{F}(T_1)$, $\mathbf{G}(T_2)$
  - Control points:
    - $\mathbf{F}(T_1)$: $f(a, a)$, $f(a, b)$, $f(b, b)$, $f(a, c)$, $f(c, c)$, $f(b, c)$
    - $\mathbf{G}(T_2)$: $g(d, d)$, $g(d, b)$, $g(b, b)$, $g(d, c)$, $g(c, c)$, $g(b, c)$

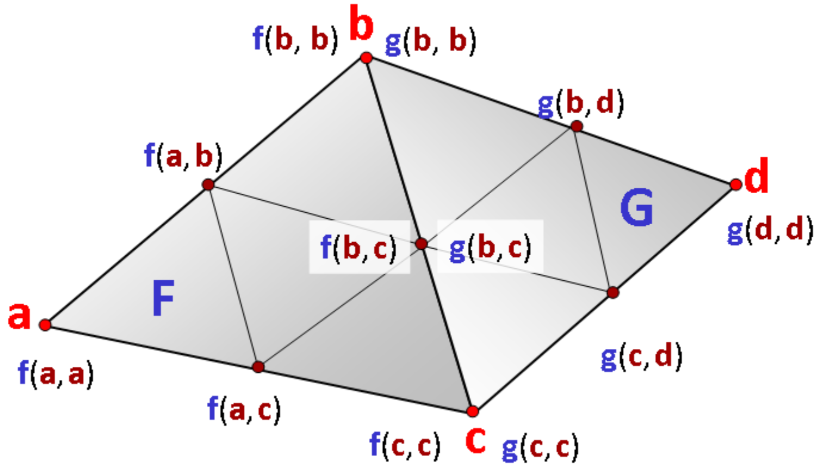
Continuity

Situation:



Continuity

C^0 Continuity:



- The points on the boundary have to agree:
  $\mathbf{f}(\mathbf{b},\mathbf{b}) = \mathbf{g}(\mathbf{b},\mathbf{b})$
  $\mathbf{f}(\mathbf{b},\mathbf{c}) = \mathbf{g}(\mathbf{b},\mathbf{c})$
  $\mathbf{f}(\mathbf{c},\mathbf{c}) = \mathbf{g}(\mathbf{c},\mathbf{c})$
- Proof: Let $\mathbf{x} := \beta\mathbf{b} + \gamma\mathbf{c},\ \beta + \gamma = 1$

$$\mathbf{f}(\mathbf{x},\mathbf{x}) = \beta\,\mathbf{f}(\mathbf{b},\mathbf{x}) + \gamma\,\mathbf{f}(\mathbf{c},\mathbf{x})$$

$$= \beta^2\,\underset{\mathbf{g}(\mathbf{b},\mathbf{b})}{\underset{\|}{\mathbf{f}(\mathbf{b},\mathbf{b})}} + 2\beta\gamma\,\underset{\mathbf{g}(\mathbf{b},\mathbf{c})}{\underset{\|}{\mathbf{f}(\mathbf{b},\mathbf{c})}} + \gamma^2\,\underset{\mathbf{g}(\mathbf{c},\mathbf{c})}{\underset{\|}{\mathbf{f}(\mathbf{c},\mathbf{c})}}$$

$$= \beta^2\,\mathbf{g}(\mathbf{b},\mathbf{b}) + 2\beta\gamma\,\mathbf{g}(\mathbf{b},\mathbf{c}) + \gamma^2\,\mathbf{g}(\mathbf{c},\mathbf{c})$$

$$= \beta\,\mathbf{g}(\mathbf{b},\mathbf{x}) + \gamma\,\mathbf{g}(\mathbf{c},\mathbf{x}) = \mathbf{g}(\mathbf{x},\mathbf{x})$$

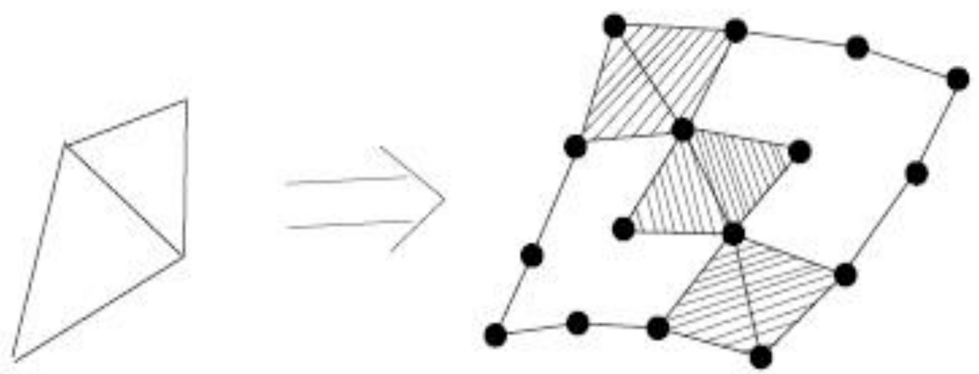
Continuity

C¹ Continuity:

- We need C^0 continuity.

In addition:

- Points at hatched quadrilaterals are coplanar
- Hatched quadrilaterals are an affine image of the same parameter quadrilateral



Curves on Surfaces, trimmed NURBS

Quad patch problem:

- All of our shapes are parameterized over rectangular or triangular regions
- General boundary curves are hard to create
- Topology fixed to a disc (or cylinder, torus)
- No holes in the middle
- Assembling complicated shapes is painful
  - Lots of pieces
  - Continuity conditions for assembling pieces become complicated
  - Cannot use C^2 B-Splines continuity along boundaries when using multiple pieces

Curves on Surfaces, trimmed NURBS

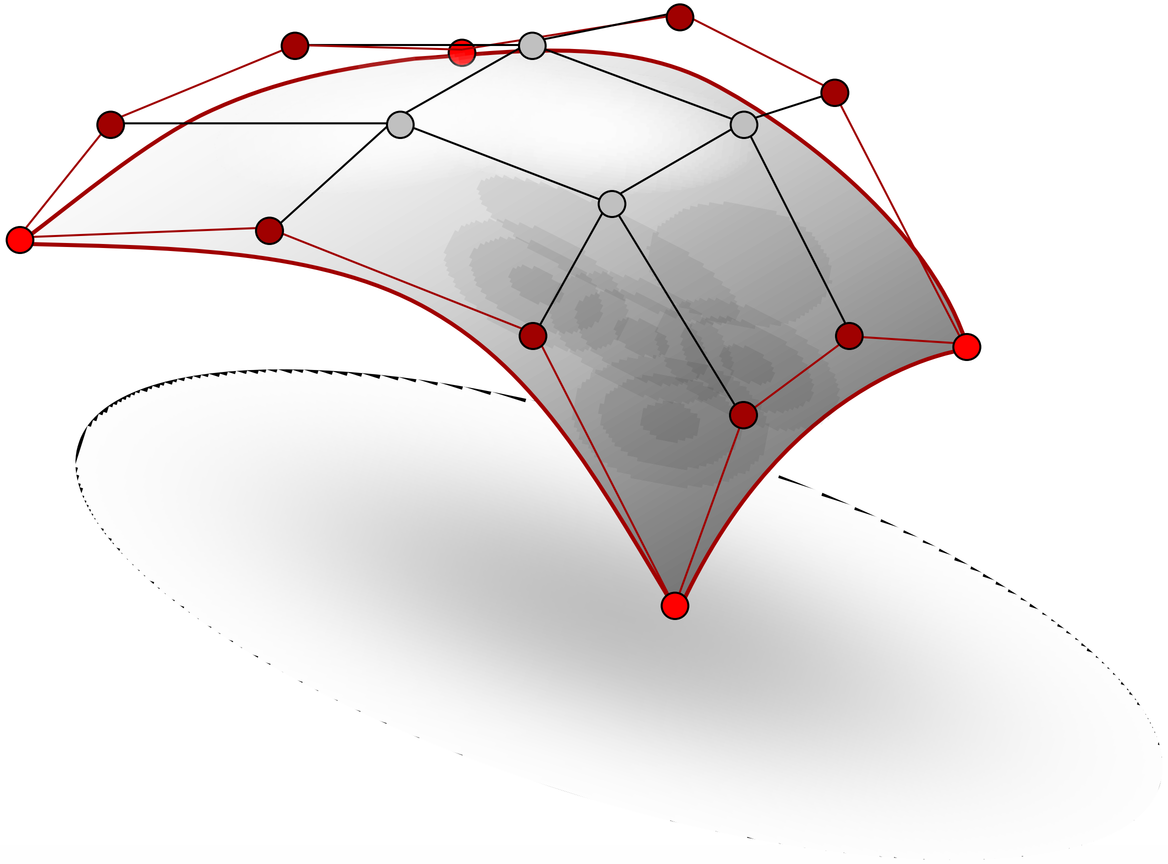
Consequence:

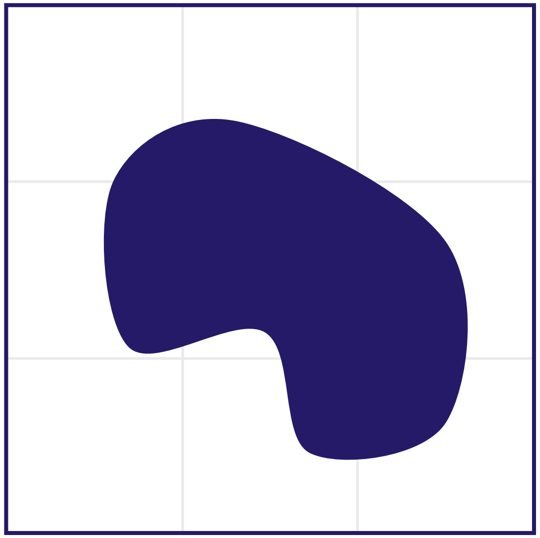
- We need more control over the parameter domain
- One solution is *trimming* using *curves on surfaces (CONS)*
- Standard tool in CAD: *trimmed NURBS*

Basic idea:

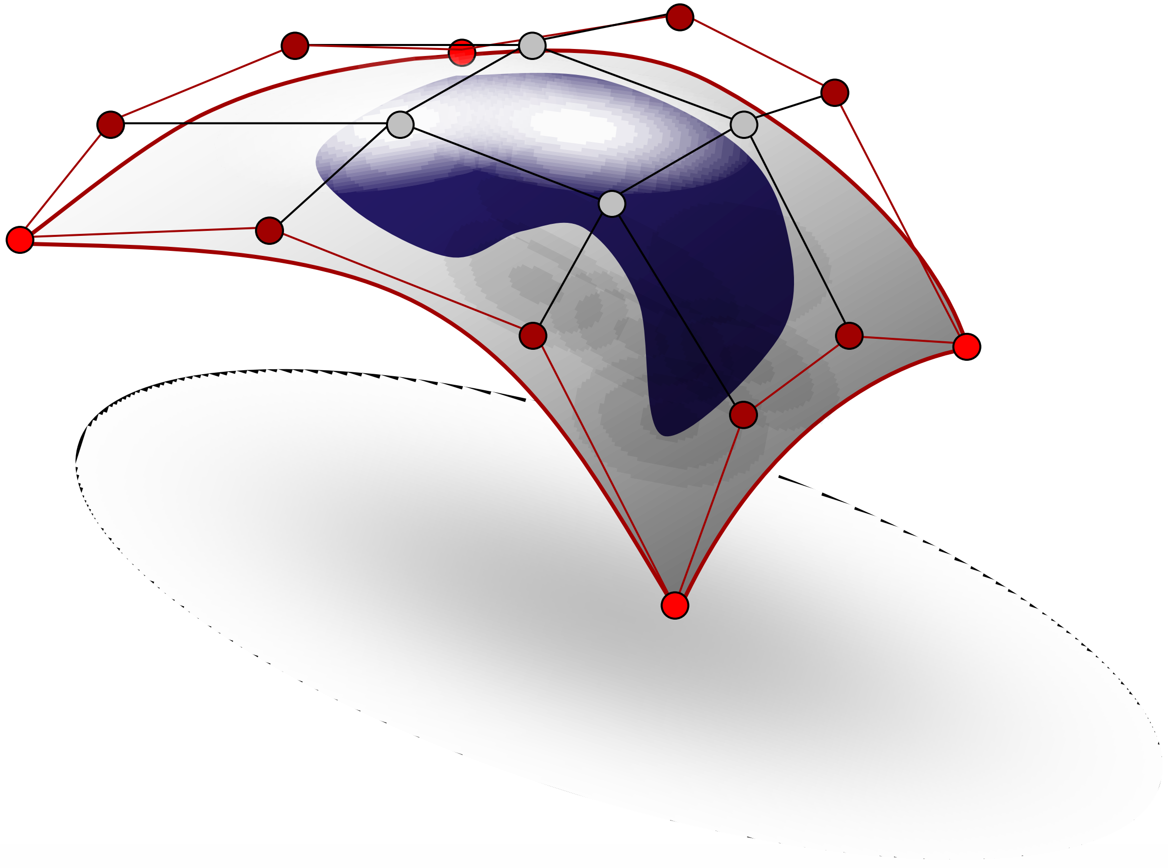
- Specify a curve in the parameter domain that encapsulates one (or more) pieces of area
- Tessellate the parameter domain accordingly to cut out the trimmed piece (rendering)

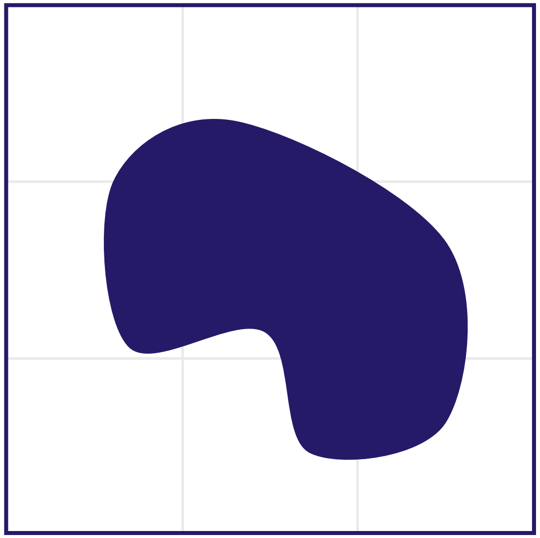
Curves-on-Surfaces (CONS)



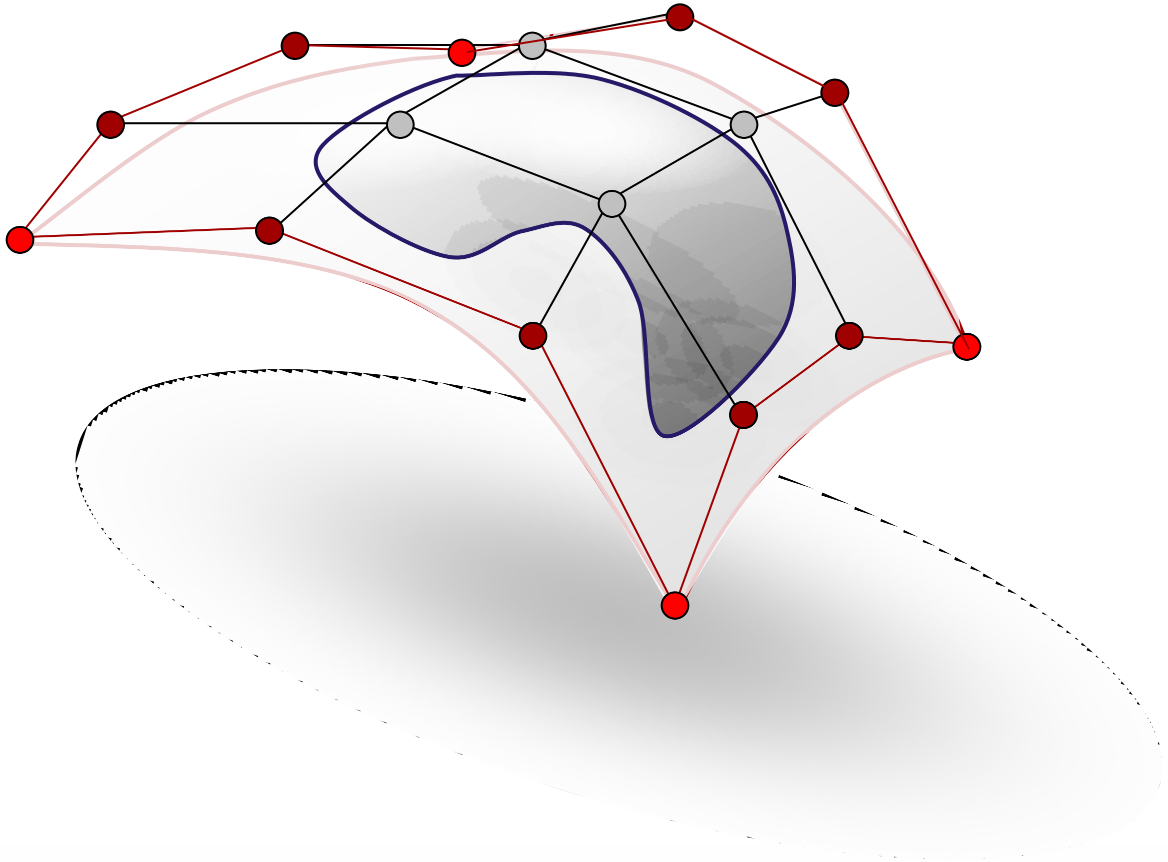


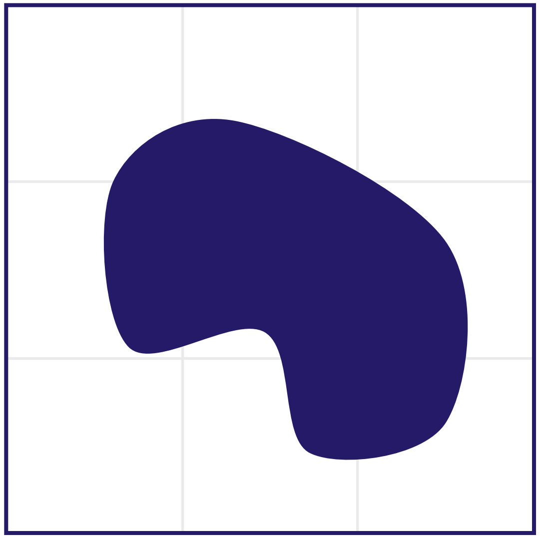
Curves-on-Surfaces (CONS)





Curves-on-Surfaces (CONS)





Summary

- Bezier Curves
  - de Casteljau algorithm
  - Bernstein form
- Bezier Splines
- Bezier Tensor Product Surfaces
- Bezier Total Degree Surfaces